On Hedetniemi's conjecture and the colour template scheme

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Abstract

Hedetniemi's conjecture states that the chromatic number of a product of graphs is the minimum of the chromatic numbers of the factors. El-Zahar and Sauer [2] have shown that the chromatic number of a product of 4-chromatic graphs is 4. In this paper, we investigate to which extent their methods can be adapted to the case of 5-chromatic graphs. Our main result is that if G and H are connected graphs of chromatic number at least five that both contain odd wheels as subgraphs, then their product also has chromatic number at least five. However, we also provide counterexamples to El-Zahar and Sauer's conjecture on the structure of n-chromatic subgraphs of products of n-chromatic graphs. This implies that the method of El-Zahar and Sauer will not be sufficient to prove all cases of Hedetniemi's conjecture for chromatic numbers larger than 4.

1 Introduction

The product of two graphs G and H is the graph $G \times H$ whose vertices are all pairs (u, v) with $u \in V(G)$, $v \in V(H)$, and whose edges are the pairs $[(u, v), (u', v')]$ with $[u, u'] \in E(G)$, and $[v, v'] \in E(H)$. The chromatic number of a product of graphs is the object of a long standing conjecture:

Conjecture 1 (Hedetniemi [3]) *For any graphs G, H,*

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}.$$

A homomorphism of G to H is a mapping $f : V(G) \to V(H)$ such that for each edge $[x, y]$ of G, $[f(x), f(y)]$ is an edge of H. If G admits a homomorphism to H, then obviously $\chi(G) \leq \chi(H)$. It is easy to see that $G \times H$ admits homomorphisms to both G and H (the projections are homomorphisms), hence $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$. The inherent difficulty of conjecture 1 lies in finding a lower bound for the chromatic number of $G \times H$. Following El-Zahar and Sauer [2], we know that this problem is equivalent to finding upper bounds to the chromatic number of certain exponential graphs. For two graphs G and H, the exponential graph G^H has all the functions from the vertex-set of H to that of G as vertices, and two of these functions f, g are joined by an edge if $[f(u), g(v)] \in E(G)$ for all $[u, v] \in E(H)$.

Proposition 2 ([2]) *Hedetniemi's conjecture is equivalent to the statement that K_n^G is n-chromatic whenever $\chi(G) > n$.*

In general it is hard to find colourings of exponential graphs that provide meaningful bounds. An alternative is to use some fixed exponential graph K_n^T as a template to colour a factor of a product. The results obtained by this method fit the following mold:

Colour template scheme Let G, H be connected graphs such that an n-chromatic graph T admits homomorphisms to both G and H. If $\chi(G \times H) \leq n$, then $\chi(G) \leq n$ or $\chi(H) \leq n$.

Colour template scheme is an idea which may or may not work for proving special cases of Hedetniemi's conjecture. To make the idea work, one needs to find appropriate graphs T as the templates. A good example is a result of Duffus, Sands and Woodrow [1] and Welzl [5] asserts that the complete graph K_n satisfies this scheme: Let G, H be connected graphs both containing the complete graph K_n. If $\chi(G \times H) \leq n$, then $\chi(G) \leq n$ or $\chi(H) \leq n$. For $n = 3$, the complete graph K_3 is also the smallest odd cycle. El-Zahar and Sauer showed that all odd cycles are similar templates: Let G, H be connected graphs both containing odd cycles. If $\chi(G \times H) \leq 3$, then $\chi(G) \leq 3$ or $\chi(H) \leq 3$. In other words, the chromatic number of the product of two 4-chromatic graphs is 4.

In this paper, we investigate further applications of this template scheme, more specifically to the case of 4-chromatic products of graphs. Our main result in this direction is the result of Section 3 which shows that all odd wheels are similar templates. It is an application of the colouring techniques introduced by El-Zahar and Sauer [2]. On the other hand, an example presented in Section 4 shows that the colour template scheme does have limitations, and new methods need to be introduced for a complete proof of Hedetniemi's conjecture. The next section provides an overview of the proof techniques involved with the colour template scheme.

2 An example

In this section, we let T be the Cayley graph $\mathrm{Cay}(\mathbf{Z}_8, \{\pm 1, \pm 2\})$, that is, the vertices of T are the integers modulo 8 and there is an edge between vertices u and v if $u - v \in \{\pm 1, \pm 2\}$.

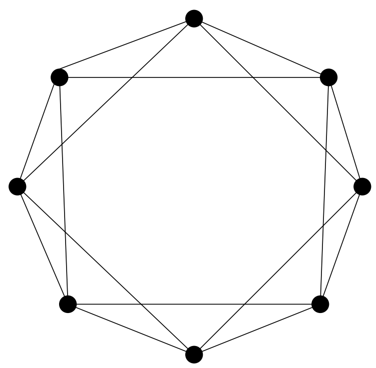

Figure 1: $T = \mathrm{Cay}(\mathbf{Z}_8, \{\pm 1, \pm 2\})$

Theorem 3 *Let G, H be connected graphs such that T admits homomorphisms into both G and H. If $\chi(G \times H) = 4$, then*

$$\min\{\chi(G), \chi(H)\} = 4.$$

This will allow us to introduce the standard proof techniques involved in the colour template scheme.

Let $\phi_G : T \mapsto G, \phi_H : T \mapsto H$ be homomorphisms. Then any proper vertex colouring $\gamma : G \times H \mapsto K_4$ naturally induces homomorphisms $\psi_G : G \mapsto K_4^T$ and $\psi_H : H \mapsto K_4^T$; where for $u \in V(G)$, $\psi_G(u)$ is the map $f_u : T \mapsto K_4$ defined by $f_u(v) = \gamma(u, \phi_H(v))$ and for $v \in V(H)$, $\psi_H(v)$ is the map $g_v : T \mapsto K_4$ defined by $g_v(u) = \gamma(\phi_G(u), v)$. Since G and H are connected, their images under ψ_G and ψ_H respectively are connected.

Our task is to show that at least one of $\psi_G(G)$ and $\psi_H(H)$ lies in a 4-colourable component of K_4^T. This will conclude the proof, as we will then have $\chi(G) \leq \chi(\psi_G(G)) \leq 4$ or $\chi(H) \leq \chi(\psi_H(H)) \leq 4$.

However note that K_4^T itself is not 4-colourable. Indeed, $\chi(T) = 4$, and the proper 4-colourings of T are loops in K_4^T, whence K_4^T admits no proper colourings whatsoever. A preliminary result concerns the structure of these loops in K_4^T.

Lemma 4 *The proper* 4*-colourings of* T *are isolated vertices of* K_4^T*.*

Proof. Let f be a proper colouring of T and suppose that $f(0) = 0$. Then we have $f(i) \neq 0$ for $i = \pm 1, \pm 2$. If f has a neighbour g different from itself, we can assume without loss of generality that $g(0) \neq 0$. This leaves only two colours to be used by f on the neighbours of 0, so we may assume that $f(1) = 1 = f(-2)$ and $f(2) = 2 = f(-1)$. This means that the vertices $3, 4, 5$ of T must be coloured by the colours 0 and 3. This is impossible since f is a proper colouring and $\{3, 4, 5\}$ is a triangle in T.

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Suppose that the image $\psi_G(G)$ of G under ψ_G contains a loop. By Lemma 4, since G is connected, $\psi_G(G)$ consists of a single vertex which is a loop. Let v be a vertex of T. Then for $u \in V(T)$, $g_{\phi_H(v)}(u) = \gamma(\phi_G(u), \phi_H(v)) = f_{\phi_G(u)}(v)$. By the previous result, $f_{\phi_G(u)}(v)$ does not depend on $\phi_G(u)$, whence $g_{\phi_H(v)}$ is a function with a constant value. Thus, the elements of $\phi_H(T)$ are mapped to constant functions by ψ_H, whence $\psi_H(H)$ is contained in the connected component of the constants in K_4^T. We conclude that at least one of $\psi_G(G)$ and $\psi_H(H)$ is contained in a connected component of K_4^T which is not an isolated loop.

Lemma 5 *All the connected components of* K_4^T *except the isolated loops (that is, the proper colourings) are* 4*-colourable.*

Proof. Consider the colouring scheme where a function f in K_4^T is assigned colour $a \in \{0, 1, 2, 3\}$ if one of the following conditions hold

(i) $f(i) = a$ for at least three values $i \in \mathbf{Z}_8$

(ii) For some $i \in \mathbf{Z}_8$ we have $f(i) = f(i + 2) = a$.

If g is adjacent to f and f is assigned colour a by this colouring scheme, then there cannot exist two adjacent vertices $i, j \in \mathbf{Z}_8$ such that $g(i) = g(j) = a$, because f uses the colour a on some neighbour of i or j; therefore g is not assigned colour a. Thus, the colouring scheme does not assign the same colour to adjacent vertices, though some vertices remain uncoloured.

Let f be an uncoloured vertex of K_4^T. Then f uses the four colours $0, 1, 2, 3$ exactly twice. If f is a proper colouring of K, then f is isolated by the previous lemma. Otherwise, we have $f(i) = f(i+1)$ for some i. It is then easy to see that if f is not an isolated vertex of K_4^T, we must have $f(i+4) = f(i+5)$. Then, for any neighbour g of f we have $g(i) = g(i+1) = f(i+4)$ and $g(i+4) = g(i+5) = f(i)$. Hence f lies in a bipartite component of K_4^T.

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This concludes the proof of Theorem 3.

In summary, the proof consists of two main steps. Starting with a colouring γ of $G \times H$, the first step (given in Lemma 4 and the following paragraph) is to decide whether to use gamma to colour the factor G or the factor H. The second step is the colouring of the appropriate factor using the proof of Lemma 5. The result of the next section will follow the same steps, although the details are more intricate. Informally, we call an n-chromatic graph T an n-colour template if the above proof can be carried out.

3 Odd wheels

The odd wheel W_{2k+1} is the graph obtained by adding a universal vertex u to the odd cycle C_{2k+1} on the vertices $0, 1, \dots, 2k$. The main result of this section is the following:

Theorem 6 *Assume G and H are connected graphs and there exists an odd wheel W_{2k+1} which admits hoomomorphisms to both G and H. If $\chi(G \times H) = 4$, then*

$$\min\{\chi(G), \chi(H)\} = 4.$$

Note that any image of an odd wheel under a homomorphism contains an odd wheel, so the hypothesis that W_{2k+1} admits homomorphisms into both G and H is equivalent to the hypothesis that both G and H contain odd wheels as subgraphs.

We first recall some results of [2]. Let f be a 3-colouring of C_{2k+1}. A vertex i of C_{2k+1} is said to be *fixed* by f if $f(i-1) \neq f(i+1)$ (where addition is modulo $2k+1$). The *parity* of f is the parity of the number of vertices that are fixed by f.

Lemma 7 ([2])

- *The parity of a function $f \in K_3^{C_{2k+1}}$ is the same as the parity of the number of vertices i of C_{2k+1} such that $f(i-1), f(i), f(i+1)$ are all distinct.*
- *If f is a proper 3-colouring of C_{2k+1}, then f has odd parity.*
- *If f and g are adjacent in $K_3^{C_{2k+1}}$, then f and g have the same parity.*
- *Let $f_0, \ldots, f_{2\ell}$ be an odd cycle of $K_3^{C_{2k+1}}$, and g the element of $K_3^{C_{2\ell+1}}$ defined by $g(i) = f_i(0), i = 0, \ldots, 2\ell$. Then the parity of g is opposite to that of $f_0, \ldots, f_{2\ell}$.*

Let f be an element of $K_4^{W_{2k+1}}$. If f has a neighbour g, then the colour $g(u)$ is not used by f on the vertices of C_{2k+1}. This means that the restriction of f to C_{2k+1} uses at most 3 colours and may be assigned a parity.

Lemma 8 *Let f and g be adjacent vertices of $K_4^{W_{2k+1}}$. If the restriction of f to C_{2k+1} is odd and the colour $f(u)$ is different from those used by f on C_{2k+1}, then $g(u) = f(u)$, $g(C_{2k+1}) = f(C_{2k+1})$ and the restriction of g to C_{2k+1} is also odd.*

Proof. By Lemma 7, there exists a vertex i of C_{2k+1} such that $f(i-1), f(i)$ and $f(i+1)$ are all distinct. Therefore, $g(u) \notin f(C_{2k+1})$ whence $g(u) = f(u)$. Also, since $f(u) \notin g(C_{2k+1})$, the restrictions of f and g to C_{2k+1} use the same three colours, and by Lemma 7, they must have the same parity.

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Definition 9 A connected component G of $K_4^{W_{2k+1}}$ is called *odd* if for $f \in V(G)$ the restriction of f to C_{2k+1} is odd and $f(u) \notin f(C_{2k+1})$. Otherwise G is called *even*.

Note that the proper 4-colourings of W_{2k+1} lie in odd components of $K_4^{W_{2k+1}}$. An even component of $K_4^{W_{2k+1}}$ may contain a function f whose restriction to C_{2k+1} is odd, however, in that case we must have $f(u) \in f(C_{2k+1})$.

Lemma 10 *Let G and H be connected graphs respectively containing the odd wheels W_{2k+1} and $W_{2\ell+1}$. Let $\gamma : G \times H \mapsto K_4$ be a proper vertex colouring and $\psi_G : G \mapsto K_4^{W_{2\ell+1}}$, $\psi_H : H \mapsto K_4^{W_{2k+1}}$ the homomorphisms naturally induced by γ. Then at least one of $\psi_G(G)$ and $\psi_H(H)$ lies in an even component.*

Proof. Suppose that $\psi_G(G)$ lies in an odd component of $K_4^{W_{2\ell+1}}$. Let $f_0, f_1, \ldots, f_{2k}$ be the images under ψ_G of the elements of the copy of C_{2k+1} contained in the copy of W_{2k+1} contained in G. Then the restrictions of $f_0, f_1, \ldots, f_{2k}$ to $C_{2\ell+1}$ are odd and all use the same three colours. Put $\psi_H(0) = g : W_{2k+1} \mapsto K_4$, where 0 denotes the corresponding element of the copy of $C_{2\ell+1}$ contained in the copy of $W_{2\ell+1}$ contained in H. Then for $i = 0, \ldots 2k$, $g(i) = f_i(0)$, and by Lemma 7, the restriction of g to C_{2k+1} is even, whence $\psi_H(H)$ lies in an even component of $K_4^{W_{2k+1}}$.

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Lemma 11 *The even components of $K_4^{W_{2k+1}}$ are 4-colourable.*

Proof. Consider the colouring scheme where a function f in an even component of $K_4^{W_{2k+1}}$ is assigned colour $a \in \{0, 1, 2, 3\}$ if one of the following conditions holds:

(i) $f(u) = f(i) = a$ for at least one vertex $i \in C_{2k+1}$,

(ii) $f(0) = a$ and at least one of $f(1)$, $f(2k)$ is also a.

Fact. Let X be the subgraph of the union of the even components of $K_4^{W_{2k+1}}$ that are left uncoloured by this scheme. Then X is a bipartite graph.

Proof of fact. Assume to the contrary that $f_0, \ldots, f_{2\ell}$ is an odd cycle in X. If $f_i(u) \neq f_{i+1}(u)$, then since f_i and f_{i+1} must avoid both colours $f_i(u)$ and $f_{i+1}(u)$ on C_{2k+1} whence only two colours, say a and b, are left for f_i and f_{i+1} on C_{2k+1}. If $f_i(0) = a$ then $f_{i+1}(1) = b$, $f_i(2) = a$, $f_{i+1}(3) = b$, ... Thus $f_i(j) = a$ and $f_{i+1}(j) = b$ for all $j \in \{0, \ldots, 2k\}$. But this is impossible since f_i and f_{i+1} would then have to be coloured by the above scheme.

Therefore $f_0(u) = f_1(u) = \cdots = f_{2\ell}(u)$ and the restrictions of $f_0, \ldots, f_{2\ell}$ to C_{2k+1} are even colourings with the same three colours. Let g be the 3-colouring of $C_{2\ell+1}$ defined by $g(i) = f_i(0), i = 0, \ldots, 2\ell$. By Lemma 7, g is odd whence there exists an index i such that the values $f_{i-1}(0) = g(i-1)$,

$f_i(0) = g(i)$ and $f_{i+1}(0) = g(i+1)$ are all distinct. Now $f_i(1)$ must be one of these three values. However, $f_i(1) \neq f_{i-1}(0)$ because f_i is adjacent to f_{i-1}, $f_i(1) \neq f_i(0)$ for otherwise f_i would be assigned the colour $f_i(0)$ by the above scheme, and $f_i(1) \neq f_{i+1}(0)$ because f_i is adjacent to f_{i+1}. This shows that X cannot contain an odd cycle which completes the proof of the fact.

Let X_0, X_1 be a bipartition of X. We complete our colouring scheme by assigning to a function f the colour $a \in \{0, 1, 2, 3\}$ if $a = f(0)$ and $f \in X_0$ or $a = f(1)$ and $f \in X_1$.

Let f, g be adjacent vertices of an even component of $K_4^{W_{2k+1}}$ such that f is coloured by colour $a = f(x)$ and g is coloured by colour $b = g(y)$ with $x, y \in \{u, 0, 1\}$. If $x \neq y$, then clearly $a \neq b$. If $x = y = u$, then $a \neq b$ because f uses the colour a on C_{2k+1}. We cannot have $x = y = 1$ because this implies $f, g \in X_1$. The last possibility is $x = y = 0$, but since f and g cannot both lie in X_0, we have that f uses the colour a on a neighbour of 0 or g uses the colour b on a neighbour of 0 whence $a \neq b$. Therefore this colouring scheme yields a proper 4-colouring of the even components of $K_4^{W_{2k+1}}$.

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Theorem 6 follows directly from the previous two lemmas. This result can be generalised to higher chromatic numbers: Let T be a direct sum of an odd cycle and the complete graph K_{n-3}. If G and H are connected graphs such that T admits homomorphisms into both G and H, then $\chi(G \times H) = n$ implies $\min\{\chi(G), \chi(H)\} = n$. A similar result holds if T is a direct sum of 7-cycles.

In the case of 4-chromatic products of graphs, it would be interesting to find further applications of the template scheme. Suppose $G = (V, E)$ is a graph. Let $V' = \{x' : x \in V\}$ be a copy of V. The *Mycielskian* $M(G)$ of G is a graph whose vertex set is

$$V(M(G)) = V \cup V' \cup \{u\}$$

and whose edge set is

$$E(M(G)) = E \cup \{[x', y] : [x, y] \in E\} \cup \{[u, x'] : x' \in V'\}.$$

Using ad hoc techniques, we have shown that $M(C_3)$ is a 4-colour template. It would be interesting to know if $M(C_{2k+1})$ is a 4-colour template for $k \geq 2$. Also note that the 4-colour template in Theorem 3 and the 4-colour templates in Theorem 6 are incomparable under homomorphism order. It can be shown that if G and H are connected graphs such that an odd wheel W_{2k+1} admits a homomorphism to G and $\mathrm{Cay}(\mathbf{Z}_8, \{\pm 1, \pm 2\})$ admits a homomorphism to H then $\chi(G \times H) = 4$ implies $\min\{\chi(G), \chi(H)\} = 4$. However, it is not clear that n-colour templates can always be mixed in this way. The following question also remains open:

If both T, T' are n-colour templates, is the product $T \times T'$ also an n-colour template?

4 A counterexample

In light of their result on the chromatic number of products of 4-chromatic graphs, El-Zahar and Sauer made the following conjecture:

> Let G, H be connected n-chromatic graphs and G', H' $(n-1)$-chromatic subgraphs of G and H respectively. Then the subgraph $(G' \times H) \cup (G \times H')$ of the product is n-chromatic.

In fact, their method shows that this is true when $n = 4$. Note that all applications of the colour template scheme can be refined to similar results. (For instance, the proof of Theorem 6 shows that if G and H are connected 5-chromatic graphs and G', H' are odd wheels contained respectively in G and H, then $\chi((G' \times H) \cup (G \times H')) = 5$.)

We now show that the conjecture is false. Let T be the Cayley graph $\mathrm{Cay}(\mathbf{Z}_7, \{\pm 1, \pm 2\})$. Note that K_4^T contains a path f_0, f_1, f_2, f_3 where f_0 is a proper colouring and f_3 is a constant. This path is defined by $f_i(j) = a_{ij}$ with the values being taken in the matrix A below, whose rows and columns are labeled from 0 to 3 and from 0 to 6 respectively.

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 1 & 2 & 3 \\ 0 & 1 & 2 & 0 & 0 & 2 & 3 \\ 0 & 1 & 3 & 3 & 1 & 1 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}.$$

Let G be the graph obtained from the disjoint union of K_5, T and a path p_0, p_1, p_2, p_3 by identifying p_0 to any vertex of K_5 and p_3 to the vertex 2 of T

Proposition 12 $\chi((T \times G) \cup (G \times T)) = 4.$

Proof. Define $\gamma : (T \times G) \cup (G \times T) \mapsto K_4$ by

$$\gamma(u,v) = \begin{cases} f_0(v) \text{ if } u \in V(K_5), \\ f_1(v) \text{ if } u = p_1, \\ f_2(v) \text{ if } u = p_2, \\ f_0(u) \text{ if } u \in V(T). \end{cases}$$

Since f_0, f_1, f_2, f_3 is a path from a proper colouring to the constant 2 and $f_0(2) = 2$, it is straightforward to verify that γ is a proper 4-colouring of $(T \times G) \cup (G \times T)$.

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It is quite easy to generalise this example. In fact, for any integers m, n such that $4 \leq m < n$, there exist n-chromatic connected graphs G, H and m-chromatic subgraphs G', H' of G and H respectively such that $\chi((G' \times H) \cup (G \times H')) = m$. Note that whenever T is a m-chromatic graph such that K_m^T contains a path from a proper colouring to a constant, then the connectivity of the neighbourhood complex of T (in the sense of Lovász [4]) is less than $m-2$. This suggests that if Conjecture 1 is false, then counterexamples might be found among the graphs with a chromatic number much larger than the connectivity of the corresponding neighbourhood complex.

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