MULTIPLICATIVE POSETS

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Abstract

A function f from the poset P to the poset Q is a strict morphism if for all $x, y \in P$ with $x < y$ we have $f(x) < f(y)$. If there is such a strict morphism from P to Q we write $P \longrightarrow Q$, otherwise we write $P \not\longrightarrow Q$. We say a poset M is multiplicative if for any posets P, Q with $P \not\longrightarrow M$ and $Q \not\longrightarrow M$ we have $P \times Q \not\longrightarrow M$. (Here $(p_1, q_1) < (p_2, q_2)$ if and only if $p_1 < p_2$ and $q_1 < q_2$). This paper proves that well-founded trees with height $\leq \omega$ are multiplicative posets.

Key words: posets, strict morphisms, multiplicativity, Hedetniemi's conjecture.

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1. INTRODUCTION

A function f from the poset P to the poset Q is a strict morphism if for all $x, y \in P$ with $x < y$ we have $f(x) < f(y)$. If there is such a strict morphism from P to Q we write $P \longrightarrow Q$, otherwise we write $P \not\longrightarrow Q$. If $P \longrightarrow Q$ and $Q \longrightarrow P$ we write $P \sim Q$. Note that $\sim$ is an equivalence relation. We say a poset M is multiplicative if for any posets P, Q with $P \not\longrightarrow M$ and $Q \not\longrightarrow M$ we have $P \times Q \not\longrightarrow M$, i.e. the class of posets which do not admit a strict morphism to M is closed under taking products. Here the product is defined in the strict sense, i.e. $(p_1, q_1) < (p_2, q_2)$ if and only if $p_1 < p_2$ and $q_1 < q_2$. The notion of multiplicativity of a graph and a directed graph has been introduced in [2]. It arose out of generalizations of Hedetniemi's conjecture [3] that the product of two n-chromatic graphs is again n-chromatic. If $\longrightarrow$ is interpreted as graph homomorphism, then this conjecture is equivalent to the statement that complete graphs are multiplicative, i.e. the class of graphs which do not admit a homomorphism to K_n (the complete graph of order n) is closed under taking products. It is proved in [1] that K_3 is multiplicative. In [2] it is proved that odd cycles are multiplicative and a complete classification of multiplicative oriented cycles is given in [4].

Here we investigate the multiplicativity property within the category of posets which is equivalent to the category of transitive directed graphs without loops.

A poset P is well founded if it contains no infinite descending chains. For $x \in P$, define $P(< x) = \{y \in P : y < x\}$ and $P(> x) = \{y \in P : y > x\}$. A poset P is called a tree if for all $x \in P$ $P(< x)$ is a chain. If P is a well founded poset and $x \in P$, then the height of x, $ht(x)$, is defined inductively as follows:

$$ht(x) = sup\{ht(y) + 1 : y \in P(< x)\}, \text{ and } ht(x) = 0 \text{ if } P(< x) = \emptyset.$$

$$\text{The height of } P, ht(P) = \sup\{ht(x) + 1 : x \in P\}.$$

This paper proves that well founded trees of height $\leq \omega$ are multiplicative. i.e. if T is a well founded tree with $ht(T) \leq \omega$, then for any posets P and Q, $P \not\longrightarrow T$ and $Q \not\longrightarrow T$ implies that $P \times Q \not\longrightarrow T$.

2. STRICT MORPHISMS FROM POSETS TO TREES

Given a well founded tree T with $ht(T) \leq \omega$. In order to prove that T is multiplicative, we first need to characterize those posets which admit a strict morphism to T. Suppose P is an arbitrary poset. If P is not well founded or P is well founded but of height $> \omega$, then it is obvious that $P \not\longrightarrow T$. So the only non-trivial case is when P is well founded and $ht(P) \leq \omega$. We denote this class of posets by $\mathcal{B}$. i.e. $\mathcal{B} = \{P : P \text{ is a well founded poset and } ht(P) \leq \omega\}$. If T contains an infinite chain $< t_0, t_1, \ldots >$, then $P \longrightarrow T (\forall P \in \mathcal{B})$ since the map $x \mapsto t_{ht(x)}$ is a strict morphism. Therefore the only non-trivial case is that T contains no infinite chains. We denote this family of trees by $\mathcal{T}$, i.e. $\mathcal{T} = \{T : T$ is a well founded tree with $ht(T) \leq \omega$ and T contains no infinite chain$\}$.

If a poset P contains no infinite chain, then its dual P^* is well founded. In this case we define the *depth* of an element $x \in P, d(x)$ to be the height of x in P^*, and $d(P) = ht(P^*)$.

Lemma 2.1. Suppose T_1, $T_2 \in \mathcal{T}$, then $T_1 \longrightarrow T_2$ if and only if $d(T_1) \leq d(T_2)$.

Proof. The "only if" part is obvious. We prove the "if" part by induction on $\alpha = d(T_1)$. Denote then the statement of lemma 2.1 by $P(\alpha)$. We will show that for all $\alpha < \beta$ $P(\alpha)$ implies $P(\beta)$.

Let $d(T_1) = \beta$ and M_0 the set of minimal elements of T_1, i.e. $M_0 = \{x \in T_1 : T(< x) = \emptyset\}$. For each $x \in M_0$, we have $d(x) < \beta$. Since $d(T_2) \geq d(T_1) = \beta$, there exists $y \in T_2$ such that $d(y) \geq d(x)$. Define $\phi : M_0 \longrightarrow T_2$ so that $d(\phi(x)) \geq d(x)$.

Now $T_1 = M_0 \cup (\cup\{(T_1(> x)) : x \in M_0\})$. For each $x \in M_0$, $T_1(> x)$ is a tree with $d(T_1(> x)) = d(x) < \beta$. Since $d(\phi(x)) \geq d(x)$ for each $x \in M_0$, $d(T_2(> \phi(x))) = d(\phi(x)) \geq d(x)$. By the induction hypothesis, for each $x \in M_0$, there exists a strict morphism $\psi_x : T_1(> x) \longrightarrow T_2(> \phi(x))$.

Now we claim that $\psi : T_1 \longrightarrow T_2$ defined by

$$\psi(y) = \begin{cases} \phi(y), & \text{if } y \in M_0, \\ \psi_x(y), & \text{if } y \in T_1(> x),\ x \in M_0. \end{cases}$$

is a strict morphism from T_1 to T_2.

Suppose $x, y \in T_1$ and $x < y$, then $y \notin M_0$. If $x \in M_0$, then $y \in T_1(> x)$. So $\psi(y) = \psi_x(y) \in T_2(> \psi(x))$ and $\psi(x) = \psi(x)$. Thus $\psi(x) < \phi(y)$.

If $x \notin M_0$, let $x_0 \in M_0$ such that $x_0 < x$, then $x_0 < y$. Thus $x, y \in T_1(> x_0)$. Thus $\psi(x) = \psi_{x_0}(x)$ and $\psi(y) = \psi_{x_0}(y)$. Since ψ_{x_0} is a strict morphism, so $\psi(x) < \psi(y)$.

Definition 2.1 Suppose $P \in \mathcal{B}$, we define a relation "$\ll_P$" on P as follows:

For any $x, y \in P$, $x \ll_P y$ if there exists a path $X = < x_0, x_1, \ldots, x_n >$ in the comparability graph of P from x to y such that $ht(x_i) \geq ht(x)$.

Define $x \simeq_P y$ if $x \ll_P y$ and $y \ll_P x$.

Obviously "$\simeq_P$" is an equivalence relation. We usually write $\ll_P$ and $\simeq_P$ as $\ll$ and $\simeq$ respectively if there is no confusion.

Definition 2.2. Let $T(P) = P/\simeq$ be the quotient set. We define an order $<$ on $T(P)$ as follows:

For $[x], [y] \in T(P)$, $[x] < [y]$ if and only if $x \ll y$ and $[x] \neq [y]$.

This relation is clearly well defined since, for any $x' \in [x]$, $y' \in [y]$ we have $x \ll y \Leftrightarrow x' \ll y'$.

Lemma 2.2. $(T(P), <)$ is a tree and $P \longrightarrow T(P)$.

Proof. It is easy to see that the relation $<$ defined on $T(P)$ is a transitive. Now we will show that for $[x] \in T(P)$, $T(P)(< [x])$ is a chain.

Suppose $[y] < [x]$ and $[z] < [x]$ and $[y] \neq [z]$. Then $y \ll x$ and $z \ll x$. By definition, there exists a path P_y from y to x such that $\min\{ht(v) : v \in P_y\} = ht(y)$. Also there exists a path P_z from z to x such that $\min\{ht(v) : v \in P_z\} = ht(z)$.

Without loss of generality, suppose $ht(y) \leq ht(z)$. Now the union of P_y and the inverse of P_z is a path that witnesses $y \ll z$. Since $[y] \neq [z]$, we have $[y] < [z]$.

Therefore $T(P)$ is a tree.

Let $\psi : P \longrightarrow T(P)$ be the induced mapping, i.e. $\psi(x) = [x]$. Then obviously $x < y$ implies $[x] < [y]$. So ψ is a strict morphism from P to $T(P)$.

Lemma 2.3. If $P \in \mathcal{B}$ is a tree, then $T(P) = P$.

Proof. If P is a tree, then for any $x, y \in P$, $x \ll y$ if and only if there is an increasing chain from x to y. Hence $[x] = \{x\}$ for all $x \in V(P)$ and $[x] < [y]$ if and only if $x < y$.

Lemma 2.4. Suppose $P \in \mathcal{B}$ and $T' \in \mathcal{B}$ is a tree such that $P \longrightarrow T'$, then $T(P) \longrightarrow T'$.

Proof. Let ϕ be a strict morphism from P to T', then define $\psi : T(P) \longrightarrow T'$ as follows:

For each $[x] \in T(P)$, choose $y \in [x]$ such that $ht(\phi(y)) = \min\{ht(\phi(v)) : v \in [x]\}$. And put $\psi([x]) = \phi(y)$.

We first show that this definition of $\psi(x)$ does not depend upon the choice of y, i.e. $\phi(y) = \phi(z)$ if $y, z \in [x]$, and $ht(\phi(y)) = ht(\phi(z)) = \min\{ht(\phi(v)) : v \in [x]\}$.

Since $y \simeq z$, there is a path X in the comparability graph of P from y to z such that $ht(z) = ht(y) = \min\{ht(v) : v \in X\}$.

For any $v \in X$, if $ht(v) = ht(y)$, then $v \in [x]$ and $ht(\phi(v)) \geq ht(\phi(y))$. If $ht(v) > ht(y)$, then there exists $v' \in P$, $v' < v$, such that $ht(v') = ht(y)$, and obviously $v' \in [x]$ and therefore $ht(\phi(v')) \geq ht(\phi(y))$. Now $\phi(v') < \phi(v)$, hence $ht(\phi(v)) > ht(\phi(v')) \geq ht(\phi(y))$.

So in any case, $ht(\phi(y)) = ht(\phi(z)) = \min\{ht(\phi(v)) : v \in X\}$. Thus $< \phi(v) : v \in P >$ is a path in the comparability graph of T' from $\phi(y)$ to $\phi(z)$ and for all $v \in P$, $ht(\phi(v)) \geq ht(\phi(y)) = ht(\phi(z))$. Since T' is a tree, we have $\phi(y) = \phi(z)$.

Now we prove ψ is a strict morphism.

Suppose $[x], [y] \in T(P)$ and $[x] < [y]$, then there exists a path X from x to y such that $ht(x) = \min\{ht(v) : v \in X\}$. By the same argument as in the paragraph above, we can show that $< \psi([v]) : v \in X >$ is a path from $\psi([x])$ to $\psi([y])$ such that $ht(\psi([v])) \geq ht(\psi([x]))$ for all $v \in X$, and furthermore if $ht(v) > ht(x)$, we have $ht(\psi([v])) > ht(\psi([x]))$. Since T' is a tree, we have $\psi([x]) < \psi([v])$ or $\psi([v]) = \psi([x])$. Furthermore if $ht(v) > ht(x)$, we have $\psi([x]) < \psi([v])$. Therefore $\psi([x]) < \psi([y])$. So ψ is a strict morphism.

Corollary. Suppose $T \in \mathcal{T}$, $P \in \mathcal{B}$, and $T(P) \in \mathcal{T}$. Then $P \longrightarrow T$ if and only if $d(T(P)) \leq d(T)$.

Lemmas 2.2 and 2.3 show that $T(P)$ is in some sense the smallest tree among those trees which are strict morphic images of P.

3 MULTIPLICATIVITY OF TREES

In the following we prove the main result of this paper:

Theorem 3.1. Well founded trees with height $\leq \omega$ are multiplicative posets.

Suppose T is a well founded tree with $ht(T) \leq \omega$. If T contains an infinite chain or T is of finite height, then it is easy to prove that T is multiplicative. In the following we assume that T is a well founded tree with $ht(T) = \omega$ and T contains no infinite chains.

To prove that T is multiplicative, we need to show that for any posets P and Q with $P \not\longrightarrow T$ and $Q \not\longrightarrow T$ we have $P \times Q \not\longrightarrow T$. If both P and Q are not well founded then $P \times Q$ is not well founded. Hence $P \times Q \not\longrightarrow T$.

Lemma 3.2. Let P, Q be partially ordered sets. If P is well-founded, then so is $P \times Q$ and, for any $(x, y) \in P \times Q$, $ht((x, y)) \leq ht(x)$ and this is equality if Q contains an increasing chain $\{y_\beta : \beta < ht(x)\}$ below y of order type $ht(x)$.

Proof. Clearly $P \times Q$ has no infinite descending chain and is therefore well-founded. We prove $ht((x, y)) \leq ht(x)$ by induction on $ht((x, y))$. Suppose that $ht((x, y)) = \alpha$. Then for any $\beta < \alpha$ there is $(x_\beta, y_\beta) < (x, y)$ at height β. By the induction hypothesis, $ht(x_\beta) \geq \beta$. Since $x_\beta < x$ this implies that $ht(x) \geq \alpha$.

We prove the last part by induction on $\alpha = ht(x)$. Suppose Q contains an increasing chain $\{y_\beta : \beta < \alpha\}$ below y of order type α. For each $\beta < \alpha$ there is $x_\beta < x$ with $ht(x_\beta) = \beta$. By the induction hypothesis, $ht((x_\beta, y_\beta)) = \beta$, and since $((x_\beta, y_\beta)) < (x, y)$ it follows that $\alpha = ht(x) \geq ht((x, y)) \geq \alpha$.

It is obvious from the proof of the above lemma that if $P_1 = < x_1, x_2, x_3, \ldots >$ is an infinite descending chain and P_2 is a well founded poset, then for any $x_i \in P_1$, $y \in P_2$, with $ht(y) < \omega$, we have $ht((x_i, y)) = ht(y)$.

If P and Q are both well founded and of height $> \omega$, then $ht(P \times Q) > \omega$ by lemma 3.2, hence $P \times Q \not\longrightarrow T$. Also if Q contains an infinite descending chain and P is well founded with height $> \omega$, then $P \times Q$ is well founded and of height $> \omega$ and therefore $P \times Q \not\longrightarrow T$. In order to complete the proof of theorem 3.1, we need to consider the following cases:

(1): $P, Q \in \mathcal{B}$,

(2): $P \in \mathcal{B}$ and Q is not well founded;

(3): $P \in \mathcal{B}$ and Q is well founded and $ht(Q) > \omega$.

We first discuss the case $P, Q \in \mathcal{B}$.

Lemma 3.3. Suppose $P, Q \in \mathcal{B}$, and both $T(P)$, $T(Q)$ contain infinite increasing chains, then $T(P \times Q)$ contains an infinite increasing chain.

Proof. Suppose $C_1 =< [a_1], [a_2], \cdots >$ is an infinite increasing chain of $T(P)$, $C_2 =< [b_1], [b_2], \cdots >$ is an infinite increasing chain of $T(Q)$. Without loss of generality, assume that $ht([a_i]) = i = ht([b_i])$. Now we show $(a_i, b_i) \ll (a_{i+1}, b_{i+1})$. Let $X_1 =< x_1, x_2, \cdots, x_n >$ be a path that witnesses $a_i \ll a_{i+1}$, $X_2 = < y_1, y_2, \cdots, y_m >$ be a path that witnesses $b_i \ll b_{i+1}$. In path X_1, if $x_i < x_{i+1} < x_{i+2}$ or $x_i > x_{i+1} > x_{i+2}$ for some i, we can omit x_{i+1} from X_1. If $x_{n-1} > x_n = a_{i+1}$, then since $ht(a_{i+1}) = i + 1$ there is a point $v \in P(< a_{i+1})$ with $ht(v) = i$. Replace the segment $< x_{n-1}, x_n >$ of X_1 by $< x_{n-1}, v, x_n >$, the resulting path still witnesses that $a_i \ll a_{i+1}$. Therefore we can assume that $x_1 < x_2 > x_3 < x_4 > \cdots < x_n$ and n is then even. Similarly we assume that m is even and $y_1 < y_2 > y_3 < y_4 > \cdots < y_m$. Assume that $m \geq n$, then it is easy to check that $X = <(x_1, y_1), \cdots, (x_n, y_n), (x_{n-1}, y_{n+1}), (x_n, y_{n+2}), \cdots, (x_n, y_m)>$ is a path that witnesses $(a_i, b_i) \ll (a_{i+1}, b_{i+1})$ by using Lemma 3.2.

Therefore $C = ([(a_1, b_1)], [(a_2, b_2)], \cdots)$ is an infinite increasing chain of $T(P \times Q)$.

Lemma 3.4. Suppose $P, Q \in \mathcal{B}$. If $T(P)$ contains an infinite increasing chain and $T(Q) \in \mathcal{T}$ (i.e. $T(Q)$ contains no infinite chains), then $T(P \times Q) \in \mathcal{T}$ and $d(T(P \times Q)) = d(T(Q))$.

Proof. First of all, we have $P \times Q \longrightarrow Q \longrightarrow T(Q)$. So $T(P \times Q) \longrightarrow T(Q)$, hence $T(P \times Q) \in \mathcal{T}$ and $d(T(P \times Q)) \leq d(T(Q))$.

Let $C = ([a_1, [a_2], \cdots)$ be an infinite increasing chain in $T(P)$ with $ht([a_i]) = i$. We will show for any $[y] \in T(Q)$ with $ht([y]) = i$, $d([(a_i, y)]) \geq d([y])$ which implies $d(T(P \times Q)) \geq d(T(Q))$.

We prove this by induction on $d([y])$. If $d([y]) = 0$, it is obviously true.

Suppose this is true for all y with $d([y]) < \alpha$ and $y_0 \in Q$, $d([y_0]) = \alpha$, $ht([y_0]) = i$. Then for any $[y] \in T(Q)$ such that $[y_0] < [y]$, we have $ht([y]) > ht([y_0]) = i$. Thus $a_{ht(y)} \gg a_i$. Similar to the proof of lemma 3.3, we can show that $(a_i, y_0) \ll (a_{ht(y)}, y)$, hence $[(a_i, y_0)] < [(a_{ht(y)}, y)]$ (because $[(a_i, y_0)] \neq [(a_{ht(y)}, y)]$).

Now $d([(a_i, y_0)]) = \sup\{d([(a, y)]) + 1 : [(a_i, y_0)] < [(a, y)]\} \geq \sup\{d([a_{ht(y)}, y)]) : y_0 < y\} \geq \sup\{d([y]) : y_0 < y\} = d(y_0)$.

Lemma 3.5. Suppose $P, Q \in \mathcal{B}$, and $T(P), T(Q) \in \mathcal{T}$, then $T(P \times Q) \longrightarrow T(P) \times T(Q)$ and $T(P) \times T(Q) \longrightarrow T(P \times Q)$.

Proof. For each $[x] \in T(P)$ fix a representative $x' \in [x]$ and similarly, for each $[y] \in T(Q)$ fix $y' \in [y]$. Now consider the map $f : T(P) \times T(Q) \longrightarrow T(P \times Q)$ given by: $f(([x], [y])) = [(x', y')]$ for any $[x] \in T(P)$ and $[y] \in T(Q)$.

We prove that f is a strict morphism.

Suppose that $([x], [y]), ([a], [b]) \in T(P) \times T(Q)$, $([x], [y]) < ([a], [b])$, $f(([x], [y])) = [(x', y')]$ and $f(([a], [b])) = [(a', b')]$ where $x' \in [x]$, $y' \in [y]$, $a' \in [a]$ and $b' \in [b]$ are the fixed representatives of $[x], [y], [a]$ and $[b]$ respectively.

By definition of the product, we have $[x] < [a]$ in $T(P)$ and $[y] < [b]$ in $T(Q)$. i.e. $[x] \neq [a]$, and $x \ll a$, $[y] \neq [b]$ and $y \ll b$. Hence $ht(x) < ht(a)$, $ht(y) < ht(b)$.

Now by definition, we have $x' \ll x$, $a \ll a'$, $y' \ll y$, $b \ll b'$, $ht(x) = ht(x')$, $ht(a) = ht(a')$, $ht(y') = ht(y)$, $ht(b) = ht(b')$. Therefore $x' \ll a'$, $y' \ll b'$ and $ht(x') < ht(a')$, $ht(y') < ht(b')$. Let $X_1 = <x_1, x_2, \cdots, x_n>$ be a path of P from x' to a' which witnesses that $x' \ll a'$, $X_2 = <y_1, y_2, \cdots, y_m>$ be a path of Q from y' to b' which witnesses that $y' \ll b'$. Similar to the proof of lemma 3.3, we can assume that n, m are even, $n \leq m$, $x_1 < x_2 > x_3 < \cdots > x_{n-1} < x_n$ and $y_1 < y_2 > y_3 < \cdots > y_{m-1} < y_m$.

Let $X = <(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n), (x_{n-1}, y_{n+1}), (x_n, y_{n+2}), (x_{n-1}, y_{n+3}), \cdots, (x_n, y_m)>$. Obviously X is a path of $P \times Q$ from (x', y') to (a', b'). Since $ht((x_i, y_i)) = \min\{ht(x_i), ht(y_i)\} \geq \min\{ht(x'), ht(y')\} = ht((x', y'))$ by lemma 3.2, X is a path which witnesses that $(x', y') \ll (a', b')$.

Since $ht((x',y')) < ht((a',b'))$, we have $[(x',y'))] \neq [(a',b')]$. Therefore $[(x',y'))] < [(a',b')]$. Thus f is a strict morphism.

To prove $T(P \times Q) \longrightarrow T(P) \times T(Q)$, we define the mapping $\xi : T(P \times Q) \longrightarrow T(P) \times T(Q)$ as follows:

Given $[(x,y)] \in T(P \times Q)$, suppose $ht((x,y)) = \min\{ht(x), ht(y)\} = \alpha$. Fix any $(x',y') \in [(x,y)]$. Let a be any (fixed) element of $P(< x') \cup \{x'\}$ with $ht(a) = \alpha$, let b be any (fixed) element of $Q(< y') \cup \{y'\}$ with $ht(b) = \alpha$. Put $\xi([(x,y)]) = ([a],[b])$. Now we show that ξ is a strict morphism.

Suppose $[(x,y)] < [(u,v)]$, and $\xi([(x,y)]) = ([a],[b])$, $\xi([(u,v)]) = ([c],[d])$. By the definition of ξ, there exists $(x',y') \in [(x,y)]$, such that $a \in P(< x') \cup \{x'\}$, $b \in Q(< y') \cup \{y'\}$ and $ht(a) = ht(b) = ht((x',y')) = ht((x,y)) = \min\{ht(x'), ht(y')\} = \alpha$, similarly there exists $(u',v') \in [(u,v)]$, such that $c \in P(< u') \cup \{u'\}$, $d \in Q(< y') \cup \{y'\}$ and $ht(c) = ht(d) = ht((u',v')) = ht((u,v)) = \min\{ht(u'), ht(v')\} = \beta$.

Since $[(x,y)] < [(u,v)]$ in $T(P \times Q)$, we know $\alpha < \beta$.

Let $X_1 = <(x_1,y_1),(x_2,y_2),\cdots,(x_n,y_n)>$ be a path of $P \times Q$ which witnesses that $(x',y') \ll (x,y)$; let $X_2 = <(w_1,z_1),(w_2,z_2),\cdots,(w_m,z_m)>$ be a path of $P \times Q$ which witnesses that $(x,y) \ll (u,v)$; let $X_3 = <(u_1,v_1),(u_2,v_2),\cdots,(u_k,v_k)>$ be a path of $P \times Q$ which witnesses that $(u,v) \ll (u',v')$. Then $ht(x_i,y_i) = \min\{ht(x_i), ht(y_i)\} \geq ht((x',y')) = \alpha$, $ht((w_j,z_j)) = \min\{ht(w_j), ht(z_j)\} \geq ht((x,y)) = \alpha$, $ht((u_t,v_t)) = \min\{ht(u_t), ht(v_t)\} \geq ht((u,v)) = \beta > \alpha$. Therefore $W_1 = <a, x_1, x_2, \cdots, x_n, w_2, \cdots, w_m, u_2, \cdots, u_k, c>$ is a path of P which witnesses that $a \ll c$ (it is possible that $a = x_1 = x'$, and/or $c = u_k = u'$). Similarly $W_2 = <b, y_1, y_2, \cdots, y_n, z_2, \cdots, z_m, v_2, \cdots, v_k, d>$ is a path of Q which witnesses that $b \ll d$.

Since $ht(a) = \alpha < \beta = ht(c)$, we know $[a] \neq [c]$, hence $[a] < [c]$ in $T(P)$. Similarly, $[b] < [d]$ in $T(Q)$. Therefore $([a],[b]) < ([c],[d])$ in $T(P) \times T(Q)$ and ξ is a strict morphism.

Corollary. Suppose $P, Q \in \mathcal{B}$, and $T(P), T(Q) \in \mathcal{T}$, then $T(P \times Q), T(T(P) \times T(Q)) \in \mathcal{T}$ and $d(T(P \times Q)) = d(T(T(P) \times T(Q)))$.

Proof. It is obvious that $T(P) \times T(Q)$ contains no infinite chains, hence $T(P \times Q)$ contains no infinite chains. Since $T(P) \times T(Q) \longrightarrow T(P \times Q)$, we have $T(T(P) \times T(Q)) \longrightarrow T(P \times Q)$ by lemma 2.4. Now $T(P \times Q) \longrightarrow T(P) \times T(Q)$ implies that $T(P \times Q) \longrightarrow T(T(P) \times T(Q))$ and hence $d(T(P \times Q)) = d(T(T(P) \times T(Q)))$ by lemma 2.1.

Lemma 3.6. Suppose $T_1, T_2 \in \mathcal{T}$, then $T(T_1 \times T_2) \in \mathcal{T}$ and $d(T(T_1 \times T_2)) = \min\{d(T_1), d(T_2)\}$.

Proof. By the corollary of lemma 3.5, we have $T(T_1 \times T_2) \in \mathcal{T}$. To show that $d(T_1 \times T_2) = \min\{d(T_1), d(T_2)\}$, it is enough to show for all $x \in T_1$, for all $y \in T_2$, $d([(x,y)]) = \min\{d(x), d(y)\}$. This can be proved (similar to the proof or lemma 3.2) by induction on $\min\{d(x), d(y)\}$.

As an immediate consequence of lemmas 3.3, 3.4, 3.5 and 3.6, we have

Corollary 3.7. If $P, Q \in \mathcal{B}, T \in \mathcal{T}$ and $ht(T) = \omega$ and $P \not\longrightarrow T$, $Q \not\longrightarrow T$, then $P \times Q \not\longrightarrow T$.

Now we consider the case $Q \in \mathcal{B}$ and P is not well founded. Obviously $P \times Q \in \mathcal{B}$.

Lemma 3.8. Suppose $T(Q)$ contains an infinite increasing chain. Then $T(P \times Q)$ contains an infinite increasing chain.

Proof. Let $X = < [y_1], [y_2], [y_3], \cdots >$ be an infinite increasing chain of $T(Q)$. Let $C = < c_0, c_1, c_2, \cdots >$ be an infinite descending chain of P.

By lemma 3.2, for all $y \in Q$, $ht(c_i, y) = ht(y)$.

We claim that $(c_1, y_1) \ll (c_1, y_2) \ll (c_1, y_3) \ll \cdots$, hence $X' = < [(c_1, y_1)], [(c_1, y_2)], [(c_1, y_3)], \cdots >$ is an infinite increasing chain of $T(P \times Q)$.

Let $X_i = < a_1, a_2, \cdots, a_n >$ be a path of Q which witnesses $y_i \ll y_{i+1}$. Similar to the proof of lemma 3.3, we can assume that n is even and $a_1 < a_2 > a_3 < \cdots < a_n$. Also we assume that $n \geq 4$ (otherwise we let X_i go back and forth twice).

Let $X'_i = < (c_1, a_1), (c_0, a_2), (c_2, a_3), (c_1, a_4), (c_2, a_5), \cdots, (c_1, a_n) >$. Then X'_i is a path of $P \times Q$ from (c_1, y_i) to (c_1, y_{i+1}). Since $ht((c_i, a_j)) = ht(a_j)$, we know X'_i witnesses that $(c_1, y_i) \ll (c_1, y_{i+1})$.

Therefore the claim, hence the lemma, is proved.

Lemma 3.9. If $T(Q) \in \mathcal{T}$, then $T(P \times Q) \in \mathcal{T}$ and $d(T(P \times Q)) = d(T(Q))$.

Proof. Since $P \times Q \longrightarrow Q$, we have $T(P \times Q) \longrightarrow T(Q)$. Hence $T(P \times Q) \in \mathcal{T}$ and $d(T(P \times Q)) \leq d(T(Q))$.

Let $C = < c_0, c_1, c_2, \cdots >$ be an infinite descending chain of P.

As in the proof of lemma 3.8, $ht((c_i, y)) = ht(y)$ for $y \in Q$. Suppose $y \ll y'$ and $ht(y) < ht(y')$ in Q. By the same argument as in the proof of lemma 3.8, we can show that $(c_2, y) \ll (c_2, y')$.

Now we show by induction that for any $[y] \in T(Q)$, $d([(c_2, y)]) \geq d([y])$.

If $d([y]) = 0$, this is obviously true.

Suppose $d([(c_2, y')]) \geq d([y'])$ for any $[y'] \in T(Q)$ with $d([y']) < \alpha$ and $[y] \in T(Q)$, $d([y]) = \alpha$.

By definition, for any $\beta < \alpha$, there exists $[y'] \in T(Q)(> [y])$, such that $\alpha > d([y']) \geq \beta$. Since $(c_2, y) \ll (c_2, y')$ and $ht((c_2, y)) = ht(y) < ht(y') = ht((c_2, y'))$, we have $[(c_2, y')] \in T(P \times Q)(> [(c_2, y)])$. Hence $d([(c_2, y)]) \geq \beta + 1$. Therefore $d([(c_2, y)]) \geq \alpha = d([y])$.

Combining lemma 3.8 and lemma 3.9 (and using lemma 2.1), we have proved the case $Q \in \mathcal{B}$ and P is not well founded.

Now we discuss the case $Q \in \mathcal{B}$ and P is well founded but of height $> \omega$.

Lemma 3.10. If $T(Q)$ contains an infinite increasing chain, then $T(P \times Q)$ contains an infinite increasing chain.

Proof. Since P is of height $> \omega$, there exists $x_0 \in P$ such that $ht(x_0) = \omega$. By lemma 3.2, for all $y \in Q$, $ht((x_0, y)) = ht(y)$. Also for all $y \in Q$, for all $x \in P(< x_0)$ with $ht(x) \geq ht(y)$, we have $ht((x, y)) = ht(y)$ by lemma 3.2.

Let $Y = < [y_1], [y_2], [y_3], \cdots >$ be an infinite increasing chain of $T(Q)$. Without loss of generality, we can assume that $ht(y_i) = i$.

Since $ht(x_0) = \omega$, we know that for any integer i, $\exists x_i \in P(< x_o)$ such that $ht(x_i) = i$.

We claim that $Y' = < [(x_1, y_1)], [(x_2, y_2)], [(x_3, y_3)], \cdots >$ is an infinite increasing chain of $T(P \times Q)$.

Let $X_i = < a_1, a_2, \cdots, a_n >$ be a path of Q which witnesses $y_i \ll y_{i+1}$. We can assume that n is even and $a_1 < a_2 > a_3 < \cdots < a_n$. Also we assume that $n \geq 4$ (otherwise we let X_i go back and forth twice).

Then $X' = < (x_i, a_1), (x_0, a_2), (x'_i, a_3), (x_{i+1}, a_4), (x'_i, a_5), \cdots, (x_{i+1}, a_n) >$ is a path of $P \times Q$ from (x_i, y_i) to (x_{i+1}, y_{i+1}), where $x'_i \in P(< x_{i+1})$ and $ht(x'_i) = i$. Since $ht((x'_i, a_j)) = \min\{ht(x'_i), ht(a_j)\} = i$, we know X' witnesses that $(x_i, y_i) \ll (x_{i+1}, y_{i+1})$.

Lemma 3.11. If $T(Q)$ contains no infinite chains, then $T(P \times Q) \in \mathcal{T}$ and $d(T(P \times Q)) = d(T(Q))$.

Proof. Since $P \times Q \longrightarrow Q \longrightarrow T(Q)$, we have $T(P \times Q) \longrightarrow T(Q)$. Hence $T(P \times Q) \in \mathcal{T}$ and $d(T(P \times Q)) \leq d(T(Q))$.

To see that $d(T(P \times Q)) \geq d(T(Q))$, it is enough to show that $\forall y \in H$, $\exists x \in G$ such that $d([(x, y)]) \geq d([y])$.

Let $x_0 \in P$ be an element such that $ht(x_0) = \omega$. For $y \in Q$ with $ht(y) = i$, let $x \in P(< x_0)$ with $ht(x) = i$. We claim that $d([(x, y)]) \geq d([y])$.

Similarly we prove this claim by induction on $d([y])$.

If $d([y]) = 1$, this is obviously true.

Suppose the claim is true for any $y' \in Q$ with $d([y']) < \alpha$ and $y \in Q$, $d([y]) = \alpha$.

Suppose $ht(y) = i$, $x \in P(< x_0)$ and $ht(x) = i$.

By definition, for any $\beta < \alpha$, $\exists [y'] \in T(Q)(> [y])$ such that $\alpha > d([y']) \geq \beta$.

Let $x' \in P(< x_0)$ such that $ht(x') = ht(y') = j > i$. By the induction hypothesis, $d([(x', y')]) \geq \beta$.

Now we show that $[(x', y')] \in T(P \times Q)(> [(x, y)])$. Since $ht((x', y')) = j > i = ht((x, y))$, it is enough to show that $(x, y) \ll (x', y')$.

Since $[y'] \in T(Q(> [y])$, we have $y \ll y'$.

Let $X =< a_1, a_2, \cdots, a_n >$ be a path of Q which witnesses $y \ll y'$. We can assume that $n \geq 4$ is even and $a_1 < a_2 > a_3 < \cdots < a_n$. Then $X' =< (x, a_1), (x_0, a_2), (x', a_3), (x', a_4), (x'', a_5), (x', a_6), (x'', a_7), \cdots, (x', a_n) >$ is a path of $P \times Q$ from (x, y) to (x', y'), where $x'' \in P(< x')$ and $ht(x'') \geq i$. Since $ht((x'', a_j)) = \min\{ht(x''), ht(a_j)\} \geq i$, we know X' witnesses that $(x, y) \ll (x', y')$.

Therefore $[(x', y')] \in T(P \times Q)(> [(x, y)])$. This implies that $d([(x, y)]) \geq \beta + 1 (\forall \beta < \alpha)$. Hence $d([(x, y)]) \geq \alpha = d([y])$.

As a consequence of Lemma 10 and Lemma 11, the case $Q \in \mathcal{B}$ and P is well founded but of height $> \omega$ is proved. Therefore the proof of Theorem 3.1 is completed.

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