A survey on Hedetniemi's conjecture

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Abstract

More than 30 years ago, Hedetniemi made a conjecture which says that the categorical product of two n-chromatic graphs is still n-chromatic. The conjecture is still open, despite many different approaches from different point of views. This article surveys methods and partial results; and discuss problems related to or motivated by this conjecture.

1 Introduction

Suppose G and H are simple finite graphs. The product $G \times H$ of G and H has vertex set $V(G \times H) = \{(g,h) : g \in V(G) \text{ and } h \in V(H)\}$ and edge set $E(G \times H) = \{(g,h)(g',h') : gg' \in E(G) \text{ and } hh' \in E(H)\}$. This product is called the *categorial product* or the *tensor product* in the literature. As this is the only product we shall discuss in this paper, we shall simply call it the product.

Given an n-coloring c of the graph G, it is straightforward to verify that the mapping $c'((g,h)) = c(g)$ is an n-coloring of the product $G \times H$. Therefore, $\chi(G \times H) \le \chi(G)$. Similarly, we have $\chi(G \times H) \le \chi(H)$, and hence

$$\chi(G \times H) \le \min\{\chi(G), \chi(H)\}. \tag{1}$$

Hedetniemi's conjecture [22] asserts that the equality holds in (1) for all graphs G and H.

Conjecture 1 *For any finite simple graphs G and H,*

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}.$$

This conjecture is also known as the Lovász-Hedetniemi's conjecture [35], or the product conjecture.

For a positive integer n, we define $C(n)$ to be the following statement:

$C(n)$: *If graphs G and H are not n-colorable, then their product $G \times H$ is not n-colorable.*

In light of inequality (1), Hedetniemi's conjecture is equivalent to say that $C(n)$ is true for all $n \geq 1$.

Hedetniemi's conjecture has enjoyed considerable attention. Many authors tried many different approaches to this conjecture. Just like any other difficult conjecture, all the efforts trying to settle it bring up more problems than solutions. We shall survey methods and partial results; and discuss questions related to or motivated by Hedetniemi's conjecture.

2 Exponential graphs

One approach to Hedetniemi's conjecture is the *exponential graph method*, which was first used in [15] to the study of Hedetniemi's conjecture, although the exponential graphs were studied much earlier [6, 37]. Given a graph G and an integer n, the *exponential graph* K_n^G has as vertices all the mappings from $V(G)$ to $V(K_n) = \{1, 2, \cdots, n\}$. Two such mappings f and g are adjacent in K_n^G if for every edge $ab \in E(G)$ we have $f(a) \neq g(b)$. It follows from the definition that K_n^G has no loop if and only if $\chi(G) > n$.

We define a *homomorphism* from a graph G to another graph H as a mapping $h : V(G) \to V(H)$ such that for any edge xy of G the images $f(x)f(y)$ is an edge of H. We write $G \preceq H$ to indicate that there is a homomorphism from G to H. Then the set of all finite graphs forms a partially ordered set. An important property of this partial order is that $G \preceq H$ implies that $\chi(G) \leq \chi(H)$. Indeed, any n-coloring c of H induces an n-coloring c' of G through a homomorphism h from G to H, defined as $c'(x) = c(h(x))$.

For any graph G, let $\mathcal{S}_n(G)$ be the set of graphs H such that $\chi(G \times H) \leq n$. It turns out that the exponential graph K_n^G is the largest element of $\mathcal{S}_n(G)$ with respect to the order of homomorphism. First we show that $K_n^G \in \mathcal{S}_n(G)$.

Lemma 1 *For any graph G and for any integer n, the graph $G \times K_n^G$ is n-colorable.*

Proof. It is straightforward to verify that the coloring $c : V(G \times K_n^G) \to \{1, 2, \cdots, n\}$ defined as $c((x, f)) = f(x)$ is an n-coloring of $G \times K_n^G$. ∎

Next we shall show that K_n^G is the maximum element of $\mathcal{S}_n(G)$.

Lemma 2 *If H is a graph with $\chi(G \times H) \leq n$, then $H \preceq K_n^G$.*

Proof. If c is a coloring of $G \times H$ with n colors $1, 2, \cdots, n$, then the mapping $f : V(H) \to V(K_n^G)$ defined as $(f(x))(y) = c(x, y)$ for all $y \in V(G)$ is a homomorphism of H to K_n^G. ∎

Combining Lemmas 1 and 2, we have the following result:

Theorem 1 *Given two graphs G and H. The following are equivalent:*

1. *The product $G \times H$ is n-colorable.*
2. *H admits a homomorphism to K_n^G.*
3. *G admits a homomorphism to K_n^H.*

Using Theorem 1, we conclude that statement $C(n)$ is equivalent to the following statement $C'(n)$.

$C'(n)$: *If $\chi(G) > n$, then K_n^G is n-colorable.*

Indeed if $C'(n)$ is false, then there is a graph G with $\chi(G) > n$ such that $\chi(K_n^G) > n$. Then we have two graphs G and K_n^G, none of which is n-colorable and yet their product is n-colorable, contrary to $C(n)$. On the other hand, if $C(n)$ is false, then there are graphs G and H such that both G and H are not n-colorable, but $G \times H$ is n-colorable. Then H admits a homomorphism to K_n^G, and hence K_n^G is not n-colorable, contrary to $C'(n)$.

The exponential graph method was used in [15] to show that $C(3)$ is true. In other words, El-Zahar and Sauer proved the following result:

Theorem 2 *If G and H are not 3-colorable, then their product $G \times H$ is also not 3-colorable.*

Theorem 2 is probably the most significant progress in the investigation of Hedetniemi's conjecture. We note that $C(1)$ is trivial. The validity of $C(2)$ was established in [22] by observing that the product of two odd cycles contains an odd cycle. Theorem 2 was published a decade ago, and yet no attempts to generalize it has succeeded. The proof was quite complicated. However, we shall still give a complete proof of Theorem 2, because it is important and also because we would like to discuss possible ways to extend it. The proof is essentially the one in [15], with some re-arrangements and simplification to make it shorter.

Proof of Theorem 2. We first consider the exponential graphs $K_3^{C_n}$, where C_n denotes the cycle of n vertices, say $v_1, v_2, \cdots, v_n$. For a vertex f of $K_3^{C_n}$, a vertex v_i of C_n is called a *fixed point* of f if $f(v_{i-1}) \neq f(v_{i+1})$.

Claim 1. *For two adjacent vertices f_1 and f_2 of $K_3^{C_n}$, where n is odd, the number of fixed points of f_1 has the same parity as the number of fixed points of f_2.*

Let f_1 and f_2 be two adjacent vertices in $K_3^{C_n}$. Consider the product $C_n \times K_2$ which has vertices $\{(v_i, a_j) : i = 1, 2, \cdots, n \text{ and } j = 1, 2\}$, and which is isomorphic to an even cycle C_{2n}. It is straightforward to verify that the mapping $f \in V(K_3^{C_n \times K_2})$ defined as $f(c_i, a_j) = f_j(c_i)$ is a proper coloring of $C_n \times K_2$, and that the number of fixed points of f is the sum of the fixed points of f_1 and f_2. Therefore to prove the claim, it suffices to show that f has an even number fixed points. However it can be proved easily by induction on the number of vertices that any proper 3-coloring of an even cycle has an even number of fixed points. This completes the proof of Claim 1.

Suppose H is a graph with $\chi(H) \geq 4$. Let $f \in V(K_3^H)$ be a non-isolated vertex of K_3^H. We shall show that

Claim 2. *There is an odd cycle, say C_n, of H such that f has an even number of fixed points on C_n.*

Let $X = \{x \in V(H) : \exists y \in V(H), xy \in E(H), f(x) = f(y)\}$. We claim that X contains an odd cycle. Otherwise X can be partitioned into two independent subsets X_1 and X_2. Let g be any vertex adjacent to f in K_3^H. It is straightforward to verify that the mapping c defined as

$$c(x) = f(x) \text{ for } x \in V(H) - X_1 \text{ and } c(x) = g(x) \text{ for } x \in X_1$$

is a proper 3-coloring of H, contrary to the assumption that $\chi(H) \geq 4$.

Let $C_n = (v_1, v_2, \cdots, v_n)$ be an odd cycle of X. We shall show that f has an even number of fixed points on C_n. Consider monochromatic intervals $(v_i, \cdots, v_{i+k})$ of C_n. Such an interval contributes 2 fixed points if $k \geq 1$. If $k = 0$ then it contributes 0 or 1 fixed point depending on whether $f(v_{i-1}) = f(v_{i+1})$ or not. Therefore if f has an odd number fixed points on C_n, then there are three consecutive vertices, say (v_{i-1}, v_i, v_{i+1}), of the cycle C_n which get different colors from f. Since g is adjacent to f, we conclude that $g(v_i) \neq f(v_{i-1}), f(v_{i+1})$, and hence $g(v_i) = f(v_i)$. However this is a contradiction, as there exists a vertex y of H such that $v_{i+1}y \in E(H)$ and $f(y) = f(v_i)$. This completes the proof of Claim 2.

Assume contrary to Theorem 2 that there are connected graphs G and H with $\chi(G) \geq 4$ and $\chi(H) \geq 4$ and that $\chi(G \times H) \leq 3$. Let f be a proper 3-coloring of $G \times H$. For each vertex $x \in V(G)$, let $f_x \in V(K_3^H)$ be defined as $f_x(y) = f(x, y)$. Similarly, for each $y \in V(H)$, let $f_y \in V(K_3^G)$ be defined as $f_y(x) = f(x, y)$. It is easy to see that $xx' \in E(G)$ implies that $f_x f_{x'} \in E(K_3^H)$. Hence f_x is a non-isolated vertex of K_3^H for each $x \in G$. Therefore, there is an odd cycle C_n of H such that f_x has an even number of fixed points on C_n. As adjacent vertices in K_3^H have the same parity of fixed points on the cycle C_n, we conclude that for all $x \in V(G)$, f_x have even number of fixed points on C_n. Similarly, there is an odd cycle, say C_m, of G such that for all y, f_y have even number of fixed points on C_m.

Suppose C_n has vertex set $\{v_1, v_2, \cdots, v_n\}$ and C_m has vertex set $\{u_1, u_2, \cdots, u_m\}$. Let M_i be the number of fixed points of f_{v_i} on C_m, and let N_j be the number of fixed points of f_{u_j} on C_n. As all M_i and N_j are even, and nm is odd, it follows that

$$mn - \sum_{i=1}^{m} M_i - \sum_{j=1}^{n} N_j$$

is odd.

Figure 1 below shows some of the edges of $C_n \times C_m$. Let

$$Q_{i,j} = \{(v_{i-1}, u_j), (v_i, u_{j+1}), (v_{i+1}, u_j), (v_i, u_{j-1})\}$$

be the quadrilateral as shown in Figure 1. Then the edges of $C_n \times C_m$ is partitioned into k disjoint "cycles" of quadrilaterals, each of the form $Q_{i,j}Q_{i+1,j+1}\cdots$, where $k = \gcd(m,n)$. It is easy to verify that if a quadrilateral $Q_{i,j}$ is colored by 3 colors, then either v_i is a fixed point of f_{u_j} or u_j is a fixed point of f_{v_i}, and hence it contributes 1 to the sum $\Sigma M_i + \Sigma N_j$; and if $Q_{i,j}$ is colored by 2 colors, then neither v_i is a fixed point of f_{u_j} nor u_j is a fixed point of f_{v_i} and hence it contributes 0 to the sum $\Sigma M_i + \Sigma N_j$. Therefore $mn - \Sigma M_i - \Sigma N_j$ is equal to the number of quadrilaterals $Q_{i,j}$ that are colored by 2 colors.

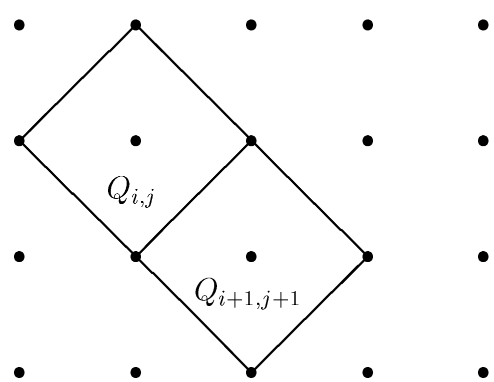

Fig. 1.

We define an orientation of the edges of $C_n \times C_m$ such that the arrows goes from colors 1 to 2, 2 to 3 and 3 to 1. It is easily verified that if $Q_{i,j}$ is colored by 3 colors then the opposite sides of $Q_{i,j}$ have parallel orientations, and if $Q_{i,j}$ is colored by 2 colors then the opposite sides of $Q_{i,j}$ have opposite orientations.

Consider a "cycle" $Q_{i,j}Q_{i+1,j+1}\cdots$ of quadrilaterals. "Adjacent" quadrilaterals share a common edge. The number of quadrilaterals in this sequence in which the opposite sides are oriented in opposite directions is equal to the number of times that the "common" edges change directions in the sequence, which is even as it starts and finishes at the same edge. Therefore $mn-\Sigma M_i-\Sigma N_j$ is even, contrary to our previous conclusion. This completes the proof of Theorem 2. ∎

This proof made extensive use of the structure of odd cycles, which are the only critical 3-chromatic graphs. One difficulty in generalizing this proof is that we know almost nothing about the structure of critical n-chromatic graphs for $n \geq 4$.

The above argument actually proved a stronger statement, namely it is proved that if C_n is an odd cycle of G and C_m is an odd cycle of H, then $(C_n \times H) \cup (G \times C_m)$ is not 3-colorable, provided that G and H are not 3-colorable.

Based on this observation, the following conjecture was proposed in [15]:

Conjecture 2 *Let G and H be connected graphs which are not n-colorable. Let G' and H' be n-chromatic subgraphs of G and H respectively. Then $(G \times H') \cap (G' \times H)$ is not n-colorable.*

It is trivial that Conjecture 2 implies Conjecture 1.

A crucial concept in the proof of Theorem 2 is the parity of the number of fixed points of an element $f \in V(K_3^{C_n})$. It seems that what really matters is which component of K_n^H contains the element f in consideration. We denote by $C(K_n^H)$ the component of K_n^H which contains the constant mappings. (Note that the n constant mappings $f_i(v) = i$ for $1 \leq i \leq n$ form a complete subgraph.) We propose the following conjecture corresponding to Claim 2.

Conjecture 3 *Suppose $\chi(H) > n$ and that f is contained in a component of K_n^H which has chromatic number at least n. Then there is a subgraph H' of H with $\chi(H') = n$ and that $f|H' \in C(K_n^{H'})$. Here $f|H'$ is the restriction of f to H'.*

Corresponding to the argument of counting the 2-colored quadrilaterals, we propose the following conjecture for the general case.

Conjecture 4 *Suppose G and H are n-chromatic graphs. Let f be a proper n-coloring of $G \times H$. For $x \in V(G)$, let $f_x \in V(K_n^H)$ be defined as $f_x(y) =$*

$f(x,y)$. Similarly, we define $f_y \in V(K_n^G)$ for each $y \in V(H)$. Then either $f_x \notin C(K_n^H)$ or $f_y \notin C(K_n^G)$.

Theorem 3 *For any integer $n \geq 1$, if Conjectures 3 and 4 are true, then statement $C(n)$ is true.*

Proof. Assume to the contrary that statement $C(n)$ is false, and that Conjectures 3 and 4 are true. Let G and H be connected graphs such that $\chi(G) > n$ and $\chi(H) > n$ and that $G \times H$ is n-colorable. Let f be a proper n-coloring of $G \times H$. Because Conjecture 3 is true, there are subgraphs G' and H' of G and H respectively such that for $\chi(G') = \chi(H') = n$ and that for each $x \in V(G')$, $f_x \in C(K_n^{H'})$ and for each $y \in V(H')$, $f_y \in C(K_n^{G'})$. However this is in contrary to Conjecture 4. ∎

The exponential graph method can be used to simplify the proofs of many other special cases of Hedetniemi's Conjecture.

For example, the following result proved by Turzik [46] can be easily proved by using exponential graphs.

Theorem 4 *If $\chi(G) > n$ and for every pair of edges e_i and e_j there exists an edge e adjacent to both of them, then for any graph H with $\chi(H) > n$, the product $G \times H$ is not n-colorable.*

By Theorem 1, Theorem 4 is equivalent to the statement that K_n^G is n-colorable. This is quite obvious. Indeed, for any $f \in V(K_n^G)$, let $e = ab$ be an edge of G such that $f(a) = f(b)$ (such an edge exists because $\chi(G) > n$). Then the coloring c defined as $c(f) = f(a) = f(b)$ is a proper n-coloring of K_n^G.

Suppose G is a connected graph with $\chi(G) > n$. One important property of K_n^G is that K_n^G contains a unique complete subgraph of order n, namely the subgraph induced by the n constant mappings. To prove this fact, we let $f_1, f_2, \cdots, f_n$ be the vertices of a complete subgraph of K_n^G. Let H be a critical $(n+1)$-chromatic subgraph of G. We claim that for each $i = 1, 2, \cdots, n$, for each $x \in H$, there is a vertex $y \in H$ such that $f_i(x) = f_i(y)$. If this is not true, say, $f_1(x) \neq f_1(y)$ for any neighbour y of x, then we partition $H - x$ into n independent subsets $V_1, V_2, \cdots, V_n$ and define a coloring f of H as follows:

$$f(x) = f_1(x),\ f(y) = f_i(y) \text{ if } y \in V_i.$$

Then f is a proper n-coloring of H, contrary to our assumption that $\chi(H) = n + 1$. From this we deduce that $f_i(x) \neq f_j(x)$ for any x and any $i \neq j$. It follows then that $f_i(x) = f_i(y)$ for any edge xy of H. Thus f_i are constant mappings on H, and since G is connected, we conclude that f_i are constant mappings on G.

Using this fact, the proofs of the following results of Duffus, Sands and Woodrow [13], of Welzl [48], and of Burr, Erdös and Lovász [11] can be simplified.

Theorem 5 ([13, 48]) *Let G and H be connected graphs with $\chi(G) > n$ and $\chi(H) > n$. If each of G and H contains a clique of size n, then $\chi(G \times H) > n$.*

Proof. Assume to the contrary that $G \times H$ is n-colorable, and that f is an n-coloring of $G \times H$. Suppose $x_1, \cdots, x_n$ are the vertices of a clique of G, and that $y_1, \cdots, y_n$ are the vertices of a clique of H. Then $f_{x_1}, \cdots, f_{x_n}$ induces a complete subgraph of K_n^H, and $f_{y_1}, \cdots, f_{y_n}$ induces a complete subgraph of K_n^G. Therefore $f(x_i, y)$ is independent of y, and $f(x, y_i)$ is independent of x, which is an obvious contradiction. ∎

Theorem 6 ([11]) *If $\chi(G) > n$ and every vertex of G is contained in a clique of size n, then for any graph H with $\chi(H) > n$, the product $G \times H$ has chromatic number greater than n.*

Proof. Assume to the contrary that $G \times H$ is n-colorable, and that f is an n-coloring of $G \times H$. Then $\alpha : G \to K_n^H$ defined as $\alpha(x) = f_x$ is a homomorphism from G to K_n^H. Therefore f_x are constant mappings. Then $c(x) = f_x(y)$ is a proper n-coloring of G, contrary to our assumption that $\chi(G) > n$. ∎

3 Hajós' construction

Another approach to Hedetniemi's conjecture, first used by Duffus, Sands and Woodrow [13], is by using a theorem of Hajós concerning the construction of graphs with chromatic number greater than n.

Suppose that G and H are simple graphs, $ab \in E(G)$ and $uv \in E(H)$. The *Hajós sum* of G and H (with respect to ab and uv), denoted by $G \oplus H$, is the graph obtained from the disjoint union of G and H by contracting a and u into a single vertex, deleting the edges ab and uv and adding the edge bv.

If G and H are graphs of chromatic number greater than n, then it is easy to see that $G \oplus H$ is also of chromatic number greater than n. Adding vertices and edges to a graph G or contracting two non-adjacent vertices of G will not decrease its chromatic number. Hajós [18] proved that all graphs with chromatic number at least $n+1$ can be constructed from copies of K_{n+1} by these operations.

Theorem 7 *Let $\mathcal{G}_n$ be the set of graphs which contains K_{n+1} and which is closed under the following operations:*

1. *adding edges and vertices;*
2. *contracting non-adjacent vertices;*
3. *Hajós sums.*

Then $\mathcal{G}_n$ contains all graphs of chromatic number greater than n.

Fix an integer n, a graph G of chromatic number greater than n is called *persistent* if for all $(n+1)$-chromatic graphs H we have $\chi(G \times H) = n+1$. By Theorem 1, a graph G is persistent if and only if K_n^G is n-colorable.

Then statement $C(n)$ is equivalent to say that all graphs G with $\chi(G) > n$ are persistent. Since every graph G with $\chi(G) > n$ can be constructed from copies of K_{n+1} by the three operations above, it suffices to prove that K_{n+1} is persistent, and that all the three operations preserve persistency, in order to establish Hedetniemi's conjecture.

It is trivial that K_{n+1} is persistent, and that the operation of adding edges and vertices preserves persistency, and the operation of contraction also preserve persistency. However it is unknown whether or not the operation of Hajós sum preserves persistency.

To overcome this difficulty, some properties that are stronger than persistency were studied. Usually such stronger properties are preserved by Hajós sum. However, such properties may not be preserved by contraction. By using such an approach, Duffus, Sands and Woodrow [13] proved that if a graph G is constructed from copies of K_{n+1} by performing the three operations in such a way that all contractions are performed after all the Hajós sums then G is persistent. This result was generalized by Sauer and Zhu in [44], where the class of known persistent graphs is considerably extended.

A graph G is called *strongly persistent* if G is persistent and the Hajós sum of G with any other persistent graph H is still persistent. It was proved in [44] that if G is constructed from copies of K_{n+1} by Hajós sums, adding vertices and edges and at most one contraction then G is strongly persistent.

Theorem 8 *Suppose G is obtained from copies of K_{n+1} by means of Hajós sums, adding vertices and edges and at most one contraction then G is strongly persistent. Moreover, the Hajós sum of two strongly persistent graphs is a strongly persistent graph.*

Theorem 8 was proved by considering some properties of graphs that are stronger than persistency. For a graph H, let $L(K_n^H)$ be the graph obtained from K_n^H by deleting the loops.

Let G be a graph of chromatic number greater than n and let a be a vertex of G. We denote by $G - a$ the subgraph of G induced by $V(G) \setminus \{a\}$. A vertex a is said to have property $(*)$ if $(1)^*$ and $(2)^*$ hold.

$(1)^*$ The graph $L(K_n^{G-a})$ is n-colorable.

$(2)^*$ If ff and fg are edges of K_n^{G-a}, then $f = g$.

A graph G is said to have property $(*)$ if every vertex of G has property $(*)$.

For an edge ab of G, we denote by $G - ab$ the subgraph of G obtained from G by removing the edge ab. An edge ab of G is said to have property $(**)$ if $(1)^{**}$ and $(2)^{**}$ hold.

$(1)^{**}$ The graph $L(K_n^{G-ab})$ is n-colorable.

$(2)^{**}$ If ff and fg are edges of K_n^{G-ab}, then $f = g$.

The graph G is said to have property $(**)$ if every edge of G has property $(**)$.

It turns out that each of the properties $(*)$ and $(**)$ are "stronger" than that of being strongly persistent.

Lemma 3 *Suppose $\chi(G) > n$. If G has property $(*)$ or property $(**)$, then G is strongly persistent.*

Moreover, it was shown in [44] that both property $(*)$ and property $(**)$ are preserved by Hajós sum.

Lemma 4 *If both G and H have property $(*)$ (resp., $(**)$), then the Hajós sum $G \oplus H$ also has property $(*)$ (resp., $(**)$).*

As it is straightforward to verify that the complete graph K_{n+1} has property $(*)$. it follows that if G is constructed from copies of K_{n+1} by Hajós sums then G has property $(*)$.

The property $(*)$ is not preserved by contraction of non-adjacent vertices. However, if there is only one contraction, then the resulting graph is not very bad. It may not have property $(*)$, but it was shown that such a graph has property $(**)$. This was proved by observing the following relation between $(*)$ and $(**)$:

Lemma 5 *If G is a graph of chromatic number greater than n and a is a vertex of G with property $(*)$, then for any other vertex b of G with $ab \in E(G)$, the edge ab has property $(**)$.*

It is easy to see that if G has property $(*)$, then after one contraction, for each edge ab of G, at least one of the vertices a and b have property $(*)$. Therefore the resulting graph has property $(**)$. It is trivial that the addition of edges and vertices preserve property $(*)$ as well as property $(**)$. This shows that graphs obtained from copies of K_{n+1} by means of Hajós sums, adding vertices and edges and at most one contraction are strongly persistent. As Hajós sum is associative, it follows that the Hajós sum of two strongly persistent graphs is still strongly persistent. This completes the proof of Theorem 8

We note that there are examples that shows that properties $(*)$ and $(**)$ are not preserved by contraction.

4 Multiplicativity

A proper n-coloring of a graph G is equivalent to a homomorphism from G to the complete graph K_n. In this language, Hedetniemi's conjecture says that if both G and H do not admit homomorphism to K_n then their product does not admit a homomorphism to K_n.

It is natural that we may replace K_n by an arbitrary graph M and ask whether or not the same statement is true. We say a graph G is M-colorable if G admits a homomorphism to M. We say a graph M is *multiplicative* if for any graphs G and H both not M-colorable, their product $G \times H$ is also not M-colorable. In other words, the property of being not M-colorable is preserved by production. Hedetniemi's conjecture says that complete graphs are multiplicative.

The concept of multiplicativity of general graphs and digraphs were first studied in [42], where the term "productivity" was used. The name "multiplicativity" was first used in [19]. It is hoped that in studying the multiplicativity, and non-multiplicativity, of other graphs, we will gain insights relevant for Hedetniemi's conjecture. Whether or not such a hope will turn into reality remains to be seen. However, the problem of multiplicativity of general graphs and digraphs seems to be an interesting problem by itself and has been studied extensively. Indeed, many results suggest that homomorphism of graphs is the proper way of viewing coloring problems.

Theorem 2 in Section 2 is equivalent to the statement that K_3 is multiplicative. This result was generalized by Häggkvist, Hell, Miller and Neumann Lara. It was proved in [19] that odd cycles are multiplicative.

Theorem 9 *Let C be an odd cycle. If neither G nor H admits a homomorphism to C, then $G \times H$ does not admit a homomorphism to C.*

The proof of Theorem 9 is closely parallel to the proof of Theorem 2.

We remark that the odd cycles, the complete graphs K_3, K_2, K_1 are the only known graphs which are multiplicative, and which are cores (i.e., does not admit a homomorphism to any of its proper subgraphs).

It is easy to construct non-multiplicative graphs. Take two graphs G and H such that none of which admits a homomorphism to the other (such graphs are abundant, for example, the Grötzsch graph and K_3). Then the product $G \times H$ is non-multiplicative. Indeed, neither G nor H admits a homomorphism to $G \times H$, and yet $G \times H$ admits a homomorphism to $G \times H$. However, we do not know other non-multiplicative graphs. There are not many useful tools for testing which graph is multiplicative.

The multiplicativity and non-multiplicativity of digraphs were investigated by many authors. It was proved by Poljak and Rödl [41] that the complete digraphs of order at least 3 are not multiplicative. In other word, the "directed" version of Hedetniemi's conjecture is false.

Theorem 10 *For any integer $n \geq 3$, there are digraphs G and H, neither of which is n-colorable, and yet their product is n-colorable.*

The proof is by construction. For $n = 3$, we may take G the transitive tournament of order 4, and take H the digraph obtained from G by reversing the orientation of the edge from the top to the bottom, i.e., $V(H) = \{1, 2, 3, 4\}$ and $E(H) = \{(i, j), i < j \text{ and } (i, j) \neq (1, 4)\} \cup \{(4, 1)\}$. Neither G nor H is 3-colorable, however $G \times H$ is 3-colorable. For $n \geq 4$, the digraphs G and H with $\chi(G) = \chi(H) > n$ and $\chi(G \times H) = n$ are constructed similarly.

For oriented paths P (i.e., P is obtained from an undirected path by assigning an orientation to each edge) which are cores, it was shown in [50, 58] that P is multiplicative if and only if P is a directed path (i.e., all the edges have the same orientation).

The multiplicativity of directed cycles was first studied in [42], where it was proved that directed cycles of prime length is multiplicative, and directed cycles of length which is not a prime power is non-multiplicative. It was conjectured in [42] that directed cycles of prime power length is multiplicative.

This conjecture was confirmed in [19], where deep theorems from homology were used. A simple combinatorial proof of this conjecture was given in [57].

For general oriented cycles (i.e., digraphs obtained from an undirected cycle by assigning an orientation to each edge), the problem turned out to be more difficult. A complete classification of multiplicative oriented cycles was given in [32]. To state the classification, we need some technical definitions.

Let B_n, S_n and T_n be the digraphs in Figure 2.

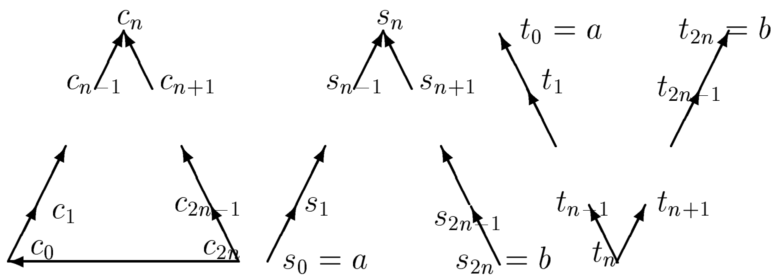

Fig. 2 The digraphs B_n, S_n and T_n

We now inductively define the class of $\mathcal{C}$-cycles:

1. Each B_n is a $\mathcal{C}$-cycle.
2. Let C be a $\mathcal{C}$-cycle and let v be a vertex of out-degree 2 (respectively in-degree 2). Then there are two maximal directed paths P and P' starting (respectively, ending) at v, say of lengths $l \leq l'$. Let $m \leq l$ be an integer. Replace v by S_m (respectively, by T_m), identifying a with the beginning of P and b with the beginning of P'. The resulting digraph C' is also a $\mathcal{C}$-cycle.
3. There are no other $\mathcal{C}$-cycles.

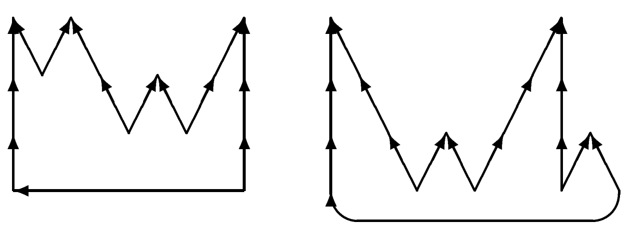

Fig. 3. Examples of C-cycles

Theorem 11 *Let C be an oriented cycle which is a core. Then C is multiplicative if and only if either C is a directed cycle of prime power length or C is a $\mathcal{C}$-cycle.*

The proof of Theorem 11 is very long and complicated. It relies on characterizations of digraphs which admit homomorphism to the oriented cycles, and these characterizations are simply stated and easily verified. Indeed, based on a generalization of these characterizations, it was shown in [34] that the H-coloring problem is polynomial time decidable for any unbalanced oriented cycle H. Here an oriented cycle is unbalanced if the number of forward edges is not equal to the number of backward edges along an arbitrary traversal of the cycle. The H-coloring problem is the decision problem with an arbitrary digraph (or undirected graph when H is undirected) G as an instance, and the question is whether or not G is H-colorable. The K_n-coloring problem is just to decide whether a given instance is n-colorable or not.

There are some other digraphs which are known to be multiplicative, including transitive tournaments [2], a transitive tournament followed by a directed path [55]. There are also other digraphs which are known to be non-multiplicative, including all those digraphs obtained from a transitive tournament by deleting edges which is neither a transitive tournament nor a transitive tournament followed by a directed path [55].

All the present known proofs of multiplicativity of a digraph H relies on good characterizations of those digraphs G which admit homomorphisms to H. Such characterizations usually provide easy polynomial algorithms for the H-coloring problem. We must say that it is unlikely that such method can be applied to many digraphs, and is perhaps not appliable to any non-bipartite undirected graphs, as the H-coloring problem is NP-complete for any non-bipartite undirected graph H [25]. For most digraphs H, it seems unlikely that there are simple characterizations of those digraphs which admit a homomorphism to H. Therefore, it would be more interesting if we can prove the multiplicativity of a digraph (resp., undirected graph) H, for which the H-coloring problem is NP-complete.

At present time, the only known undirected graph H which are multiplicative and for which that H-coloring problems are NP-complete are the odd cycles. We do not know any directed graph H (except those symmetric digraphs equivalent to odd cycles) which is multiplicative, and for which the H-coloring problem is NP-complete.

This makes the following two digraphs D and D' very interesting, and also seems challenging.

Both D and D' have three vertices $\{1, 2, 3\}$, D has arcs $\{12, 23, 31, 13\}$ and D' has arcs $\{12, 23, 32, 31, 13\}$. Thus D and D' have 4 and 5 arcs, respectively. The digraph with 3 vertices and 6 arcs is the complete digraph of order 3. As we discussed earlier, this digraph is non-multiplicative. All digraphs with 3 vertices and at most 3 arcs are known to be multiplicative. Also we know that both D-coloring problem and D'-coloring problem are NP-complete [1].

Another interesting digraph is the one obtained by identifying a vertex of a directed triangle with a digon. To be precise, this directed graph has vertex set $\{1, 2, 3, 4\}$ and edge set $\{12, 23, 31, 34, 43\}$. We also know that for this digraph H, the H-coloring problem is NP-complete. This suggests that it is unlikely that there is a simple characterization of those digraphs G which are H-colorable.

For non-multiplicative graphs or digraphs H, Duffus and Sauer [14] introduced a parameter that "measures" the "degree" to which H fails to be multiplicative. We say two graphs (or digraphs) G and G' are *homomorphically equivalent*, denoted by $G \sim G'$, if G admits a homomorphism to G' and G' admits a homomorphism to G. Obviously $\sim$ is an equivalence relation. Denote by $\mathbf{FG}$ the set of equivalence classes of finite graphs (or digraphs) under the equivalence relation $\sim$. We shall use any member of an equivalence class in $\mathbf{FG}$ to denote that class, and do not distinguish a graph and an equivalence class of graphs in $\mathbf{FG}$. For two classes $G, H \in \mathbf{FG}$, we write $G \preceq H$ if G admits a homomorphism to H. Then it is straightforward to verify that $\mathbf{FG}$ together with the relation $\preceq$ form a distributive lattice, with the product as the *meet*, and the disjoint union as the *join*.

It is straightforward to verify that for any graph (or digraph) M, the set $\{M^H : H \in \mathbf{FG}\}$ contains at least two elements, one is a single vertex with a loop, and the other is M. It follows easily from Theorem 1 that a graph (or digraph) M is multiplicative if and only if the set $\{M^H : H \in \mathbf{FG}\}$ contains no other elements. Therefore, the size of the set $\{M^H : H \in \mathbf{FG}\}$ maybe taken as the measure of "degree" to which H fails to be multiplicative.

We do not know any graph or digraph M for which the set $\{M^H : H \in \mathbf{FG}\}$ is infinite. On the other hand, we do not know whether or not the set $\{K_n^H : H \in \mathbf{FG}\}$ is finite, while Hedetniemi's conjecture says that this set contains exactly two elements.

5 The function $f(n)$ and $g(n)$

In the study of Hedetniemi's conjecture, Poljak and Rödl defined the functions $f(n)$ and $g(n)$ as follows:

$$f(n) = \min\{\chi(G \times H) : G \text{ and } H \text{ are undirected graphs with } \chi(G) = \chi(H) = n\},$$

$$g(n) = \min\{\chi(G \times H) : G \text{ and } H \text{ are digraphs with } \chi(G) = \chi(H) = n\}.$$

Inequality (1) shows that $f(n) \leq n$ and Hedetniemi's conjecture says that $f(n) = n$. It is straightforward to verify that $g(n) \leq f(n)$, and Theorem 10 shows that $g(n) < n$ for all $n \geq 4$. We also know that $f(1) = g(1) = 1, f(2) = g(2) = 2, f(3) = g(3) = 3$. Besides these, we know very little about the functions. A very annoying fact is that we do not know whether the functions $f(n)$ and $g(n)$ are bounded, or go to infinity when n goes to infinity. However we have the following interesting result:

Theorem 12 *The function $g(n)$ is either bounded by 3 or goes to infinity; the function $f(n)$ is either bounded by 9 or goes to infinity.*

This theorem is a generalization of a similar result of Poljak and Rödl [41], where the corresponding numbers are 4 and 16 instead of 3 and 9. The generalization is straightforward. I obtained this generalization in 1990, and learned afterwards from Duffus that the generalization was obtained earlier by Schelp, and learned from Hell that it was also obtained independently by Poljak [40], although at that time none of them published it.

We shall first prove that $g(n)$ is either bounded by 4 or goes to infinity.

We need a few definitions. For a digraph D, two arcs x and y of D are called consecutive arcs if the terminal vertex of x is the initial vertex of y. We shall denote by $\partial(D)$ the digraph which has vertices all the arcs of D and two arcs x and y of D are connected by an arc in $\partial(D)$ if x and y are consecutive arcs. Let $c(D)$ be the arc-chromatic number of D, which is the minimum integer n such that the arcs of D can be colored by n colors in such a way that consecutive arcs receive distinct colors. In other words, $c(D)$ is the chromatic number of $\partial(D)$. We denote by D^{-1} the digraph obtained from D by reversing the directions of all the arcs.

It follows easily from the definition that

(1) $c(D) = \chi(\partial(D))$,

(2) $\partial(D_1 \times D_2) = \partial(D_1) \times \partial(D_2)$,

(3) $\partial(D^{-1}) = (\partial(D))^{-1}$.

Furthermore, it was proved in [21] that for any digraph D, we have

(4) $\min\{k|\ 2^k \geq \chi(D)\} \leq c(D) \leq \min\{k|\ \chi(D) \leq \binom{k}{\lfloor k/2 \rfloor}\}$.

Suppose $g(n)$ is bounded. Let c be the smallest upper bound. Since $g(n)$ is non-decreasing, we conclude that there is an integer n_0 such that fro all $n \geq n_0$, $g(n) = c$.

Let $n_1 = 2^{n_0}$, and let D_1 and D_2 be digraphs with $\chi(D_1) = \chi(D_2) = n_1$ and that $\chi(D_1 \times D_2) = c$. Then by (4), we have

$$\chi(\partial(D_1)) \geq n_0 \text{ and } \ \chi(\partial(D_2)) \geq n_0.$$

Hence $\chi((\partial(D_1)) \times (\partial(D_2))) = \chi(\partial(D_1 \times D_2)) \geq c$.

Then by (4), we have

$$\chi(D_1 \times D_2) > \binom{c-1}{\lfloor (c-1)/2 \rfloor}.$$

Therefore we have

$$c > \binom{c-1}{\lfloor (c-1)/2 \rfloor},$$

which implies that $c \leq 4$.

Now we shall prove that $c \neq 4$. Otherwise suppose $g(n) = 4$ for all $n \geq n_0$. Let $n_1 = 2^{n_0}$ and $n_2 = 2^{n_1}$. Let D_1 and D_2 be digraphs with $\chi(D_1) = \chi(D_2) = n_2$ and that $\chi(D_1 \times D_2) = 4$. The same argument as above shows that

$$\chi(\partial(\partial(D_1 \times D_2))) \geq 4.$$

However, we shall prove that for any digraph D, if $\chi(D) \leq 4$, then $\chi(\partial(\partial(D))) \leq 3$. Let $\overrightarrow{K_4}$ be the complete digraph of order 4, i.e., $\overrightarrow{K_4}$ has vertex set $\{1, 2, 3, 4\}$ and edge set $\{ij : i \neq j, 1 \leq i, j \leq 4\}$. If D is 4-colorable, then D admits a homomorphism to $\overrightarrow{K_4}$. Hence $\partial(\partial(D))$ admits a homomorphism to $\partial(\partial(\overrightarrow{K_4}))$. Therefore $\chi(\partial(\partial(D))) \leq \chi(\partial(\partial(\overrightarrow{K_4})))$. So it suffices to prove that $\partial(\partial(\overrightarrow{K_4}))$ is 3-colorable.

The following elegant coloring was given by Schelp, which I learned from Duffus.

Each element of $\partial(\partial(\overrightarrow{K_4}))$ is a pair of consecutive arcs of $\overrightarrow{K_4}$, which can be represented by a triple $ijk : i \neq j, j \neq k, 1 \leq i, j, k \leq 4$. Two such triples $ijk, i'j'k'$ are adjacent if and only if $i' = j$ and $j' = k$. It is then straightforward to verify that the coloring c defined below is a proper 3-coloring of $\partial(\partial(\overrightarrow{K_4}))$:

$$c(ijk) = \begin{cases} j, & \text{if } \ j \neq 4, \\ s, & \text{if } \ j = 4 \text{ and } \ s \in \{1, 2, 3\} - \{i, k\}. \end{cases}$$

This completes the proof of the statement that $g(n)$ is either bounded by 3 or goes to infinity.

Now we shall prove that $f(n)$ is either bounded by 9 or goes to infinity.

Let $h(D_1, D_2) = \max\{\chi(D_1 \times D_2), \chi(D_1 \times D_2^{-1})\}$ and let $h(n) = \min\{h(D_1, D_2) : \chi(D_1) = \chi(D_2) = n\}$.

The proof of the statement that $g(n)$ is either bounded by 3 or goes to infinity applies to $h(n)$ as well. In other words, the function $h(n)$ is either bounded by 3, or goes to infinity. To show that $f(n)$ is either bounded by 9 or goes to infinity, it suffices to observe that

$$h(n) \leq f(n) \leq (h(n))^2.$$

Indeed, if we let $\widetilde{D}$ denote the symmetric digraph which is obtained by replace each arc of D by two opposite arcs. Then we have the following inequality:

$$f(n) \leq \chi(\widetilde{D}_1 \times \widetilde{D}_2) \leq \chi(D_1 \times D_2)\chi(D_1 \times D_2^{-1}) \leq (h(n))^2.$$

The inequality $h(n) \leq f(n)$ is trivial. This completes the proof of Theorem 12.

Finally we remark that $f(n)$ is bounded if and only if the set $\{K_m^H : H \in \mathbf{FG}\}$ is infinite for any integer $m \geq 10$ (cf. the definition and comments at the end of the previous section). Similarly, $g(n)$ is bounded if and only if the set $\{\overrightarrow{K}_m^H : H \in \mathbf{FG}\}$ is infinite for any integer $m \geq 4$.

6 General multiplicative structures and variations of Hedetniemi's conjecture

Since multiplicativity involves only homomorphisms, we can define multiplicative element in an arbitrary category. Different aspects of this categorical setting of Hedetniemi's conjecture are well known, and are explicitly studied in [14, 58].

Let $\mathcal{L}$ be a relational language, and let A and B be models of $\mathcal{L}$. A mapping h from A to B is a homomorphism if it preserves all the relations, i.e., for any n-ary relation $R \in \mathcal{L}$ and for each sequence $a_1, a_2, \cdots, a_n$ of A, $R(a_1, a_2, \cdots, a_n)$ implies $R(h(a_1), h(a_2), \cdots, h(a_n))$. We write $A \preceq B$ if there is a homomorphism from A to B. Then $\preceq$ defines a partial order on the set of all $\mathcal{L}$ models, as the relation of homomorphism is transitive.

Denote by $\mathcal{M}_{\mathcal{L}}$ the category of $\mathcal{L}$ models under homomorphisms of $\mathcal{L}$ structures. Two structures A and B are *homomorphically equivalent*, written as $A \sim B$, if A admits a homomorphism to B and B admits a homomorphism to A. Obviously the relation "$\sim$" is an equivalence relation. We shall denote by $Sim(A)$ the equivalent class of $\mathcal{M}_{\mathcal{L}}/\sim$ that contains A, i.e., all the models B for which $A \sim B$.

The product $A \times B$ of two structures is defined similarly as the (categorical) product of graphs. Namely for a sequence $(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)$, we have $R((a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n))$ if and only if $R(a_1, a_2, \cdots, a_n)$ and $R(b_1, b_2, \cdots, b_n)$. Let $A + B$ be the disjoint union of A and B. It is clear that if $A \sim A'$ and $B \sim B'$, then $A \times B \sim A' \times B'$ and $A + B \sim A' + B'$.

It is straightforward to verify that $\mathcal{M}_{\mathcal{L}}$ with the relation "$\preceq$" form a distributive lattice, with $A \vee B = A + B$ and $A \wedge B = A \times B$.

Given two models A and B of $\mathcal{L}$, the exponential A^B is the $\mathcal{L}$-model defined on all the mappings from B to A and for a sequence $f_1, f_2, \cdots, f_n$ of mappings from B to A, we have $R(f_1, f_2, \cdots, f_n)$ if and only if for any sequence $b_1, b_2, \cdots, b_n$ of elements of B, $R(b_1, b_2, \cdots, b_n)$ implies

$R(f_1(b_1), f_2(b_2), \cdots, f_n(b_n))$.

Let $\mathbf{C}$ be a subcategory of $\mathcal{M}_{\mathcal{L}}$ which is closed under product, sum and exponential. An element A of $\mathbf{C}$ is called multiplicative if for all elements G and H in $\mathbf{C}$, $G \not\preceq A$ and $H \not\preceq A$ implies that $G \times H \not\preceq A$. In other words, A is *meet irreducible* in $\mathbf{C}$.

Some of the results concerning multiplicativity of graphs and digraphs remains valid for general structures. For example, we have the following generalization of Theorem 1:

Theorem 13 *A structure A is multiplicative in category $\mathbf{C}$ if and only if for any $B \in \mathbf{C}$ such that $B \not\preceq A$, we have $A^B \preceq A$.*

The general categorical setting of Hedetniemi's conjecture provides a wide range view of the problem. By choosing special categories, one may discuss some special multiplicative structures. For example, multiplicative partial orders was discussed in [45], where the "homomorphism" of two partially ordered sets A and B is a mapping h from A to B which preserves the strict order relation, i.e., $x < y$ implies that $h(x) < h(y)$. The following result was proved in [45]:

Theorem 14 *Suppose A is a well-founded partially ordered set (i.e., A contains no infinite decreasing chains), and that for any $x \in A$, the set $\{y \in A : y < x\}$ forms a chain. If the supremum of the length of an increasing chain is at most ω, then A is multiplicative, in the category of partially ordered set (with homomorphisms defined as above).*

The multiplicativity of hypergraphs was studied in [59]. Hedetniemi's conjecture was generalized to hypergraphs in [59].

The chromatic number of a hypergraph $H = (V, E)$ is the minimum integer n such that there is an n-coloring of the vertices of H such that there are no monochromatic hyper-edges. The product of two hypergraphs G and H are defined as follows:

The product $G \times H$ has vertex set $V(G) \times V(H)$, and a subset $e = \{(x_1, y_1), \cdots, (x_k, y_k)\}$ of $V(G \times H)$ is a hyper-edge if and only if the set $\{x_1, \cdots, x_k\}$ is a hyper-edge of G, and that $\{y_1, \cdots, y_k\}$ is a hyper-edge of H. Here the x_i's and y_i's need not be distinct. The following conjecture was proposed in [59]:

Conjecture 5 *Suppose G and H are hypergraphs. Then*

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}.$$

Conjecture 5 was confirmed in a few special cases. The results, as well as the proofs, in [59] are parallel to Theorems 4 and 5 of Section 2.

Another generalization of Hedetniemi's conjecture, which is perhaps more interesting, is the following one concerning the circular chromatic numbers.

The circular chromatic number $\chi_c(G)$ of a graph G, introduced by Vince in 1988 (under the name "the star chromatic number"), is a natural generalization of the chromatic number of a graph.

For a pair of positive integers k and d, a (k,d)-coloring of a graph G is a mapping c of $V(G)$ to the set $\{0, 1, \cdots, k-1\}$ such that for any adjacent vertices x and y of G, $d \leq |c(x) - c(y)| \leq k - d$. The *circular chromatic number* $\chi_c(G)$ of a graph G is the infimum of the ratios k/d for which there exists a (k,d)-coloring of G.

It is easy to see that a $(k,1)$-coloring of a graph G is just the ordinary k-coloring of G. Therefore, $\chi_c(G) \leq \chi(G)$ for any G. On the other hand, it was proved in [49] that $\chi_c(G) > \chi(G) - 1$. Thus if we know the circular chromatic number of a graph G then $\chi(G)$ is just the ceiling of $\chi_c(G)$. However, two graphs of the same chromatic number may have different circular chromatic number. In this sense, $\chi_c(G)$ is a refinement of $\chi(G)$, and it contains more information about the structure of the graph than $\chi(G)$ does.

It was shown in [49] that for any finite graph G, $\chi_c(G)$ is rational, and conversely for any rational $r \geq 2$, there is a graph G with $\chi_c(G) = r$. It seems that all the problems concerning the chromatic number remains interesting for the circular chromatic number. Similar to the chromatic number, it is straightforward to verify that $\chi_c(G \times H) \leq \min\{\chi_c(G), \chi_c(H)\}$. It was conjectured in [60] that the equality holds for all graphs G and H.

Conjecture 6 *For any rational number r and any graphs G and H, if $\chi_c(G) > r$ and $\chi_c(H) > r$, then $\chi_c(G \times H) > r$.*

This conjecture is stronger than Hedetniemi's conjecture. An interesting observation in [60] is that this conjecture is true for infinitely many rationals r, namely we have the following result:

Theorem 15 *For any integer k, if $\chi_c(G) > 2 + 1/k$ and $\chi_c(H) > 2 + 1/k$ then $\chi_c(G \times H) > 2 + 1/k$.*

Proof. A graph G has circular chromatic number $\chi_c(G) > 2 + 1/k$ if and only if G does not admit a homomorphism to the odd cycle C_{2k+1}. Since the odd cycles are multiplicative (Theorem 9), the conclusion follows. ∎

Note that the equivalence of Hedetniemi's conjecture, Statement $C(n)$, is only verified for $n = 1, 2, 3$.

The version of Hedetniemi's conjecture for infinite graphs was studied by Hajnal [17], who proved that the chromatic number of the product of two $\aleph_1$-chromatic graphs can be countable.

Instead of discussing the product of two graphs, we can also discuss the product of many graphs (note that the product is associative). Of course, it does not make any difference if Hedetniemi's conjecture is generalized to the product of finitely many graphs. However, for the product of infinitely many graphs, we note that the product of odd cycles C_{2k+1} $(k = 1, 2, \cdots)$ contains no odd cycles, and hence is 2-colorable.

Finally we mention a relation between uniquely n-colorable graphs and Hedetniemi's conjecture, which was first studied by Duffus, Sands and Woodrow [13].

A graph G is uniquely n-colorable if there is a unique partition of $V(G)$ into n independent sets. The following result was proved in [16]:

Theorem 16 *If G is a connected graph with $\chi(G) > n$, then $K_n \times G$ is uniquely n-colorable.*

Duffus, Sands and Woodrow proposed the following two conjectures concerning unique colorability, which are generalizations of Theorem 16, and each of which implies Hedetniemi's conjecture.

Conjecture 7 *For all uniquely n-colorable graphs G and H, each proper n-coloring of $G \times H$ is induced by G or by H, i.e., if $\phi : V(G \times H) \mapsto \{1, 2, \ldots, n\}$ is a proper n-coloring of $G \times H$, then either $\phi((g, h)) = \phi((g, h'))$ for all $g \in V(G)$ or $\phi((g, h)) = \phi((g', h))$ for all $h \in V(H)$.*

Conjecture 8 *For all uniquely n-colorable graphs G and all connected graphs H with $\chi(H) > n$, $G \times H$ is uniquely n-colorable.*

It was proved in [13] that Conjecture 7 implies Conjecture 8, and Conjecture 8 implies Statement $C(n)$. Hence each of the Conjectures 7 and 8 is stronger than Hedetniemi's conjecture.

The following conjecture was proposed in [58]:

Conjecture 9 *For all uniquely n-colorable graphs G, $L(K_n^G)$ is n-colorable.*

The graph $L(K_n^G)$ was defined in Section 3. It was shown in [58] that Conjecture 9 is weaker than Conjecture 8, but stronger than Hedetniemi's conjecture.

It was also shown in [58] that Conjecture 7 holds for integer n if and only if for any uniquely n-colorable graphs G, the graph $L(K_n^G)$ is n-colorable and that $L(K_n^G)$ contains no uniquely n-colorable subgraphs other than the complete graph of order n induced by the constant mappings.

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