Acyclic and Oriented Chromatic Numbers of Graphs

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Abstract

The oriented chromatic number $\chi_o(\vec{G})$ of an oriented graph $\vec{G} = (V, A)$ is the minimum number of vertices in an oriented graph $\vec{H}$ for which there exists a homomorphism of $\vec{G}$ to $\vec{H}$. The oriented chromatic number $\chi_o(G)$ of an undirected graph G is the maximum of the oriented chromatic numbers of all the orientations of G. This paper discusses the relation between the oriented chromatic number and the acyclic chromatic number and some other parameters of a graph. We shall give a lower bound for $\chi_o(G)$ in terms of $\chi_a(G)$. An upper bound for $\chi_o(G)$ in terms of $\chi_a(G)$ was given by Raspaud and Sopena. We also give an upper bound for $\chi_o(G)$ in terms of the maximum degree of G. This upper bound improves an earlier upper bound of the same kind. We shall show that this upper bound is not far from being optimal.

1 Introduction

A homomorphism f of a digraph $\vec{G} = (V, A)$ to a digraph $\vec{H} = (V', A')$ is a mapping of V to V' such that for any arc (x, y) of $\vec{G}$, the image $(f(x), f(y))$ is an arc of $\vec{H}$. Homomorphisms of undirected graphs are defined similarly. Homomorphisms of digraphs and undirected graphs have been studied as a generalization of graph coloring in the literature [6, 7, 8, 13]. It is easy to see that a k-coloring of an undirected graph G is equivalent to a homomorphism of G to the complete graph K_k. Therefore the chromatic number $\chi(G)$ of an undirected graph G is equal to the minimum integer k such that there is a homomorphism of G to K_k, or equivalently $\chi(G)$ is equal to the minimum number of vertices of an undirected graph H for which there exists a homomorphism of G to H.

We call those digraphs which contain no opposite arcs *oriented graphs*. Generalizing the above definition of chromatic number to oriented graphs, we define the oriented chromatic number of an oriented graph $\vec{G}$ to be the minimum number of vertices in an oriented graph $\vec{H}$ for which there exists a homomorphism of $\vec{G}$ to $\vec{H}$. We denote this number by $\chi_o(\vec{G})$.

The oriented chromatic number of an oriented graph $\vec{G}$ can also be defined as follows: For an integer k, a k-coloring of $\vec{G}$ is an assignment c of $\vec{G}$ such that for every arc (x, y) of $\vec{G}$ the following is true: (1) $c(x) \neq c(y)$; (2) there is no arc (z, t) of $\vec{G}$ for which $c(t) = c(x)$ and $c(z) = c(y)$. The oriented chromatic number $\chi_o(\vec{G})$ of $\vec{G}$ is the minimum k for which there exists a k-coloring of $\vec{G}$.

For an undirected graph G, let $\mathcal{O}_G$ be the set of all orientations of G. We define the oriented chromatic number $\chi_o(G)$ of G to be the maximum of $\chi_o(\vec{G})$ over all orientations $\vec{G}$ of G, i.e., $\chi_o(G) = \max\{\chi_o(\vec{G}) : \vec{G} \in \mathcal{O}_G\}$.

Graph coloring problems usually can be transformed into graph orientation problems. For example, a graph G is k-colorable if and only if there is an acyclic orientation of G in which the longest directed path has length at most k. The class of perfect graphs can be characterized by the existence of "good" orientations. The star-chromatic number of a graph, the fractional chromatic number of a graph G can also be obtained by finding some optimal orientations of G. Recently, Galvin [4] solved the problem of list chromatic index of bipartite multigraphs by showing the existence of certain orientations of the line graph of a bipartite multigraph. All such efforts are to find certain optimal orientations of a graph. In contrast to this, the concept of the oriented chromatic number of a graph deals with the 'worst' orientation of a graph. It is natural to ask how 'bad' could be an orientation of a graph G (here the 'badness' is measured by the oriented chromatic number), if the graph G itself has some 'good' property such as having small chromatic number, bounded genus, bounded maximum degree, bounded acyclic chromatic number, bounded arboricity or bounded treewidth, etc.

Such problems were first studied by B. Courcelle [3]. It was shown in [3] that every planar oriented graph with $d^-(x) \leq 3$ for every vertex x has a *semi-strong* oriented coloring using at most $4^3 \times 3^{63}$ colors. (cf. [3] or [14] for the definition of semi-strong oriented coloring). This result was improved by Raspaud and Sopena [14] who showed that every planar graph has oriented chromatic number at most 80, and every planar oriented graph with $d^-(x) \leq 3$ for every vertex x has a *semi-strong* oriented coloring using at most 320 colors. It was also proved in [14] that if a graph G has acyclic chromatic number k then it has oriented chromatic number at most $k \cdot 2^{k-1}$. For graphs with bounded treewidth, Sopena [15] showed that any partial k-tree has oriented chromatic number at most $(k+1)2^k$. It was also shown in [15] that any graph with maximum degree k has oriented chromatic number at most $(2k-1)2^{2k-2}$. For graphs of treewidths 2 and 3, their oriented chromatic numbers are at most 7 and 16, respectively.

We shall also study the relation between the oriented chromatic number and other parameters of a graph, including the chromatic number, the arboricity, the acyclic chromatic number, the maximum degree, the genus, etc. First we note that it was shown by Albertson and Berman [1] that any graph of genus $n > 0$ can be acyclically colored with $4n + 4$ colors. Combining this result with the above mentioned result of Raspaud and Sopena [14], we obtain:

Theorem 1 *Any graph of genus $n > 0$ has oriented chromatic number at most $(4n+4)2^{4n+3}$.*

For the relation between the oriented chromatic number and the chromatic number of a graph, we note that it follows from the definition that the oriented chromatic number of a graph G is at least the chromatic number of G. However, bipartite graphs may have arbitrarily large oriented chromatic numbers. Indeed, let $G = (A, B, E)$ be the complete bipartite graph with two parts $A = \{a_1, a_2, \cdots, a_n\}$ and $B = \{b_1, b_2, \cdots, b_n\}$. We orient the edge (a_i, b_j) from a_i to b_j if $i \le j$ and from b_j to a_i if $i > j$. It is straightforward to verify that this orientation of G has oriented chromatic number $2n$.

The arboricity of a graph G is the minimum number k such that the edges of G can be decomposed into k forests. It was shown by Nash-Williams [11] that for any graph G the arboricity $arb(G)$ of G is equal to the maximum of $\lceil e(H)/(v(H)-1)\rceil$ over all subgraphs H of G where $v(H), e(H)$ are the number of vertices and the number of edges of H respectively. For the relation between the oriented chromatic number and the arboricity of a graph, we note here that graphs of arboricity 2 could have arbitrarily large oriented chromatic number (cf. Observation 2 in Section 2). Of course, forests have oriented chromatic number at most 3, cf. [14].

On the other hand, we shall show in Section 2 that any graph with oriented chromatic number k has arboricity at most $\log_2 k + k/2$ (cf. Theorem 2).

The main result of this paper, Theorem 4 proved in Section 2, gives an upper bound for the acyclic chromatic number of a graph G in terms of the oriented chromatic number of G. An upper bound for the oriented chromatic number of a graphs G in terms of the acyclic chromatic number was given by Raspaud and Sopena [14] as mentioned above. Therefore any class of graphs has bounded acyclic chromatic number if and only if it has bounded oriented chromatic number.

In Section 5, we consider the relation between the oriented chromatic number of a graph and its maximum degree. We shall give an upper bound for the oriented chromatic number of a graph in terms of its maximum degree, which improves the previous known upper bound [15]. We shall show that our upper bound is not far from being optimal.

Given a class $\mathcal{C}$ of graphs, we say that $\mathcal{C}$ is χ_a-bounded if there exists an integer n such that $\chi_a(G) \le n$ for all $G \in \mathcal{C}$. Similarly we say $\mathcal{C}$ is χ_o-bounded if there exists an integer n such that $\chi_o(G) \le n$ for all $G \in \mathcal{C}$. It follows from the result of Raspaud and Sopena [14] and the result in Section

2 that a class of graph is χ_a-bounded if and only if it is χ_o-bounded. A few classes of graphs, such as graphs of genus at most m, graphs of maximum degree at most k, etc. are known to be χ_a-bounded. In Section 4, we shall construct some more classes of graphs which are χ_a-bounded.

2 Acyclic chromatic number

The acyclic chromatic number $\chi_a(G)$ of a graph G is the least integer n for which there is a coloring of the vertices of G with n colors in such a way that each color class is an independent set, and the union of any two color classes induces a forest. In this section, we discuss the relation among the oriented chromatic number, the arboricity and the acyclic chromatic number of a graph. We have observed in the previous section that graphs of arboricity 2 could have arbitrarily large oriented chromatic number. We now show that graphs of bounded oriented chromatic number must have bounded arboricity.

Theorem 2 *If a graph G has oriented chromatic number k, then the arboricity $arb(G)$ of G is at most $\lceil \log_2 k + k/2 \rceil$.*

This theorem will easily follow from the observation below:

Observation 1 *If a graph G' has n vertices, m edges and $\chi_o(G') \leq k$, then*

$$2^{\binom{k}{2}} k^n \geq 2^m.$$

That is,

$$\log_2 k \geq m/n - \binom{k}{2}/n. \tag{1}$$

Proof. Consider G' as a labelled graph. Then there are 2^m different orientations of G'. On the other hand, each k-coloring, i.e., each partition of the vertices of G' into k classes, is compatible with at most $2^{\binom{k}{2}}$ orientations of edges of G', and the number of possible k-colorings is less than k^n. Therefore $2^m \leq 2^{\binom{k}{2}} \cdot k^n$. ∎

We now prove Theorem 2. By Nash-Williams' Theorem, it suffices to show that for any subgraph G' of G, $e(G')/(v(G')-1) \leq \log_2 k + k/2$. Thus we let G' be an arbitrary subgraph of G. If $v(G') \leq k$, then $e(G')/(v(G')-1) \leq v(G')/2 \leq k/2$. Now we assume that $v(G') > k$. Since $\chi_o(G') \leq \chi_o(G) \leq k$, it follows from Observation 1 that $\log_2 k \geq \frac{e(G')}{v(G')} - \frac{k(k-1)}{2v(G')} \geq \frac{e(G')}{v(G')-1} - \frac{e(G')}{v(G')(v(G')-1)} - \frac{k-1}{2} \geq \frac{e(G')}{v(G')-1} - 1/2 - k/2 + 1/2 \geq \frac{e(G')}{v(G')-1} - k/2$. Therefore $\frac{e(G')}{v(G')-1} \leq \log_2 k + k/2$. Theorem 2 is proved.

Next we give an upper bound of the acyclic chromatic number of a graph G in terms of its oriented chromatic number and its arboricity.

Theorem 3 *If a graph G has oriented chromatic number k and arboricity q, then G has acyclic chromatic number at most $k^{\lceil \log_2 q \rceil + 1}$.*

Proof. Since the arboricity of G is q, there are q forests $T_1, \ldots T_q$ covering $E(G)$.

Let $s = \lceil \log_2 q \rceil$ and let $\mathbf{a_i}$ be binary form of the number i, $i = 1, \ldots, q$. For $j = 1, \ldots, s$, let

$$M_j = \{\bigcup E(T_i) \mid \text{ the } j\text{-th digit of } \mathbf{a_i} \text{ is } 1\}.$$

Then for any $i_1 \neq i_2$, $i_1, i_2 \leq q$, there exists some M_j which contains one of $E(T_{i_1})$ and $E(T_{i_2})$, and disjoint from the other.

Let $v_1, v_2, \cdots, v_n$ be an arbitrary enumeration of the vertices of G. We say v_i is less than (or proceeds) v_j if $i < j$. Let $\vec{G}_0$ be the orientation of G in which an edge (v_i, v_j) is oriented from v_i to v_j if and only if $i < j$. For $j = 1, \ldots, s$, let $\vec{G}_j$ be the orientation of G obtained from $\vec{G}_0$ by reversing the orientations of those edges contained in M_j. Since the graph G has oriented chromatic number k, for each $j = 0, 1, \cdots, s$, there is an oriented coloring φ_j of $\vec{G}_j$ with k colors.

We claim that the coloring φ^* of the vertices of G, defined as $\varphi^*(v) = (\varphi_0(v), \varphi_1(v), \ldots, \varphi_s(v))$, is an acyclic coloring of G. Obviously adjacent vertices of G are colored with distinct colors by φ^*. Suppose φ^* is not an acyclic coloring, then there exists a 2-colored cycle $C = [c_1, c_2, \cdots, c_{2m}]$. Then C is 2-colored in each of the coordinate colorings φ_j for $j = 0, 1, \cdots, s$.

Observe that if three vertices x, y, z form a directed path $x \to y \to z$, then all the three vertices must be colored with distinct colors in any oriented coloring. Therefore the edges of the cycle C form alternating cycles in all the orientations $\vec{G}_j$ of G. In particular, the edges of C forms an alternating cycle in $\vec{G}_0$. Thus without loss of generality, we may assume that $c_1 < c_2 > c_3 < c_4 \cdots c_{2m-1} < c_{2m} > c_1$ (cf. the definition of $\vec{G}_0$).

Since each T_j is a forest, there are two adjacent edges of C, say $(c_i, c_{i+1}), (c_{i+1}, c_{i+2})$, that belongs to distinct forests. Suppose $(c_i, c_{i+1}) \in E(T_a)$ and $(c_{i+1}, c_{i+2}) \in E(T_b)$. As we have observed before, there is an M_j such that M_j contains all the edges of T_a and contains no edge of T_b. We assume that $(c_{i+1}, c_{i+2}) \in M_j$ and $(c_i, c_{i+1}) \notin M_j$. Then the orientation of the edge (c_{i+1}, c_{i+2}) in $\vec{G}_j$ is the reverse of its orientation in $\vec{G}_0$, while the orientation of the edge (c_i, c_{i+1}) in $\vec{G}_j$ is the same as its orientation in $\vec{G}_0$. This implies that the two edges (c_i, c_{i+1}) and (c_{i+1}, c_{i+2}) form a directed path in $\vec{G}_j$, contrary to our observation in the previous paragraph. This proves that φ^* is indeed an acyclic coloring of G. The number of colors used by φ^* is $k^{\lceil \log_2 q \rceil + 1}$. Therefore the acyclic chromatic number of G is at most $k^{\lceil \log_2 q \rceil + 1}$. ∎

Combining Theorem 2 and Theorem 3, we obtain an upper bound for the acyclic chromatic number of a graph in terms of its oriented chromatic number:

Corollary 1 *If a graph G has oriented chromatic number k, then its acyclic chromatic number is at most $k^{\lceil \log_2(\lceil \log_2 k \rceil + k/2) \rceil + 1}$.*

However a more careful analysis of the proofs of the above two theorems yields a better upper bound.

Theorem 4 *Let G be a graph with the oriented chromatic number $o(G) \le k$. If $k \ge 4$, then $\chi_a(G) \le k^2 + k^{3 + \lceil \log_2 \log_2 k \rceil}$.*

Proof. Let t to be the maximum real number such that there exists $G' \subseteq G$ with $|V(G')| \ge k^2$ and

$$|E(G')| \ge t|V(G')|. \tag{2}$$

Let G'' be the biggest subgraph of G with $|E(G'')| > t|V(G'')|$. By the definition of t,

$$|V(G'')| < k^2.$$

Denote $G_0 = G - G''$. Clearly, $\chi_a(G) \leq \chi_a(G_0) + k^2$.

By the maximality of G'', we have

$$|E(H)| \leq t|V(H)| \qquad \text{for each } H \subset G_0.$$

If $t \leq \frac{|V(H)|-1}{2}$, then $|E(H)| \leq (t+1/2)(|V(H)|-1)$. If $t > \frac{|V(H)|-1}{2}$, then $\frac{|V(H)|}{2} < t + \frac{1}{2}$. Thus $|E(H)| \leq \frac{(|V(H)|-1)|V(H)|}{2} \leq (t+1/2)(|V(H)|-1)$. Therefore

$$|E(H)| \leq (t+1/2)(|V(H)|-1) \qquad \text{for each } H \subset G_0,$$

and by Nash-Williams' theorem, there exist $q = \lceil t + 1/2 \rceil$ forests $T_1, \ldots T_q$ covering $E(G_0)$.

Similarly to the proof of Theorem 3, we can show that $\chi_a(G_0) \leq k^{s+1}$.

It follows from Observation 1 that $\log_2 k \geq t - 1/2$. Therefore

$$s = \lceil \log_2(\lceil t + 1/2 \rceil) \rceil \leq \lceil \log_2(1 + \lceil \log_2 k \rceil) \rceil \leq 2 + \log_2 \log_2 k.$$

Hence $\chi_a(G) \leq k^2 + k^{3+\lceil \log_2 \log_2 k \rceil}$. ∎

We close this section with two observations:

Observation 2 [10] *Graphs of arboricity 2 could have arbitrarily large acyclic chromatic numbers.*

We just recall here the construction proposed in [10]. Let G be the graph obtained from K_n by replacing each edge of K_n by $n-1$ parallel paths of length 2. It is easy to see that G has arboricity 2. To see that G has acyclic chromatic number at least n observe that if the vertices of G are colored by $n-1$ colors, then two of the vertices of K_n are colored by the same color. Among the $n-1$ paths of length 2 connecting these two vertices there are two paths whose middle vertices are colored with the same color. Therefore we obtain a 2-colored cycle C_4.

Observation 3 *Graphs of acyclic chromatic number k could have oriented chromatic number greater than $2^{k-1}-1$.*

Let $G=(V,E)$ be a graph such that $V=V_1\cup V_2\cup\cdots\cup V_k$, where $V_1,V_2,\cdots V_k$ are disjoint sets of cardinality p, $G(V_i)$ has no edges for each i, and $G(V_i\cup V_j)$ is a path of length $2p-1$ for each $\{i,j\}$. Then $v(G)=kp, e(G)=\binom{k}{2}(2p-1)$. It follows from the construction that $\chi_a(G)=k$. Suppose the oriented chromatic number of G is t. Then by Observation 1

$$\log_2 t \geq \binom{k}{2}(2p-1)/kp - \binom{t}{2}/kp. \tag{3}$$

$$\log_2 t \geq k-1-\frac{k-1}{2p} - \binom{t}{2}/kp. \tag{4}$$

When p is sufficiently large, we have $t>2^{k-1}-1$.

3 Maximum degree and oriented chromatic number

It is known that graphs of maximum degree k have acyclic chromatic number at most $\mathcal{O}(k^{\frac{4}{3}})$, [2]. Since graphs of acyclic chromatic number m have oriented chromatic number at most $m2^{m-1}$, (cf. [14]), it follows that graphs of maximum degree k have oriented chromatic number at most $\mathcal{O}(k^{\frac{4}{3}})2^{\mathcal{O}(k^{\frac{4}{3}})}=2^{\mathcal{O}(k^{\frac{4}{3}})}$. A better upper bound for the oriented chromatic number in terms of the maximum degree was proved in [15]. It was shown in [15] that graphs of maximum degree k have oriented chromatic number at most $(2k-1)2^{2k-2}$. We shall prove in this section that graphs of maximum degree k have oriented chromatic number at most $2k^2 2^k$. This upper bound seems to be not too far from the optimal upper bound. We shall show that for each integer $k>1$, there is a graph of maximum degree k whose oriented chromatic number is at least $2^{k/2}$.

Theorem 5 *If G is a graph of maximum degree k, then its oriented chromatic number $\chi_o(G)$ is at most $2k^2 2^k$.*

To prove this theorem, we shall first prove the existence of a tournament which satisfies certain property. Suppose $\vec{G}$ is an oriented graph, and that $I = \{x_1, ..., x_i\}$ is a subset of $V(\vec{G})$, and that v is vertex of $\vec{G}$ which is adjacent to each vertex in I. We denote by $F(I, v, \vec{G})$ the vector $\mathbf{a}$ of length i, where the j-th coordinate of $\mathbf{a}$ equals 1 if (x_j, v) is an arc of $\vec{G}$ and equals -1 if (v, x_j) is an arc of $\vec{G}$.

Lemma 1 *Let $k \geq 5$ be an integer. There exists a tournament $T = (V, A)$ on $t = 2k^2 2^k$ vertices with the following property:*
for each i, $0 \leq i \leq k$, for each $I \subset V$ with $|I| = i$, and for each ± 1-vector $\mathbf{a}$ of length i, there exist at least $1 + (k-i)(k-1)$ vertices v in $V \setminus T$ with $F(I, v, T) = \mathbf{a}$.

Proof. Consider a random tournament $T = (V, A)$ on t vertices, where for each pair $\{v, w\}$ of vertices of T, the events that $(v, w) \in A$ and that $(w, v) \in A$ are complementary and equiprobable, and for distinct pairs of vertices corresponding events are independent.

Let $I \subset V$, $|I| = i$ and $\mathbf{a}$ be a ± 1-vector of length i. Let $\mathbf{P}(i, I, \mathbf{a})$ denote the probability of the event $R(i, I, \mathbf{a})$ that the number of vertices in $V \setminus T$ with $F(I, v, T) = \mathbf{a}$ is at most $(k-i)(k-1)$.

For a fixed $v \in V \setminus T$, the probability that $F(I, v, T) = \mathbf{a}$ is equal to 2^{-i} and for distinct $v, w \in V \setminus T$, the events that $F(I, v, T) = \mathbf{a}$ and that $F(I, w, T) = \mathbf{a}$ are independent. Hence

$$\mathbf{P}(i, I, \mathbf{a}) = \sum_{j=0}^{(k-i)(k-1)} \binom{t-i}{j} 2^{-ij} (1 - 2^{-i})^{t-i-j} <$$

$$< (1 - 2^{-i})^t \sum_{j=0}^{(k-i)(k-1)} \frac{t^j}{j!} (1 - 2^{-i})^{-i-j} 2^{-ij} <$$

$$< 2e^{-t2^{-i}} \sum_{j=0}^{(k-i)(k-1)} t^j < e^{-t2^{-i}} t^{(k-i)(k-1)+1}.$$

The probability $\mathbf{P}(T)$ that at least one of $R(i, I, \mathbf{a})$ occurs is at most

$$\sum_{i=0}^{k} \sum_{\{I \subset V | |I|=i\}} \sum_{\mathbf{a}} < \sum_{i=0}^{k} \binom{t}{i} 2^i e^{-t2^{-i}} t^{(k-i)(k-1)+1} \leq$$

$$2 \sum_{i=0}^{k} e^{-t2^{-i}} t^{(k-i)(k-1)+1+i}. \tag{5}$$

The ratio of the $(i+1)$-th summand over i-th summand in (5) is

$$\frac{e^{t2^{-i}} t^{(k-i-1)(k-1)+1+i+1}}{e^{t2^{-i-1}} t^{(k-i)(k-1)+1+i}} = \frac{e^{t2^{-i-1}}}{t^{k-2}} \geq \frac{e^{t2^{-k}}}{t^{k-2}}.$$

But for $t = 2k^2 2^k$,

$$\frac{e^{t2^{-k}}}{t^{k-2}} = \frac{e^{2k^2}}{(k^2 2^{k+1})^{k-2}} \geq \left(\frac{e^{2k}}{k^2 2^k}\right)^k > \left(\frac{e^k}{k^2}\right)^k > 2.$$

Thus,

$$\mathbf{P}(T) < 2e^{-t2^{-k}} t^{1+k} \leq 2e^{-2k^2} \left(\frac{e 2k^2 2^k}{1+k}\right)^{1+k} \leq 2 \left(\frac{(6k)^{(k+1)/k} 2_{k+1}}{e^{2k}}\right)^k <$$

$$< 2 \left(\frac{(6k)^{1+1/k}}{e^k}\right)^k < 1/2$$

for $k \geq 5$. Consequently, there exists T for which no one of $R(i, I, \mathbf{a})$ occurs. ∎

We now prove Theorem 5. Let $\vec{G}$ be an oriented graph of maximum degree k. We need to prove that $\chi_o(G) \leq 2 \cdot k^2 \cdot 2^k$. For $k \leq 4$, the result follows from Theorem 4.1 of [15]. Assume now that $k \geq 5$. By Lemma 1, there is a tournament $T = (V, A)$ such that for each i, $0 \leq i \leq k$, for each $I \subset V$ with $|I| = i$, and for each ± 1-vector $\mathbf{a}$ of length i, there exist at least $1 + (k-i)(k-1)$ vertices v in $V \setminus T$ with $F(I, v, T) = \mathbf{a}$. We shall prove that there exists a homomorphism f of $\vec{G}$ to T. Suppose the vertices of G are $v_1, v_2, \cdots, v_n$. We shall define $f(v_1), f(v_2), \cdots, f(v_n)$ recursively such that at step m, the images $f(v_1), f(v_2), \cdots, f(v_m)$ are defined, and

(i) the partial mapping $f(v_1), f(v_2), \cdots, f(v_m)$ is a partial homomorphism; and

(ii) for each v_j with $j > m$, all the neighbours v_s of v_j with $s \leq m$ have different images under the partial mapping f.

Step 1 is trivial. Suppose that $f(v_1), f(v_2), \cdots, f(v_m)$ are defined such that (i) and (ii) hold. We shall call $v_1, v_2, \cdots, v_m$ the colored vertices, and call $f(v_1), f(v_2), \cdots, f(v_m)$ their colors. We need to define $f(v_{m+1})$ (i.e., to color v_{m+1}) so that (i) and (ii) still hold. Suppose $y_1, \ldots y_i$ are colored neighbours of v_{m+1}, and that $F(\{y_1, \ldots y_i\}, v_{m+1}, G) = \mathbf{a}$. By (ii), the set I of colors $\{f(y_1), f(y_2), \cdots f(y_i)\}$ also has cardinality i. Let K be the set of vertices w in $V(T) \setminus I$ such that $F(\{f(y_1), f(y_2), \cdots f(y_i)\}, w, T) = \mathbf{a}$. By Lemma 1, $|K| \geq 1 + (k-i)(k-1)$. Let A be the set of uncolored neighbours of v_{i+1}, and let B be the set of colored neighbours of vertices in A. Then $|A| \leq k - i$, and $|B| \leq (k-i)(k-1)$. Therefore $K \setminus f(B) \neq \emptyset$. Let $f(v_{m+1})$ be any vertex in $K \setminus f(B)$, it is straightforward to verify that (i) and (ii) still hold. This completes the proof of Theorem 5.

We close this section with an observation that the upper bound given in Theorem 5 is not too far from being optimal.

Observation 4 *For each integer $k > 1$, there exists an oriented graph $\vec{G}$ of maximum degree k for which $\chi_o(\vec{G}) \geq 2^{k/2}$.*

Proof. Let G be an k-regular graph on n vertices. Then it has $kn/2$ edges. By Observation 1, $\chi_o(G) \geq 2^{k/2} \cdot 2^{\binom{k}{2}/n}$. If n is sufficiently big, say $n \geq 100^k$, we then have $\chi_o(G) \geq 2^{k/2}$. ∎

4 χ_a-bounded classes of graphs

There are a few classes of graphs known to have bounded acyclic chromatic numbers: Albertson and Berman [1] showed that the class of graphs of bounded genus is χ_a-bounded; it is trivial that the class of graphs of bounded maximum degree is χ_a-bounded; Sopena [15] proved that the class of graphs of bounded treewidth is χ_o-bounded. We present here a method of constructing new χ_a-bounded classes of graphs from old ones.

For two graphs G_1, G_2 on the same vertex set, we denote by $G_1 + G_2$ the graph with vertex set $V(G) = V(G_1) = V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. For two classes $\mathcal{C}_1, \mathcal{C}_2$ of graphs, let $\mathcal{C}_1 + \mathcal{C}_2 = \{G_1 + G_2 : G_1 \in \mathcal{C}_1, G_2 \in \mathcal{C}_2, V(G_1) = V(G_2)\}$.

Theorem 6 *Suppose that $\mathcal{C}_1$ is a χ_a-bounded class of graphs and that $\mathcal{C}_2$ is a class of graphs of bounded maximum degree. Then $\mathcal{C}_1 + \mathcal{C}_2$ is a χ_a-bounded class of graphs.*

Proof. It suffices to show the following: Suppose G is a graph with acyclic chromatic number at most k and that G' is a graph on the same vertex set as G and has maximum degree s. Then $\chi_a(G + G') \leq k(2((2\lceil(\log_2 k + k - 1 + k \cdot 2^{k-2})\rceil + s)^2 + 2s\lceil(\log_2 k + k - 1 + k \cdot 2^{k-2})\rceil) + 1)$.

Let $v_1, v_2, \cdots, v_n$ be an ordering of the vertices of G such that for each $i = 1, 2, \cdots, n$, the vertex v_i has minimum degree in $G[\{v_1, v_2, \cdots, v_i\}]$. Since $\chi_a(G) \leq k$, we have $\chi_o(G) \leq k2^{k-1}$ (cf. Section 1). By Theorem 2, G has arboricity at most $\lceil(\log_2 k + k - 1 + k \cdot 2^{k-2})\rceil$. Let $m = \lceil(\log_2 k + k - 1 + k \cdot 2^{k-2})\rceil$. It follows from Nash-Williams' Theorem that each vertex v_i has degree at most $2m$ in the subgraph $G[\{v_1, v_2, \cdots, v_i\}]$ of G.

Let $G'' = G + G'$, let D be the digraph with vertex set $V = V(G'')$ and (v_i, v_j) be an arc of D if and only if either (v_i, v_j) is an edge of G'' and $j < i$, or there exists a vertex v_q such that

(i) $(v_i, v_q), (v_q, v_j)$ are edges of G'' and $j < q$;

(ii) either $q < i$ or (v_i, v_q) is an edge of G'.

It is straightforward to verify that each vertex of D has out-degree at most $(m + s)^2 + ms$. Then D has a vertex which has in-degree at most $(m + s)^2 + ms$, and hence has total degree at most $2((m + s)^2 + ms)$. This is also true for any subgraph of D, i.e., any subgraph of D has a vertex which has total degree at most $2((m + s)^2 + ms)$. Therefore the underline graph of D has chromatic number at most $2((m + s)^2 + ms) + 1$.

Let f_1 be an acyclic coloring of G with k colors; and let f_2 be a proper coloring of the underline graph of D with $2((m+s)^2+ms)+1$ colors. We claim that the coloring f defined as $f(x) = (f_1(x), f_2(x))$ is an acyclic coloring of G''. Suppose to the contrary that there exists a 2-colored cycle $C = [c_1, c_2, \cdots, c_{2t}]$. Then C must contain an edge not belonging to G. Suppose (c_1, c_2) is an edge of G'. Without loss of generality, we assume that $c_1 = v_i, c_2 = v_j$ and $i < j$. Suppose $c_3 = v_q$. If $j < q$, then (c_3, c_1) is an arc of D and hence c_3, c_1 are colored with distinct colors by f_2. If $j > q$, then (c_1, c_3) is an arc of D and hence c_3, c_1 are again colored with distinct colors by f_2. This is in contrary to the assumption that C is a 2-colored cycle under the coloring f. ∎

Note that considering the example used in Observation 2 (see Section 2), it is not enough to assume that the class $\mathcal{C}_2$ in Theorem 6 is of bounded genus, of bounded arboricity or of bounded treewidth.

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