A Note On Homomorphisms to Acyclic Local Tournaments

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Abstract

A homomorphism of a digraph to another digraph is an edge-preserving vertex mapping. A local tournament is a digraph in which the inset as well as the outset of each vertex induces a tournament. Thus acyclic local tournaments generalize both directed paths and transitive tournaments. In both these cases there is a simple characterization of homomorphic preimages. Namely, if H is a directed path, or a transitive tournament, then G admits a homomorphism to H if and only if each oriented path which admits a homomorphism to G also admits a homomorphism to H. We prove that this result holds for all acyclic local tournaments.

Let G and H be digraphs. A homomorphism of G to H is a mapping $f : V(G) \to V(H)$ such that $gg' \in E(G)$ implies that $f(g)f(g') \in E(H)$. If such a homomorphism exists, we say G is homomorphic to H and write $G \to H$. Otherwise we write $G \not\to H$.

Suppose H is digraph and v is a vertex of H. The inset of v is the set of all the vertices u of H such that uv is an edge of H, and the outset of v is

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the set of all vertices u of H such that vu is an edge of H. A digraph H is a local tournament if for each vertex v of H, the inset as well as the outset of v induces a tournament. Local tournaments were introduced in [1] as a natural extension of the class of tournaments, for which many results about tournaments remain valid. A local tournament H is acyclic if and only if H admits an enumeration $v_1, v_2, \ldots, v_n$ of vertices, such that whenever $v_i v_j$ is an edge of H then $i < j$ and $v_k v_l$ is also an edge of H for any $i \leq k < l \leq j$, [1]. Thus each transitive tournament and each directed path is an acyclic local tournament.

Let H be a directed path of length k. Note that any oriented path P which has net length greater than k does not admit a homomorphism to H. (An oriented path is the digraph obtained from an undirected path by orienting its edges. The net length of an oriented path is the absolute value of the difference between the number of edges oriented one way and the number of edges oriented the other way.) It is clear that if any path P of net length greater than k admits a homomorphism to G, then $G \not\to H$, since otherwise $P \to H$ by composition. Conversely, it it is not hard to prove, cf. [3, 8, 9], that $G \to H$ if and only if $P \not\to G$ for every oriented path P of net length greater than k. It is also not hard to prove, cf. [3, 8, 9], that if H is a transitive tournament with n vertices, then $G \to H$ if and only if $P \not\to G$ for all directed paths of length at least n. These two results have a common re-statement as follows: If H is either a transitive tournament or a directed path, then $G \to H$ if and only if for every oriented path P with $P \to G$ we also have $P \to H$. In this short note we wish to point out that this common generalization holds for the entire class of acyclic local tournaments.

Theorem 1 *Let W be an acyclic local tournament. Then for any digraph G we have $G \to W$ if and only if for every oriented path P with $P \to G$ we also have $P \to W$.*

Proof. The necessity is obvious. Therefore we assume that for any oriented path P with $P \to G$ we also have $P \to W$. We proceed to prove that $G \to W$.

We have noted above that there exists an enumeration $v_1, v_2, \ldots, v_n$ of the vertices of W such that whenever $v_i v_j$ is an edge of W then $i < j$ and

$v_k v_l$ is also an edge of W for any $i \leq k < l \leq j$. Using this, we define a linear order on the vertices of W as $v_i \leq v_j$ if and only if $i \leq j$.

Denote by $\mathcal{P}$ the set of all pairs (P, p) such that P is an oriented path with $P \to G$, and p is an end vertex of P. By assumption we have $P \to W$ for any $(P, p) \in \mathcal{P}$. Define $\Phi : \mathcal{P} \to V(W)$ by

$$\Phi((P, p)) = \min\{h(p) : h \text{ is a homomorphism of } P \text{ to } W\}.$$

For $(P, p) \in \mathcal{P}$, let $i(P, p)$ be the set of vertices x of G such that there exists a homomorphism $h : P \to G$ such that $h(p) = x$. Define $\Psi : V(G) \to V(W)$ by

$$\Psi(x) = \max\{\Phi((P, p)) : (P, p) \in \mathcal{P}, x \in i(P, p)\}.$$

It is easy to see that Φ and Ψ are well defined. We now show that Ψ is a homomorphism of G to W.

Suppose xy is an edge of G. There is a pair $(P, p) \in \mathcal{P}$ such that $x \in i(P, p)$ and $\Psi(x) = \Phi((P, p))$. Let P' be the oriented path obtained from P by adjoining a vertex p' and the edge pp'. It is easy to see $(P', p') \in \mathcal{P}$ and $y \in i(P', p')$. There is a homomorphism $g : P' \to W$ such that $g(p') = \Phi((P', p')) \leq \Psi(y)$. Considering the restriction of g to P, we see that $g(p) \geq \Phi((P, p)) = \Psi(x)$. Since $g(p)g(p')$ is an edge of W, we have $g(p) < g(p')$. Thus $\Psi(x) \leq g(p) < g(p') \leq \Psi(y)$.

Let (Q, q) be a pair in $\mathcal{P}$ such that $y \in i(Q, q)$ and $\Psi(y) = \Phi((Q, q))$. Let Q' be the oriented path obtained from Q by adjoining a vertex q' and the edge $q'q$. Again we have $(Q', q') \in \mathcal{P}$ and $x \in i(Q', q')$. Let g' be a homomorphism of Q' to W such that $g'(q') = \Phi((Q', q')) \leq \Psi(x)$. Considering the restriction of g' to Q, we see that $g'(q) \geq \Phi((Q, q)) = \Psi(y)$. Thus we have $g'(q') \leq \Psi(x) < \Psi(y) \leq g'(q)$. Since $g'(q')g'(q)$ is an edge of W we conclude that $\Psi(x)\Psi(y)$ is an edge of W from the property of the enumeration $v_1, v_2, \ldots, v_n$ of the vertices of W. ∎

If the conclusion of Theorem 1 holds for a digraph W, then we say that W has *path-duality*. It is proved in [7, 6] that all oriented paths, and all oriented cycles of net length one have path-duality. If a digraph W has path-duality (or a weaker property called tree-duality, [4, 5]), then there exists a polynomial algorithm for the following problem, called the W-colouring problem, [4, 5]:

Instance: A digraph G;

Question: Is G homomorphic to W?

Therefore Theorem 1 implies that the W-colouring problem is polynomial for acyclic local tournaments W. (This can also be proved by showing that W has the so called $\underline{X}$-property defined in [2]).

In closing, we wish to observe that a slight extension of the class of acyclic local tournaments fails to have path-duality. Let W be a local tournament with the enumeration $v_1, v_2, \cdots, v_n$. Call an edge $e \in E(W)$ an *inner* edge if $e = v_i v_j$ where $j \neq i + 1$, and if there exists a different edge $e' = v_{i'} v_{j'}$ such that $i' \leq i < j \leq j'$.

Theorem 2 *Suppose W is a local tournament and e is an inner edge. Then $W - e$ does not have path duality.*

Proof. Using the enumeration $v_1, v_2, \cdots, v_n$, and letting $e = v_i v_j$, we may assume that $v_i v_{j+1}$ is an edge of W. To prove that $W - e$ does not have path-duality, it suffices to construct a digraph G such that G is not homomorphic to $W - e$, and every oriented path homomorphic to G is also homomorphic to $W - e$.

We construct a digraph G as follows: Take two directed paths $P_1 = [x_1, x_2, \cdots, x_n]$ and $P_2 = [y_1, y_2, \cdots, y_n]$ of length $n - 1$, and connect them by adding a single edge $x_i y_j$.

Observe that $W - e$ contains a unique directed path of length $n - 1$ in $W - e$, namely $[v_1, v_2, \cdots, v_n]$. Thus if $h : G \to W - e$ is a homomorphism then $h(x_k) = v_k$ and $h(y_k) = v_k$ for all $1 \leq k \leq n$. However $x_i y_j$ is an edge of G and $v_i v_j$ is not an edge of W_e. Therefore $G \not\to W - e$.

It remains to show that any oriented path homomorphic to G is also homomorphic to $W - e$. Let P be an oriented path and let $h : P \to G$ be a homomorphism. Let $S = \{t \in V(P) : h(t) = y_j\}$. For a vertex $t \in S$, t has at most two neighbours, say t_1 and t_2, on the path P. The images of the neighbours of t under the homomorphism of h must be neighbours of y_j in G. Thus $h(t_1), h(t_2) \in \{y_{j-1}, y_{j+1}, x_i\}$. We partition S into three subsets S_1, S_2, S_3 as follows: If $h(t_1), h(t_2) \in \{y_{j-1}, y_{j+1}\}$ then $t \in S_1$; if $t \notin S_1$ and $h(t_1), h(t_2) \in \{y_{j-1}, x_i\}$ then $t \in S_2$; if $t \notin S_1 \cup S_2$ and $h(t_1), h(t_2) \in \{y_{j+1}, x_i\}$ then $t \in S_3$.

Define a mapping $f : V(P) \to V(W)$ as follows:

$f(t) = v_k$ if $h(t) = x_k$, or $h(t) = y_k$ and $k \neq j$;

$f(t) = v_j$ if $t \in S_1$, $f(t) = v_{j+1}$ if $t \in S_2$, and $f(t) = v_{j-1}$ if $t \in S_3$.

It is a routine exercise to verify that f is indeed a homomorphism of P to $W - e$. ∎

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