Equivalence of the one-rate model to the classical model on strictly nonblocking switching networks

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Abstract

In the one-rate(f) network, each link can carry up to f messages for some integer f. The classical model is the special case when $f = 1$. We show that a network is strictly nonblocking under the one-rate(f) model if and only if it is strictly nonblocking under the classical model.

1 Introduction

A switching network consists of a set of nodes and a set of (directed) links. Typically, an outlink of a node is the inlink of another node, and vice versa. There are two special types of nodes: the *inputs* and the *outputs*. Each input (output) is a node which has no inlink (outlink) and exactly one outlink (inlink).

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We view a network as a directed graph $G = (V, E)$, where each vertex is a node and each edge is a link. The inputs and outputs are subsets I, O of V. To emphasize the special roles of the inputs and outputs, we denote a network as $G = (V, E, I, O)$. A network is called *acyclic* if the directed graph G is acyclic, i.e., G contains no directed cycles.

Let $G = (V, E, I, O)$ and f a positive integer. The 1-rate(f) network, denoted by (G, f), is a network G together with the capacity constraint that each edge can carry up to f messages. If $f = 1$, then the 1-rate network $(G, 1)$ is the *classical model*. In other words, a classical model is a network in which each edge can carry at most 1 message. In this paper, we only consider 1-rate networks.

A *traffic* of (G, f) is a sequence of input-output pairs (i, j), where $i \in I$ and $j \in O$, of two types: a *request* is a pair (i, j) neither of i, j has appeared in more than $f - 1$ previous uncancelled requests. Namely, the pair requests a connection in the network. A *cancellation* is a previous request whose connection in the network is to be removed. A request (i, j) is *routed* if a directed i-j-path is chosen, without exceeding the capacity of the edges. So a request (i, j) can be routed in the network (which has already routed many previous requests) if and only if there exists a directed i-j-path each of whose edges has not been used more than $f - 1$ times.

A *state* S of (G, f) is a collection of (not necessarily distinct) directed paths of G joining vertices of I to vertices of O, such that each edge e is contained in at most f directed paths. Given a state S, let $S(e)$ denote the number of directed paths containing e. Then $0 \leq S(e) \leq f$. A state is *blocking* if there exists a vertex $i \in I$ and $j \in O$ such that both i and j are contained in fewer than f directed paths in S, and every directed i-j-path of G contains an edge e with $S(e) = f$. We say that (G, f) is strictly nonblocking if there is no blocking state.

The classical model is of course the dominating model in the study of switching networks. Recently, the multirate network has received increasing attention due to the popular attempt to integrate multimedia service into one network. Since the theory of the classical model is well established, it is profitable to ask how much of it can be extended to the multirate model. The 1-rate model is the simplest multirate model, but also has its own application. It is used in the *digital symmetrical matrices* in time-space switching [7, 10], and the principle of providing more links between two nodes, known as *statistical line grouping* in [8], was promoted as a major technique to cut down network blocking. On the other hand, strict nonblockingness is one of the most fundamental property of a switching network. Therefore, asking whether one model implies the other on this property can serve as a natural start to explore the relation between the classical model and the multirate model. In this paper we prove that if $G = (V, E, I, O)$ is an acyclic network, then the strict nonblockingness of a 1-rate network (G, f) is equivalent to that of the classical model $(G, 1)$.

2 Strictly nonblocking under the 1-rate(f) model implies the same for the classical model

We first prove the implication in one direction.

Theorem 1 *If (G, f) is strictly nonblocking for some positive integer f, then $(G, 1)$ is strictly nonblocking.*

Proof. It suffices to prove that if $(G, 1)$ has a blocking state, then (G, f) has a blocking state. Suppose S is a blocking state of $(G, 1)$. Let S' be the collection of directed paths of G which is obtained by duplicating f times each directed path of S. Then S' is a state of (G, f) and for each edge e of G, $S'(e) = f \times S(e)$. As S is a blocking state of $(G, 1)$, there is an input $i \in I$ and an output $j \in O$ such that none of i, j is contained in any directed path of S, and any directed i-j-path of G contains an edge e with $S(e) = 1$. Then both of i and j are contained in no directed paths of S', and every directed i-j-path of G contains an edge e with $S'(e) = f$. Therefore S' is a blocking state of (G, f). ∎

In the remainder, we shall prove the other direction, i.e., if for some integer $f \geq 1$, (G, f) has a blocking state, then $(G, 1)$ has a blocking state. Let S be a blocking state of (G, f). Then there exist $i \in I$ and $j \in O$ such that both i, j are contained in at most $f - 1$ directed path of S, and any directed i-j-path contains an edge e with $S(e) = f$. We need to construct a blocking state S' for $(G, 1)$. One may attempt to partition the directed paths in S into f classes such that

(i) Directed paths which share a edge belong to different classes.

(ii) There exists a class C not containing any directed path with end vertex i or j.

If such a partition exists, then it is easy to verify that the class C is a blocking state of $(G, 1)$. However, such a partition may not exist. Consider the following network.

The collection of directed paths $S = \{P_1, P_2, P_3, P_4\}$ in Figure 1 is a blocking state for $(G, 2)$. However, it is impossible to partition the paths into 2 classes in such a way that directed paths sharing a edge belong to different classes, because every two directed paths share a edge. Thus to construct the blocking state S' for $(G, 1)$, we need to use directed paths not contained in the collection S.

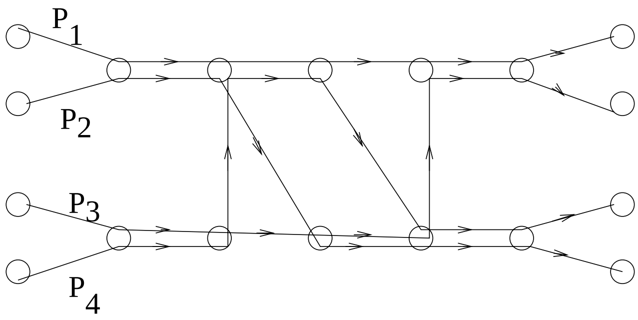

Figure 1: An example

3 Strictly nonblocking for $(G, 1)$ implies the same for $(G, 2)$

In this section, we consider the case $f = 2$.

Theorem 2 *Suppose G is acyclic. If $(G, 1)$ is nonblocking, then $(G, 2)$ is nonblocking.*

Proof. Let S be a blocking state for $(G, 2)$. Thus there exist $i \in I$ and $j \in O$ such that both i, j are contained in at most 1 directed path of S, and any directed i-j-path contains an edge e with $S(e) = 2$.

We shall construct a blocking state for $(G, 1)$. For each vertex v of G, denote by $E^+(v)$ the outlinks of v, and by $E^-(v)$ the inlinks of v. Let $E(v) = E^+(v) \cup E^-(v)$. Let

$$s^+(v) = \sum_{e \in E^+(v)} S(e) = \sum_{P \in S} |P \cap E^+(v)|,$$

$$s^-(v) = \sum_{e \in E^-(v)} S(e) = \sum_{P \in S} |P \cap E^-(v)|$$

and

$$s(v) = s^+(v) + s^-(v) = \sum_{P \in S} |P \cap E(v)|.$$

Since each directed path $P \in S$ connects a vertex of I to a vertex of O, we conclude that for each vertex $v \notin I \cup O$, $|P \cap E^+(v)| = |P \cap E^-(v)|$. Hence $s^+(v) = s^-(v)$ and $s(v) = 2s^+(v)$. Let $E_1 = \{e \in E : S(e) = 1\}$ and let $E_2 = \{e \in E : S(e) = 2\}$. Then $s(v) = |E_1 \cap E(v)| + 2|E_2 \cap E(v)|$. If $v \notin (I \cup O)$, then $s(v)$ is even, and hence $|E_1 \cap E(v)|$ is even. Let $G_1 = (V, E_1)$ be the subgraph of G induced by the edge set E_1. As each vertex of $V - (I \cup O)$

has even degree in G_1, we can decompose G_1 into edge-disjoint union of (not necessarily directed) cycles and paths, say

$$E_1 = (P_1 \cup P_2 \cup \cdots \cup P_l) \cup (C_1 \cup C_2 \cup \cdots \cup C_m),$$

where each path P_k connects two vertices of $I \cup O$. We color the edges of each P_k and C_l by two colors, a and b, as described below.

Given an undirected cycle (or a path), there are two choices for the *positive direction* of the cycle (or path). If the cycle is drawn on the plane, then either the clockwise direction, or the anticlockwise direction can be chosen as the positive direction. For a path with end vertices i and j, one can traverse the path from i to j, or from j to i. Once a positive direction is chosen, then those directed edges agree with the positive direction of the cycle (or path) are called *forward edges*, and those directed edges oppose the positive direction are called *backward edges*. For each cycle C_l, arbitrarily choose a positive direction, and color the forward edges of C_l by color a, and backward edges by color b. If P_k is a path connecting a vertex $x \in I$ to a vertex $y \in O$, then traverse P_k from x to y. If P_k is a path connecting two vertices I or two vertices of O, then arbitrarily choose a traversal of P_k, except that if i or j is an end vertex of P_k, then the edge incident to i or j must be a forward edge of P_k. Then color the forward edges of P_k by color a and color the backward edges of P_k by color b.

Let $E_a \subset E_1$ be the edges of color a and $E_b \subset E_1$ be the edges of color b. Let $B_1 = E_a \cup E_2$ and $B_2 = E_b \cup E_2$. Suppose $v \notin (I \cup O)$. Let $i_a(v)$ (respectively, $o_a(v)$) be the number of inlinks (respectively, outlinks) of v of color a, and let $i_b(v)$ (respectively, $o_b(v)$) be the number of inlinks (respectively, outlinks) of v of color b.

If P_k (or C_l) contains v, then either P_k (or C_l) contains two inlinks or two outlinks of v which are colored by distinct colors, or one outlink and one inlink of v which are colored by the same color.

Hence the sum of the number of color-a inlinks of P_k (or C_l) and the number of color-b outlinks of P_k (or C_l) is equal to the sum of the number of color-b inlinks of P_k (or C_l) and the number of color-a outlinks of P_k (or C_l). Therefore

$$i_a(v) + o_b(v) = o_a(v) + i_b(v).$$

Let $i_2(v) = |E_2 \cap E^-(v)|$ and $o_2(v) = |E_2 \cap E^+(v)|$. Then

$$s^-(v) = i_a(v) + i_b(v) + 2i_2(v)$$

and

$$s^+(v) = o_a(v) + o_b(v) + 2o_2(v).$$

As $s^+(v) = s^-(v)$, we conclude that $i_a(v) + i_2(v) = o_a(v) + o_2(v)$ and $i_b(v) + i_2(v) = o_b(v) + o_2(v)$.

Let H_1 be the directed subgraph of G induced by the edge set $E_a \cup E_2$ and H_2 the directed subgraph of G induced by the edge set $E_b \cup E_2$. Then for each vertex $v \notin (I \cup O)$, the number of inlinks of v in H_1 is $i_a(v) + i_2(v)$ and the number of outlinks of v in H_1 is $o_a(v) + o_2(v)$. So the number of inlinks of v is equal to the number of outlinks of v. As G is acyclic, H_1 is acyclic. Therefore H_1, and similarly H_2, can be decomposed into directed paths joining vertices of I to vertices of O. For $k = 1, 2$, denote by S_k the collection of directed paths which form a decomposition of H_k. For each edge e of G, $0 \leq S_k(e) \leq 1$ and $S(e) = S_1(e) + S_2(e)$. Moreover, both i abd j are not contained in any directed paths of S_2. As any directed i-j-path of G contains an edge e with $S(e) = 2$, and hence $S_2(e) = 1$. Therefore S_2 is a blocking state of $(G, 1)$. ∎

4 Strictly nonblocking for (G, f) implies the same for $(G, 1)$

In this section, we prove that the strict nonblocking of the classical model implies the strict nonblocking of the 1-rate(f) model for any $f \geq 1$.

Lemma 1 *Suppose G is acyclic. If S is a state of (G, f), then there are f states $S_1, S_2, \cdots, S_f$ of $(G, 1)$ such that for each edge e of G, $S(e) = \sum_{i=1}^{f} S_i(e)$.*

Proof. The lemma can be derived easily from a result of Little, Tutte and Younger [9] on graph flows. For the completeness of this paper, we give a direct proof. The proof is by induction on f. If $f = 1$, then there is nothing to be proved. Assume the lemma is true for $f = k - 1$, and we consider the case that $f = k$.

Let $e_i = x_i y_i$, $i = 1, 2, \cdots, m$, be all the edges of G such that $S(e_i) = k$. (Note that x_i, x_j need not be distinct, y_i, y_j need not be distinct, and x_i, y_j also need not be distinct). Let G'' be the directed graph obtained from G by identifying all the inputs and outputs, i.e., identifying all the vertices of $I \cup O$ into a single vertex v^*. Let G' be obtained from G'' as follows:
delete all the directed edges e_i, $i = 1, 2, \cdots, m$,
add two vertices x^*, y^*,
add directed edges $x_i x^*, y^* y_i$ for $i = 1, 2, \cdots, m$,
and add the directed edge $x^* y^*$.

Let $\phi : E(G') \to Z$ be defined as $\phi(e) = S(e)$ if $e \in E(G)$, and $\phi(x_i x^*) = \phi(y^* y_i) = k$, $\phi(x^* y^*) = mk$.

By definition, for each vertex u of G', $\sum_{e \in E^+_{G'}(u)} \phi(e) = \sum_{e \in E^-_{G'}(u)} \phi(e)$.

Let X be any subset of $V(G')$ such that $x^* \in X$ and $y^* \notin X$. Denote by $[X, V(G') - X]$ the set of directed edges e of G' from X to $V(G') - X$ such that $\phi(e) > 0$. Similarly, denote by $[V(G') - X, X]$ the set of directed edges of G' from $V(G') - X$ to X such that $\phi(e) > 0$. Then

$$\sum_{e \in [X, V(G')-X]} \phi(e) - \sum_{e \in [V(G')-X, X]} \phi(e) = \sum_{e \in E^+_{G'}(u), u \in X} \phi(e) - \sum_{e \in E^-_{G'}(u), u \in X} \phi(e) = 0.$$

Since $\sum_{e \in [X, V(G')-X]} \phi(e) \geq \phi(x^* y^*) = mk$, we have $\sum_{e \in [V(G')-X, X]} \phi(e) \geq mk$. However, for each $e \in [V(G') - X, X]$, $\phi(e) \leq k$. Therefore $|[V(G') - X, X]| \geq m$. Note that X is an arbitrary subset of $V(G')$ such that $x^* \in X^*$ and $y^* \notin X^*$. Hence by Menger's theorem [2, 11], there are m edge-disjoint directed paths, say $P_1, P_2, \cdots, P_m$, of G' from y^* to x^*, consisting of edges e with $\phi(e) > 0$. Note that there are exactly m directed edges to x^*, namely, the edges $x_i x^*$, and exactly m directed edges from y^*, namely, the edges $y^* y_i$, $i = 1, 2, \cdots, m$. Therefore the union of the edges $x_i y_i$ and $\cup_{i=1}^m (P_i - \{x^*, y^*\})$ induces a directed subgraph H of G'' which is the edge-disjoint union of directed cycles. Note that since G is acyclic, all the directed cycles of G'' contains the vertex v^*, and hence corresponds to a path of G connecting a vertex of S to a vertex of T. That is to say, viewed as a subgraph of G, H is the edge-disjoint union of paths joining vertices of S to vertices of T. Moreover, for each edge e of H, $\phi(e) > 0$, which is equivalent to $S(e) > 0$, and if $S(e) = k$ then e is contained in one of the paths P_i.

Let S_1 be the collection of paths of G defined above. Then S_1 is a state for $(G, 1)$. Let $S'(e) = S(e) - S_1(e)$. From the construction, we know that for each e, $0 \leq S'(e) \leq k - 1$. Moreover, for each vertex $u \notin (I \cup O)$, $\sum_{e \in E^+(v)} S'(e) = \sum_{e \in E^-(v)}$. As G is acyclic, there is a collection of directed paths joining vertices of I to vertices of O such that each edge e is contained in exactly $S'(e)$ paths. By an abuse of notation, we view S' as the collection of these paths, and hence S' is a state of $(G, k - 1)$. By induction hypothesis, there are states $S_2, \cdots, S_k$, of $(G, 1)$ such that for each edge e,

$$\sum_{i=2}^{f} S_i(e) = S'(e).$$

Hence $S_1, S_2, \cdots, S_k$ are states of $(G, 1)$ such that for each edge e,

$$\sum_{i=1}^{f} S_i(e) = S(e).$$

∎

Theorem 3 *If $(G, 1)$ is strictly nonblocking then (G, f) is strictly nonblocking for any $f \geq 1$.*

Proof. Assume (G, f) is not strictly blocking and S is a blocking state of (G, f). Then there exist $i \in I$ and $j \in O$ such that both u and v are contained in fewer than f directed paths in S, and every directed u-v-path of G contains an edge e with $S(e) = f$. By Lemma 1, there exists f states, $S_1, S_2, \cdots, S_f$, of $(G, 1)$ such that for every edge e,

$$S(e) = \sum_{k=1}^{f} S_k(e).$$

As both i and j are contained in fewer than f directed paths in S, there exists $1 \leq a, b \leq f$ such that i is not contained in any path of S_a, and j is not contained in any path of S_b. If $a = b$, then S_a is a blocking state of $(S, 1)$. Assume $a \neq b$. Then $S_a \cup S_b$ is a blocking state of $(G, 2)$. By Theorem 2, $(G, 1)$ has a blocking state. ∎

Corollary 1 *Suppose $G = (V, E, O, I)$ is an acyclic network. Then for any positive integers f, f', (G, f) is strictly nonblocking is and only if (G, f') is strictly nonblocking.*

Proof. The strictly nonblocking of (G, f) is equivalent to strictly nonblocking of $(G, 1)$, for any integer f. Hence strictly nonblocking of (G, f) is equivalent to strictly nonblocking of (G, f'). ∎

5 Some concluding remarks

Some other implications between the classical model and the multirate model are available from the literature. These involve some other notions of nonblockingness. A network is *wide-sense nonblocking* if every request can be routed provided all routing follows a given algorithm. A network is *rearrangeably nonblocking* if all requests can be routed if they are given at once (instead of the usual "sequential" model).

Let $C(n_1, r_1, m, n_2, r_2)$ denote the 3-stage clos network whose nodes are partitioned into three stages (parts):
the first stage consists of r_1 nodes each with n_1 inlinks and m outlinks,
the second stage consists of m nodes each with r_1 inlinks and r_2 outlinks,
the third stage consists of r_2 nodes each with m inlinks and n_2 outlinks ,
such that there exists a link from each stage-i node to each stage-$(i+1)$ node, but no other links between two nodes.

Clos [4] proved

Lemma 2 *$C(n_1, r_1, m, n_2, r_2)$ is strictly nonblocking under the classical model if and only if*

$$m \geq \min\{n_1 + n_2 - 1, n_1 r_1, n_2 r_2\}.$$

Hwang and Yeh, as reported in [6], proved a similar result under a model slightly more general than the 1-rate(f) model, suppose each input has capacity f_0, each output has capacity $f_0^{'}$, each link between stage 1 and stage 2 has capacity f_1, and each link between stage 2 and stage 3 has capacity f_2.

Lemma 3 *$C(n_1, r_1, m, n_2, r_2; f_0, f_0^{'}, f_1, f_2)$ is strictly nonblocking if and only if*

$$m \geq \lfloor \frac{\min\{n_1 f_1, n_2 r_2 f_2\} - 1}{f_0} \rfloor + \lfloor \frac{\min\{n_1 r_1 f_1, n_2 r_2\} - 1}{f_0} \rfloor + 1.$$

By setting $f_0 = f_0^{'} = f_1 = f_2 = f$, we obtain

Corollary 2 *$C(n_1, r_1, m, n_2, r_2)$ is strictly nonblocking under the 1-rate(f) model if and only if*

$$m \geq \min\{n_1 + n_2 - 1, n_1 r_1, n_2 r_2\}.$$

Note that the conditions in Lemmas 2 and Corollary 2 are the same. Hence

Theorem 4 *For $C(n_1, r_1, m, n_2, r_2)$, strictly nonblocking under the classical model implies the same for the 1-rate(f) model, and vice versa.*

Benes [1] proved

Lemma 4 *$C(n, 2, m, n, 2)$ is wide-sense nonblocking under the 1-rate(f) model if and only if $m \geq \lceil \frac{3n}{2} \rceil$.*

On the other hand, Fishburn et al. [5] proved

Lemma 5 *$C(n, 2, m, n, 2)$ is wide-sense nonblocking under the 1-rate(f) model if and only if $m \geq \lceil \frac{3n}{2} \rceil$.*

By comparing Lemmas 4 and 5, we obtain

Theorem 5 *Wide-sense nonblocking under the classical model does not imply the same for the 1-rate model.*

Finally, Chung and Ross [3] proved

Lemma 6 *Rearrangeably nonblocking under the classical model implies the same for the 1-rate(f) model.*

For the other direction, only special cases are proved by Slepian, as reported in [1].

Lemma 7 *$C(n_1, r_1, m, n_2, r_2)$ is rearrangeably nonblocking under the classical model if and only if $m \geq \max\{n_1, n_2\}$.*

On the other hand, Hwang and Yeh, as reported in [6], proved

Lemma 8 *$C(n_1, r_1, m, n_2, r_2; f_0, f_0', f_1, f_2)$ is rearrangeably nonblocking if and only if*

$$m \geq \max\{\frac{\min\{n_1 f_1, n_2 r_2 f_2\}}{f_0}, \frac{\min\{n_1 r_1 f_1, n_2 f_2\}}{f_0'}\}.$$

By setting $f_0 = f_0' = f_1 = f_2$, we obtain

Corollary 3 *$C(n_1, r_1, m, n_2, r_2)$ is rearrangeably nonblocking under the 1-rate(f) model if and only if $m \geq \max\{n_1, n_2\}$.*

By comparing Lemma 7 and Corollary 3, we obtain

Theorem 6 *For the 3-stage Clos network, rearrangeably nonblocking under the 1-rate(f) model implies the same for the classical model.*

Note that all these results deal with the very special 3-stage Clos networks. Chung and Ross, and us, are the only exceptions to attack the much harder general networks.

To Summarize, we have

	classical		1-rate(f)	remark
strict		$\Longrightarrow$		proved
		$\Longleftarrow$		proved
wide sense		$\Longrightarrow$		not true
		$\Longleftarrow$		possible
rearrangeable		$\Longrightarrow$		proved
		$\Longleftarrow$		possible

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