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On the Multiplicativity of Acyclic Local Tournaments

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Abstract

A homomorphism of a digraph to another digraph is an edge-preserving vertex mapping. A digraph H is said to be multiplicative if the set of digraphs which do not admit a homomorphism to H is closed under categorical product. In this paper we discuss the multiplicativity of acyclic Hamiltonian digraphs, i.e., acyclic digraphs which contains a Hamilton path. As a consequence, we give a complete characterization of acyclic local tournaments with respect to multiplicativity.

1 Introduction

Let G and H be digraphs. A homomorphism of G to H is a mapping $f : V(G) \to V(H)$ such that $gg' \in E(G)$ implies that $f(g)f(g') \in E(H)$. If such a homomorphism exists, we say G is homomorphic to H and write $G \to H$. Otherwise we write $G \not\to H$. The *product* $G \times G'$ of two digraphs G and G' has the vertex-set $V(G) \times V(G')$ and has the (directed) edges $(x, x')(y, y')$ with $xy \in E(G)$ and $x'y' \in E(G')$. A digraph H is *multiplicative* if for any two digraphs G and G', $G \not\to H$ and $G' \not\to H$ implies that $G \times G' \not\to H$. Similar definitions apply to graphs.

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The notion of multiplicative graph and digraph, introduced in [5], is motivated by a conjecture of Hedetniemi [7], which states that the chromatic number of the categorical product of two n-chromatic graphs is n. An equivalent statement of the conjecture is that the class of graphs which are not homomorphic to K_n is closed under taking the product, i.e., K_n is multiplicative. The problem of the multiplicativity of graphs and digraphs has attracted considerable attention. For undirected graphs, some results can be found in [2, 3, 4, 5, 6, 7, 10, 11], among which the principal results are that (1) K_3 is multiplicative; (2) all odd cycles are multiplicative. (The multiplicativity of K_1 and any bipartite graph are obvious). For directed graphs, some results can be found in [5, 8, 9, 10, 12, 13, 14, 15, 16, 17, 19], among which the principal results are that (1) all transitive tournaments are multiplicative [5, 10]; (2) an oriented path are multiplicative if and only if it is homomorphically equivalent to a directed path, [14]; (3) An oriented cycle C is multiplicative if and only if either C is a directed cycle of prime power length or C is a $\mathcal{C}$-cycle, [5, 8, 9, 12, 17, 19]. (4) almost all digraphs which have a Hamilton path and no directed cycle are non-multiplicative [16].

Suppose H is a digraph and v is a vertex of H. The inset of v is the set of all the vertices u of H such that uv is an edge of H, and the outset of v is the set of all vertices u of H such that vu is an edge of H. A digraph H is a local tournament if for each vertex v of H, the inset as well as the outset of v induces a tournament. Local tournaments were introduced in [1] as a natural extension of the class of tournaments, for which many results about tournaments remain valid. It is known, cf. [1], and also easy to prove, that a local tournament H is acyclic if and only if H admits an enumeration $v_0, v_1, \cdots, v_n$ of vertices, such that whenever $v_i v_j$ is an edge of H then $i < j$ and $v_k v_l$ is also an edge of H for any $i \leq k < l \leq j$. Note that each transitive tournament and each directed path is an acyclic local tournament. As noted above transitive tournaments and directed paths are multiplicative. In this paper, we discuss the multiplicativity of acyclic Hamiltonian digraphs, i.e., those acyclic digraphs which contains a Hamilton path. As a consequence, we give a complete characterization of multiplicative acyclic local tournaments, which takes the results about directed paths and transitive tournaments as special cases. In Section 4, we also consider the multiplicativity of locally transitive tournaments which contains a Hamilton cycle.

The following notion will be used in this paper: An *oriented path* $P = [p_0, p_1, \cdots, p_n]$ has a sequence of vertices $p_0, p_1, \cdots, p_n$, and edges $p_i p_{i+1}$ or

$p_{i+1}p_i$ for each $i = 0, 1, \ldots, n-1$, and there is no other edge. The *order of traversal* is starting at p_0, then $p_1, p_2, \ldots, p_{n-1}$, and ending at p_n. The order of traversal can also be specified by saying that p_0 is the initial vertex of P, written as $p_0 = i(P)$, and p_n is the terminal vertex of P, written as $p_n = t(P)$. The subpath of P, induced by vertices $p_i, p_{i+1}, \ldots, p_j$, is denoted by $P[p_i, p_j]$. The *net length* $n\ell(P)$ of P is the absolute value of difference between the number of forward edges and the number of backward edges. The *level* $\ell_P(v)$ of a vertex v in the path P is the difference between the number of edges directed forward and the number of edges directed backward on the subpath from $i(P)$ to v. An oriented path P is *minimal* if the net length of any subpath of P is not greater than the net length of P. An oriented path P is called a *forward directed path* (resp. a *backward directed path*) if all the edges are forward (resp. backward) edges. Both forward and backward directed paths are called directed paths.

2 Non-multiplciative acyclic Hamiltonian digraphs

The main theorem of this paper is a characterization of all multiplicative connected acyclic local tournaments. For the remainder of this paper, we assume that H is a connected acyclic local tournament and $v_0, v_1, \cdots, v_n$ is the enumeration of its vertices such that whenever v_iv_j is an edge of H then $i < j$ and v_kv_l is also an edge of H for any $i \leq k < l \leq j$. Note that for each i ($i = 0, 1, 2, \cdots, n-1$), v_iv_{i+1} must be an edge of H, for otherwise H is not connected. We denote by E^* the set of all other edges of H, i.e, edges v_iv_j such that $j \neq i+1$. If E^* is empty, then H is a directed path. In this case we know that H is multiplicative. Thus for the remainder of this paper, we assume that $E^* \neq \emptyset$. For two edges v_iv_j and v_kv_l of H, we say v_iv_j *dominates* v_kv_l if $i \leq k$ and $j \geq l$. An edge v_iv_j of H is *maximal* if it is not dominated by any other edge. We call H a *mermaid* if there is only one maximal edge in E^* and this edge is either v_0v_m or v_mv_n (for some $0 \leq m \leq n$).

Theorem 1 *A connected acyclic local tournament H is multiplicative if and only if it is a mermaid.*

Instead of proving Theorem 1 directly, we shall consider a more general class of digraphs, which we call *acyclic Hamiltonian digraphs*. They are digraphs H' obtained from connected acyclic local tournaments H by deleting some edges of E^*. Thus each such digraph H' is still acyclic and has a Hamilton path $[v_0, v_1, v_2, \cdots, v_n]$. We will derive in this Section some necessary conditions for an acyclic Hamiltonian digraph to be multiplicative. In the next section, we give a sufficient condition for an acyclic Hamiltonian digraph to be multiplicative. These conditions are presented as several lemmas and theorems. Our main result, Theorem 1, will follow from these results.

For the remainder of this paper, we assume that H' is an acyclic Hamiltonian digraph with vertices $v_0, v_1, v_2, \cdots, v_n$, in this order. Denote by E^* again the set of edges of H' which are not on the Hamilton path. If E^* is empty, then H' is a directed path, and hence is multiplicative [5, 10]. So we assume furthermore that E^* is not empty.

Lemma 2 *If there is a digraph G and two vertices $a, b \in G$ such that G is not homomorphic to H' but both $G \setminus a$ and $G \setminus b$ are homomorphic to H', and the minimum net length of an oriented path in G connecting a and b is at least $n+2$, then H' is non-multiplicative.*

Proof: Let G' be a directed path of length $n+1$. Then obviously $G' \not\to H'$. To prove that H' is non-multiplicative, it is enough to show that $G \times G' \to H'$. Let X be a connected component of $G \times G'$. Then $X \to G$ and $X \to G'$. Let $h : X \to G$ be a homomorphism. We must have either $a \notin h(X)$ or $b \notin h(X)$, for otherwise X would contain a path of length at least $n+2$, which contradicts the fact that $X \to G'$. Since both $G \setminus a$ and $G \setminus b$ are homomorphic to H', we have that $X \to H'$. $\Box$

Lemma 3 *If there is an edge $v_i v_j \in E^*$ with $1 \le i \le j-2 \le n-3$ such that neither $v_i v_{j+1} \in E^*$ nor $v_{i-1} v_j \in E^*$, then H' is non-multiplicative.*

Proof: We consider two cases.

Case 1: There is a k such that $v_k v_{j+1} \in E^*$ and $i < k < j$. Let $k' = min\{k : v_k v_{j+1} \in E \text{ and } i < k < j\}$. Then we claim that the following graph G satisfies the condition of Lemma 2.

Let $P = [w_0, w_1, w_2, \cdots, w_n]$ and $P' = [u_0, u_1, u_2, \cdots, u_n]$ be directed paths of lengths n. Let G be the digraph obtained from the disjoint union of P and P' by putting the edge $u_i w_{j+1}$. Let $a = w_0$ and $b = u_{n+1-k+i}$.

It is easy to see that $G \setminus a, G \setminus b \to H'$ and the minimum net length of a path of G connecting a and b is at least $n+2$. Now we prove that

$G \not\to H'$. Otherwise, let $h : G \to H'$ be a homomorphism. Then we must have $h(w_t) = v_t$ and $h(u_r) = v_r$ because a directed path of G of length n must be mapped to a directed path of length n and H' has a unique directed path of length n. This is a contradiction, as $u_i w_{j+1}$ is an edge of G and $v_i v_{j+1}$ is not an edge of H'.

Case 2: There is no k such that $k(j+1) \in E$ and $i < k < j$. Then we claim that the following graph G satisfies the condition of Lemma 2.

Let $P = [w_0, w_1, w_2, \cdots, w_n]$ and $P' = [u_0, u_1, u_2, \cdots, u_n]$ be directed paths of length n. The digraph G is obtained from the disjoint union of these two paths by connecting u_{i-1} and w_{j+1} with a directed path $[u_{i-1}, s, w_{j+1}]$ of lenth 2. Let $a = w_0$ and $b = u_n$.

Again it is easy to see that $G \setminus a, G \setminus b \to H'$ and the minimum net length of a path of G connecting a and b is at least $n + 2$. Similarly we assume $h : G \to H'$ is a homomorphism. Then we must have $h(w_t) = v_t$ and $h(u_t) = v_t$. In particular $h(w_{j+1}) = v_{j+1}$ and $h(u_{i-1}) = v_{i-1}$. It is easy to see now that $h(s)$ cannot be defined. Now the lemma follows from Lemma 2. $\Box$

Corollary 4 *If H' has a maximal edge $v_i v_j \in E$ such that neither $i = 0$ nor $j = n$, then H' is non-multiplicative.*

Lemma 5 *If H' has edges $v_i v_j, v_k v_l \in E^*$ with $0 \leq i < k < j < l \leq n$ and $v_i v_l \notin E$, then H' is non-multiplicative.*

Proof: Let $k' = \min\{t \geq i : v_t v_l \in E\}$. Since $v_i v_l \notin E^*$, we have that $k' > i$. Let $r = k' - i, q = l - j$. Let $P = [w_0, w_1, w_2, \cdots, w_n]$ and $P' = [u_0, u_1, u_2, \cdots, u_n]$ be directed paths of length n. Let G be the digraph obtained from the disjoint union of P and P' by adding an edge $u_i w_l$. Let $a = w_q$ of P and let $b = u_{n-r}$. It is a routine exercise to verify that the digraph G satisfies the condition of Lemma 2. Thus H' is non-multiplicative. $\Box$

Lemma 6 *If H' has edges $v_i v_j, v_k v_l \in E^*$ with $0 \leq i < j \leq k < l \leq n$ and $v_i v_l \notin E^*$, then H' is non-multiplicative.*

Proof: Let $q = max\{t \leq l : v_i v_t \in E^*\}$. Then $q < l$. If $k < q < l$ we are done by Lemma 5. Thus we can assume $q \leq k$ and hence we can assume that $q = j$, for otherwise we use the edge $v_i v_q$ instead the edge $v_i v_j$ in our proof. Similarly we can assume that $k = min\{t \geq i : v_t v_l \in E^*\}$. Thus by applying Lemma 5, we can assume that there is no edge $v_pv_q \in E^*$ such that $i \leq p < j$ and $l \geq q > j$; or $i \leq p < k$ and $l \geq q > k$.

Let $P = [u_0, u_1, \cdots, u_{n+1+j-l}]$, $P' = [u'_0, u'_1, \cdots, u'_{n-2}]$ and $P'' = [u''_0, u''_1, \cdots, u''_{n+1+i-k}]$ be directed paths of length $n+1+j-l$, $n-2$ and $n+1+i-k$ respectively. Let G be the digraph obtained from the disjoint union of P, P' and P'' by joining the edges u'_iu_{j+1} and $u''_iu'_{i+k+2-j}$. Let a be the initial vertex of P and let b be the terminal vertex of P''.

We now show that G satisfies the condition of Lemma 2, and hence H' is non-multiplicative. First we show that $G \not\to H'$. Suppose to the contrary that h is a homomorphism of G to H'. By considering the possible images of the path P, we see that $h(u_{j+1}) = v_t$ for some $j+1 \leq t \leq l$. Since there is no edge $v_pv_q \in E^*$ such that $i \leq p < j$ and $l \geq q > j$, and since $h(u'_i) = v_r$ for some $r \geq i$ (by considering the possible images of P'), and since $h(u'_i)h(u_{j+1})$ is an edge of H', we have $h(u'_i) = v_r$ for some $r \geq j$. This implies that $h(u'_{i+k+2-j}) = v_{t'}$ for some $t' \geq k+2$ (by considering the possible images of the subpath of P' from u'_i to $u'_{i+k+2-j}$). By considering the possible images of P'', we see that $h(u''_i) = v_s$ for some $i \leq s \leq k-1$. However $h(u''_i)h(u'_{i+k+2-j})$ is an edge of H', which is a contradiction because H' contains no edge $v_pv_q \in E^*$ such that $i \leq p < k$ and $l \geq q > k$. Thus G is not homomorphic to H'. An easy counting shows that the path of G connecting a and b has net length $n+2$. It remains to show that $G \setminus \{a\}$ and $G \setminus \{b\}$ are homomorphic to H'. We only give a homomorphism h of $G \setminus \{a\}$ to H'. Let $h(u_{j+1}) = v_j, h(u'_i) = v_i, h(u'_{i+k+2-j}) = v_{i+k+2-j}$ and $h(u''_i) = v_{i+k+2-j-1}$. It is now easy to extend the definition of h to the whole digraph $G\setminus\{a\}$ so that h is homomorphism (note that $i+k+2-j-1 \leq k-1$). A homomorphism of $G \setminus \{b\}$ can be given in a similar way. This completes the proof of this lemma. $\square$

Corollary 7 *(a). If H' is a multiplicative acyclic Hamiltonian digraph, then H' has only one maximal edge, and this edge is either of the form v_0v_j or of the form v_jv_n.*

(b). If H is a multiplicative acyclic local tournament, then H is a mermaid.

Proof: Statement (a) is an immediate consequence of Corollary 4, Lemma 5 and Lemma 6, and statement (b) is a special case of (a). $\square$

Note that statement (b) in the above corollary is the "only if" part of our main theorem. In order to complete the proof of our main theorem, it suffices to show that every mermaid is indeed multiplicative.

Corollary 8 *If H' is a multiplicative acyclic Hamiltonian digraph, $v_iv_j \notin E^*$ and $i \leq j-2$, then either for every $t, i+2 \leq t \leq j$ we have $v_iv_t \notin E^*$ or for every $t, i \leq t \leq j-2$ we have $v_tv_j \notin E^*$.*

Proof: This is also an immediate consequence of Lemma 5 and Lemma 6. □

Lemma 9 *Suppose that H' is a multiplicative acyclic Hamiltonian digraph and E^* has a maximum edge om. Then for any $k, 2 \leq k \leq m$, there is an edge $ij \in E^*$ such that $j - i = k$.*

Proof: Suppose for some k, $2 \leq k \leq m$, there is no edge $v_iv_j \in E$ such that $j - i = k$. Let C be an oriented cycle with k forward edges and 1 backward edge, and let P be a directed path of length $n+1$. Then obviously $C \not\rightarrow H'$ and $P \not\rightarrow H'$. However $C \times P \rightarrow H'$, as any non-trivial connected component of $C \times P$ is a minimal path of length $n+1$ which has a backward edge before it reaches level m (and after level 1). □

Summing up the above discussion, we see that an acyclic Hamiltonian digraph H' is not multiplicative unless H' is "very close" to a mermaid, i.e., obtained from a mermaid by removing a few edges from E^*. These results nicely complement a result in [16], which asserts that almost all acyclic Hamiltonian digraphs are non-multiplicative. Although we cannot give a complete classification of multiplicative acyclic Hamiltonian digraphs at present, we shall show in the next section that many of those acyclic Hamiltonian digraphs close to mermaids are indeed multiplicative.

3 Multiplicative acyclic Hamiltonian digraphs

In this section, we give a sufficient condition for an acyclic Hamiltonian digraph H' to be multiplicative. As a consequence of this condition, we shall prove that mermaids are multiplicative.

Again H' is an acyclic Hamiltonian digraph with vertices $v_0, v_1, \cdots, v_n$, and E^* is the set of edges of H' not on the Hamilton path. We only need to consider those acyclic Hamiltonian digraphs H' with E^* contains a unique maximal edge of the form v_0v_m, or v_mv_n. For other acyclic Hamiltonian digraphs are non-multiplicative by Corollary 7. Without loss of generality, we assume that E^* has a maximal edge v_0v_m. (If we reverse the direction of all the edges of H', which will not change the multiplicativity or non-multiplicativity, we get the case that E^* has a maximum edge of the form v_mv_n).

We denote by E' the set of pairs v_iv_j such that $i \leq j-2 \leq m-2$ and v_iv_j is not an edge of H'. In other words E' are those edges of the mermaid (with maixmal edge v_0v_m) that are missing from H'.

Theorem 10 *If there is a number $k, m-1 \geq k \geq 1$ such that $E' = \{v_jv_m : m-1 > j \geq k\}$, then G is multiplicative.*

Observe that if $k = m-1$ then $E' = \emptyset$. In this case H' is a mermaid, also any mermaid is of this form. Thus this theorem implies the "if" part of our main theorem. Therefore once this theorem is proved, our main result, Theorem 1, is also established.

Proof of Theorem 10: Let $\mathcal{P}$ be the class of oriented paths $P = p_0, p_1, \ldots, p_m \ldots, p_t$ such that $\ell(P) = n+1$ and for any $i, m \leq i \leq t-1$ we have $m-1 \leq \ell_P(p_i) \leq n$. (Note that $\ell_P(p_m) \geq m-1$ implies that $P[p_0, p_m]$ is a directed path of length m.) For $P \in \mathcal{P}$, let T_P be the oriented tree obtained from P by the following operation:

For each $p_i \in P$ with $\ell_P(p_i) = m-1$, we add a directed path of length k to P by identifying the terminal vertex of that directed path with p_i.

Let $\theta = \{T_P : P \in \mathcal{P}\}$.

Claim: For any digraph H, we have $H \to H'$ if and only if for all $T_P \in \theta$, $T_P \not\to H$.

Proof of the claim: We first see that for any $T_P \in \theta$ we have $T_P \not\to H'$. Otherwise, let $h : T_P \to H'$ be a homomorphism. We shall show by induction on i that for any $p_i \in P$, if $h(p_i) = v_j$ then $j \geq \ell_P(p_i)$, which is a contradiction because then $\ell(p_t) = n+1$. For $i \leq m$, this is certainly true. Suppose for index $i \geq m$, we have $h(p_i) = v_j$, $h(p_{i+1}) = v_q$ and $j \geq \ell_P(p_i)$, we shall show that $q \geq \ell_P(p_{i+1})$. If p_ip_{i+1} is an edge of P and then $\ell_P(p_{i+1}) = \ell_P(p_i) + 1$ and $q = j+1$. Thus $q \geq \ell_P(p_{i+1})$. If $p_{i+1}p_i$ is an edge of P, then $\ell_P(p_{i+1}) = \ell_P(p_i) - 1$. If $\ell_P(p_{i+1}) \geq m$ then since v_qv_j is an edge of H' and $j \geq \ell_P(p_i) = m+1$, we must have $q = j-1$, and hence

$q \geq \ell_P(p_{i+1})$. Now suppose $\ell_P(p_{i+1}) = m-1$. If $j \geq m+1$ then we still have $q = j-1$ and $q \geq \ell_P(p_{i+1})$. Suppose $j = m$. Then since p_{i+1} is the terminal vertex of a directed path of length k, we must have $q \geq k$ but this implies that $q = m-1$. Therefore we have $q = \ell_P(p_{i+1})$.

This proves the necessity of the condition of this claim, for if for some $T_P \in \theta$ we have $T_P \to H$ and $H \to H'$ then the composition of homomorphism will give a homomorphism of T_P to H'.

To prove the sufficiency, we first define $\mathcal{P}'$ to be the class of oriented paths $P = p_0, p_1, \ldots, p_m \ldots, p_t$ such that $t \geq m$ and for any $i, m \leq i \leq t$ we have $m-1 \leq \ell_P(p_i)$. For $P \in \mathcal{P}'$, let $t(P) = p_t$ which is the last vertex of P, and let T_P be the oriented tree obtained from P by the following operation:

For each $p_i \in P$ with $\ell_P(p_i) = m-1$, we add a directed path of length k to P by identifying the sink vertex of that directed path with p_i.

Let $\theta^* = \{T_P : P \in \mathcal{P}'\}$.

Suppose G is a digraph such that for any $T_P \in \theta$, $T_P \not\to G$. We construct a homomorphism h of G to H' as follows:

Let x be a vertex of G. If there is a $T_P \in \theta^*$ such that there is a homomorphism $f : T_P \to G$ with $f(t(P)) = x$, i.e., the terminal vertex of P is mapped to x, then we say that x is of type 1, and choose such T_P so that the net length $n\ell(P) = q$ of P is the maximum, and let $h(x) = v_q$. This is well defined because $q \leq n$ (for otherwise P has a initial segment P' which is in $\mathcal{P}$ and hence T_P has a subtree $T_{P'} \in \theta$).

Otherwise, we say that x is of type 2 and let $h(x) = v_r$ where r is the maximum length of a directed path of G terminates at x.

Observe that if x is of type 2, then $r \leq m-1$, because if there is a directed path of length m terminates at x, we can easily find a $T_P \in \theta^*$ such that $T_P \to G$ with the terminal vertex of P be mapped to x.

Now we show that h is indeed a homomorphism.

Suppose xy is an edge of G, and $h(x) = v_i, h(y) = v_j$. If x is of type 1 then obviously y is also of type 1 (simply extend the path P for x by adding an edge). Also if y is of type 1 and $j \geq m+1$, i.e., the longest P for y has net length at least $m+1$, then x is of type 1 (again simply by extending the longest P for y by adding a backward edge).

Suppose y is of type 1 and $h(y) = v_m$, i.e., the longest P for y has net length m. If there is a directed path of H' of length k which terminates at x, then we also have that x is of type 1. Let P be the longest path P for y. Then we simply extend T_P by adding a directed path of length $k+1$ by identifying the sink vertex of this directed path with the terminal vertex of P. We get the required $T_{P'}$ for x (the path P' for x is the one obtained from P by adding a backward edge.)

It is easy to see that when both x and y are of type 1 then $i = j - 1$. Thus $h(x)h(y)$ is an edge of H'.

If both x and y are of type 2, then $i < j \leq m - 1$. Thus $h(x)h(y)$ is an edge of H'.

If x is of type 2 and y is of type 1. Then by the above argument we know that $j = m$ and $i \leq k - 1$. Thus again we have that $h(x)h(y)$ is an edge of H'. This finishes the proof of the claim. $\square$

To prove that H' is multiplicative, it suffices to show that for any two paths $P, P' \in \mathcal{P}$, there is another path $P'' \in \mathcal{P}$ such that $T_{P''} \to T_P$ and $T_{P''} \to T_{P'}$. Indeed if G and G' are two digraphs not homomorphic to H', then there exist $P, P' \in \mathcal{P}$ such that $T_P \to G, T_{P'} \to G'$ (by the claim). The existence of $P'' \in \mathcal{P}$ with $T_{P''} \to T_P$ and $T_{P''} \to T_{P'}$ will imply that $T_{P''} \to G \times G'$, and hence $G \times G' \not\to H'$.

Now for two paths $P, P' \in \mathcal{P}$, it follows from the discussions in [5, 9, 14, 19], and also not difficult to verify (by double induction on the net length of P, and on the number of vertices in P and P' with level zero), that there is a path $P'' \in \mathcal{P}$ such that $P'' \to P$ and $P'' \to P'$. Let $h : P'' \to P$ and $h' : P'' \to P'$ be homomorphisms. Then if $p \in P''$ is a vertex of level $m-1$ in P'', we must have $h(p)$ and $h'(p)$ have level $m - 1$ in P and P' respectively. This implies that $T_{P''} \to T_P$ and $T_{P''} \to T_{P'}$. Therefore $T_{P''} \to T_P \times T_{P'}$. This completes the proof of Theorem 10. $\square$

4 Locally transitive tournaments

Acyclic local tournaments are necessarily locally transitive tournaments, i.e., the inset as well as the out set of each vertex induces a transitive tournament. In this section we consider another subclass of locally transitive tournaments, i.e., those locally transitive tournaments with a Hamilton cycle. The following lemma is obvious:

Lemma 11 *Let H' be a locally transitive tournament with a Hamilton cycle. Then the set of lengths of directed cycles contained in H' consists of consecutive positive integers.*

$\square$

Let $S = \{k, k+1, \ldots, n\}$ be the set of lengths of directed cycles contained in H'. Then $T = \{t_1 n + t_2(n-1) + \ldots + t_{(n-k+1)}k : t_i$ is a nonnegative integer for every $i\}$ is the set of lengths of all the directed cycles that can be homomorphically mapped to H'.

If $|S| = 1$, then H' is multiplicative if and only if n is a prime power [5, 12, 19]. Assume now that $|S| > 1$. We have the following sufficient conditions for H' to be non-multiplicative.

Theorem 12 *Let H' be a locally transitive tournament with a Hamilton cycle, let S and T be defined above. If there is a pair of coprime integers $p, q \notin T$ such that $pq \in T$, then H' is non-multiplicative.*

Proof. As noted above, T is the set of lengths of all the directed cycles that can be homomorphically mapped to H'. Let C_p and C_q be directed cycles of length p and q respectively, then $C_p \not\to H', C_q \not\to H'$, but $C_p \times C_q$ is a directed cycle of length pq, and hence is homomorphic to H'. $\square$

A pair of twin prime powers are two positive integers p and $p+1$ such that both p and $p+1$ are prime powers. Examples are $\{2, 3\}$, $\{3, 4\}$, $\{7, 8\}$, $\{8, 9\}$, $\{16, 17\}$, $\{31, 32\}$, and $\{64, 65\}$ etc.

Corollary 13 *Let H' be a locally transitive tournament with a Hamiltonian cycle and S be defined above.*

(1) If $|S| = 2$, and S is not the pair of twin prime powers, then H' is non-multiplicative;

(2) If $|S| \geq 4$ and $k \geq 4$, then H' is non-multiplicative;

Proof: The proof of (1) is obvious. For the proof of (2), choose among $k, k+1, k+2$, and $k+3$ two even integers, at least one of them is not a prime power. Suppose $k + i = pq$ where $i \leq 3$, then $p, q < k$, and so neither p nor q is in T. $\square$

For the case that $|S| = 2$, and S is a pair of twin powers, we do not have a general method to prove that the corresponding H' is non-mutiplicative. However many concrete such example digraphs H' can be proved to be non-multiplicative. For $S = \{3, 4\}$, we have $C_2, C_5 \not\to H'$ and $C_2 \times C_5 \to H'$. For $S = \{7, 8\}$, we have $C_5, C_9 \not\to H'$ and $C_5 \times C_9 \to H'$ since $45 = 3 \times 7 + 3 \times 8$. For $S = \{8, 9\}$, we have $C_3 \not\to H'$, $C_{11} \not\to H'$ and $C_{11} \times C_3 \to H'$ since $33 = 9 + 3 \times 8$. For $S = \{16, 17\}$, we have $C_3, C_{11} \not\to H'$ and $C_3 \times C_{11} \to H'$ since $33 = 17 + 16$, etc.

Conjecture 14 *Let H' be a locally transitive tournament with a Hamiltonian cycle and S be defined above.*

(1) Let $|S| = 2$. Then H' is multiplicative if and only if $S = \{2, 3\}$.

(2) Let $|S| = 3$. Then H' is multiplicative if and only if $S = \{2, 3, 4\}, \{3, 4, 5\}$.

(3) Let $|S| = 4$. Then H' is multiplicative if and only if $S = \{2, 3, 4, 5\}, \{3, 4, 5, 6\}$.

Note added in proof: It is proved by Luc Rigollet that half of the above conjecture is true, i.e., all those H' claimed to be non-multiplicative are really non-multiplicative.

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