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Multiplicativity of oriented cycles

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Abstract

We complete the characterization of those oriented cycles C which have the property that the class of digraphs not homomorphic to C is closed under taking the (categorical) product. This is related to Hedetniemi's conjecture on the chromatic number of the product of undirected graphs. Our main tool is a result concerning the existence of homomorphisms between oriented paths which preserve the initial and terminal vertices. This tool is used to prove that for those cycles C we are interested in, any two oriented paths not homomorphic to C have a common preimage also not homomorphic to C. This "Common Preimage Theorem" implies the multipicativity of our cycles.

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1 Introduction

We take as our starting point the conjecture of Hedetniemi [7], which states that the chromatic number of the product of two n-chromatic graphs is n. (The product we have in mind is the conjunction [6], also known as the categorical product [11], in which (a, b) is adjacent to (c, d) just if a is adjacent to c and b to d.) An equivalent statement of the conjecture is that the class of graphs which are not n-colourable is closed under taking the product, i.e., that the product of two graphs that are not n-colourable is also not n-colourable. Since a graph is n-colourable just if it is homomorphic to K_n, this formulation has suggested the definiton of a *multiplicative* graph M, [4], as one for which graphs non-homomorphic to M are closed under taking the product (see also [12]). Thus Hedetniemi's conjecture asserts that all complete graphs are multiplicative. It is hoped that in studying the multiplicativity, and non-multiplicativity, of other graphs, we will gain insights relevant for Hedetniemi's conjecture. Hedetniemi's conjecture has enjoyed considerable interest, cf. [1], [2], [3], [5], [11], [15], [13], [21], the principal result being that it holds for $n = 3$, [3]. This result states in our terminology that K_3 is multiplicative. It is easy to construct graphs that are not multiplicative, cf. [4], [12]; cf. also [19], [20]. It is also easy to see that all bipartite graphs are multiplicative. Other multiplicative graphs are given in [4]. In particular, it is proved there that all odd cycles are multiplicative. This was the first known infinite family of multiplicative undirected graphs. Whether or not such a family existed was posed as a question in [12]. However, not much more is known about multiplicative graphs, and general proofs of multiplicativity or non-multiplicativity are hard to come by.

Multiplicativity is a concept that applies to directed graphs as well. Since this is the context of the present paper we give the detailed definitions: a *homomorphism* of the digraph G to the digraph H is a mapping $f : V(G) \to V(H)$ such that $gg' \in E(G)$ implies $f(g)f(g') \in E(H)$. If such a homomorphism exists, we say G is homomorphic to H and write $G \to H$. Otherwise we write $G \not\to H$. The *product* $G \times H$ of digraphs G and H has the vertex-set $V(G) \times V(H)$ and has the (directed) edges $(g, h)(g', h')$ with $gg' \in E(G)$ and $hh' \in E(H)$. A digraph D is *multiplicative* if for any two digraphs G and H, $G \not\to D$ and $H \not\to D$ implies that $G \times H \not\to D$. Two graphs, G and H, are hom-equivalent if $G \to H$ and $H \to G$. Two hom-equivalent graphs are either both multiplicative or both non-multiplicative. Given an oriented cycle (respectively path) G, we choose a direction as the positive direction of G. Then an edge is a *forward edge* if the direction of the edge agrees with the positive direction. Otherwise it is a *backward edge.* The *length* $l(G)$ of the oriented cycle (respectively path) G is the number of forward edges of G minus the number of backward edges of G. Note that this notion of length coincides with the usual definition in the case of a directed cycle (respectively path). Observe that the length of a cycle (or path) could be negative, and it depends on which direction is chosen as the positive direction. However the absolute value of the length does not depend on the direction. For convenience, when we consider an oriented cycle (or path), we always assume the positive direction is chosen and fixed. Let P be the oriented path $p_0, p_1, \ldots, p_k$. If $u = p_i$ precedes $v = p_j$ in P (i.e., if $i < j$), we denote by $P[u, v]$ the subpath $p_i, p_i + 1, \ldots, p_j$ of P, and by $P[v, u]$ the path $p_j, p_{j-1}, \ldots, p_i$. We also let $P[., v] = P[p_0, v]$, and $P[u, .] = P[u, p_k]$. The *level* of a vertex v in P is the length of the subpath $P[., v]$ of P. We also define P^T to be the path $p_k, p_{k-1}, \ldots, p_0$. If P' is another oriented path $p'_0, p'_1, \ldots, p'_l$, we define the *concatenation* $P \circ P'$ to be the path $p_0, p_1, \ldots, p_k = p'_0, p'_1, \ldots, p'_l$ (in other words, we identify the last vertex of P with the first vertex of P'). An oriented path is *minimal* if it does not properly include a subpath of the same length. An oriented path $v_0, v_1, \cdots, v_k$ is *n-bounded* if all vertices have non-negative levels, the last vertex has level zero, and the maximum level of a vertex is equal to n.

It is not hard to see that all transitive tournaments are multiplicative, [12], [4]. All directed paths (and hence also all oriented paths hom-equivalent to a directed path) are multiplicative [12], cf. also [4]. All other oriented paths are non-multiplicative [18], cf. also [21]. An easy example of a non-multiplicative digraph is a directed cycle C of length mn, where m and n are relatively prime. Indeed neither the directed cycle of length m nor the directed cycle of length n are homomorphic to C, yet their product is isomorphic to C (because m and n are relatively prime). Thus any directed cycle of non-prime-power length is non-multiplicative. Directed cycles of prime power length were conjectured to be multiplicative in [12]; the case of prime length was settled there. The full conjecture was established in [4], using a deep topological result. Later elementary proofs of the conjecture have been found, [17], [8], culminating in the elegant proof in [22].

The situation for oriented cycles has proved to be more complex than for oriented paths. In particular, H. Zhou in [18] introduced a special class of cycles, called $\mathcal{C}$-cycles (defined below), and showed that among oriented cycles only the $\mathcal{C}$-cycles, the directed cycles of prime-power length, and the oriented cycles hom-equivalent to directed paths, could be multiplicative. (A simpler proof of this result is given in [21].) As discussed above, the directed cycles of prime-power length are multiplicative, and so are those oriented cycles that are hom-equivalent to directed paths. However, whether or not all $\mathcal{C}$-cycles were multiplicative, remained unknown. In this paper we shall complete the characterization by showing that all $\mathcal{C}$-cycles are indeed multiplicative. Thus we achieve a classification of oriented cycles according to their being or not being multiplicative; this classification turns out to be surprizingly complex.

To prove our result, we shall show that for a $\mathcal{C}$-cycle C and any two oriented paths P, P' not homomorphic to C there exists an oriented path P^* which is homomorphic to both P and P' but not to C. This result is called the Common Preimage Theorem. In proving the Common Preimage Theorem, we shall be dealing with a special kind of homomorphisms between oriented paths. A homomorphism of oriented paths P to Q which takes the initial vertex of P to the initial vertex of Q and the terminal vertex of P to the terminal vertex of Q is said to *carry P over Q*. We say that *P can cross Q* if there is a homomorphism which carries P over Q. Otherwise we say that *P can not cross Q*.

The following fact is equivalent to Claim 1 (page 68) of [4]:

LEMMA 1 *For any two oriented paths P and P' of length k which do not have subpaths of length $k+1$, there exists an oriented path P^* which can cross both P and P'.*

¿From the proof of this lemma, we can also see several useful consequences: if both P and P' are minimal then P^* is also minimal; if both P and P' contain exactly one minimal path of length n then P^* also contains exactly one minimal path of length n; and if both P and P' end with a minimal path of length n then P^* also ends with a minimal path of length n.

The basis of our proof of the Common Preimage Theorem is a result, we refer to as the Common Crossing Lemma, which asserts that for a suitable oriented path Q (a $\mathcal{C}$-path with $N(Q) = n$, cf. below), and any n-bounded oriented paths P, P' which cannot cross Q, there exists an n-bounded path P^* which also cannot cross Q but can cross both P and P'. This technical result is proved in the last section.

2 The Results

Let B_n, S_n and T_n be the digraphs in Figure 1.

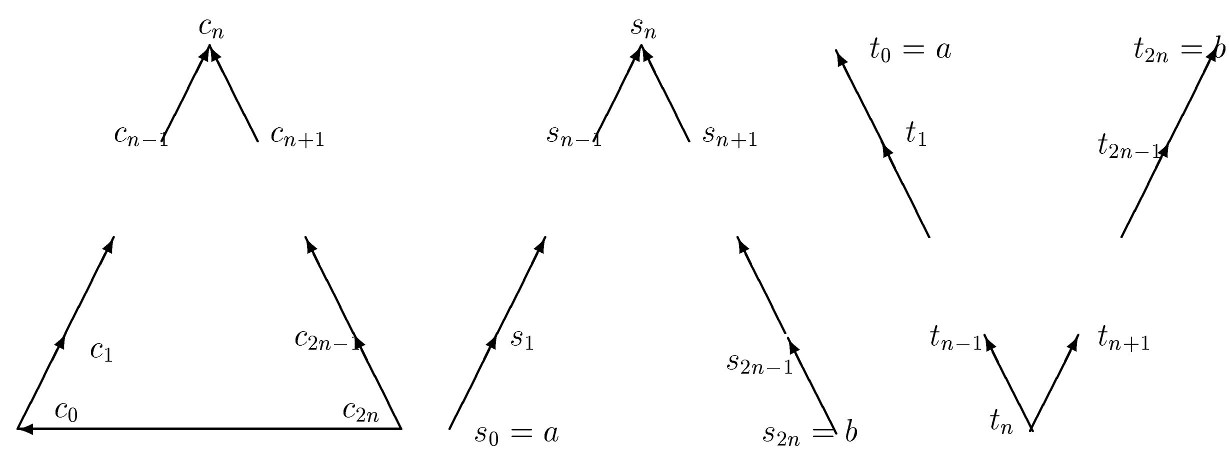

Fig. 1 The digraph B_n, S_n and T_n

We now inductively define the class of $\mathcal{C}$-cycles:

1. Each B_n is a $\mathcal{C}$-cycle.

2. Let C be a $\mathcal{C}$-cycle and let v be a vertex of out-degree 2 (respectively in-degree 2). Then there are two maximal directed paths P, P' starting (respectively ending) at v, say of lengths $l \leq l'$. Let $m \leq l$ be an integer. Replace v by S_m (respectively by T_m), identifying a with the begining of P and b with the begining of P'. The resulting digraph C' is also a $\mathcal{C}$-cycle.

3. There are no other $\mathcal{C}$-cycles.

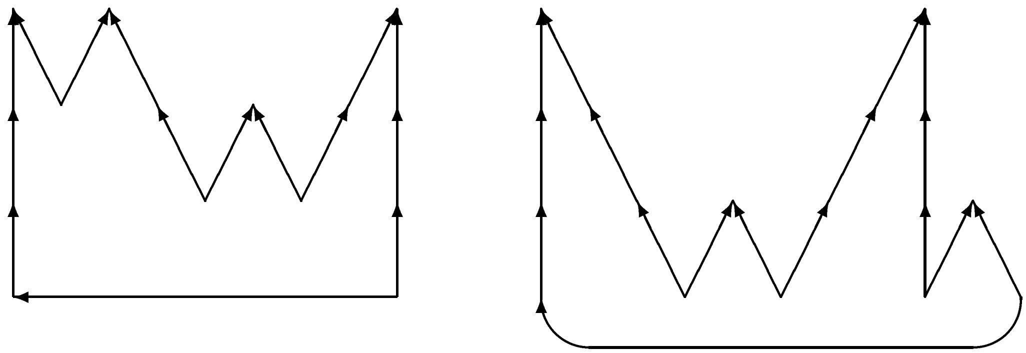

Fig. 2. Example C-cycles

By the definition of $\mathcal{C}$-cycles, we know that each $\mathcal{C}$-cycle C is constructed starting from some B_n. Thus it contains one directed path of length $n+1$ and all other directed paths are of length at most n. We denote this integer n by $N(C)$, and denote by $K(C)$ the number of maximal directed paths of C of length at least n. We always list the vertices of a $\mathcal{C}$-cycle in the positive direction as $c_0, c_1, \cdots, c_{n+1}, c_{n+2}, \cdots, c_m, c_0$ where $c_0, c_1, \cdots, c_{n+1}$ is the unique directed path of length $n+1$. Since at each step of the inductive construction of C, the path we inserted in C is of length zero, it is clear that the path $c_0, c_m, c_{m-1}, \cdots, c_{n+1}$ is of length n and it contains no paths of length greater than n.

We conclude from the above that each $\mathcal{C}$-cycle has the following property:

Property (B). C consists of two internally disjoint oriented paths, one a directed path of length $n+1$, and the other of length n and without paths of length greater than n.

(The first path is $c_0, c_1, \cdots, c_{n+1}$ and the second path is $c_0, c_m, c_{m-1}, \cdots, c_{n+1}$.)

The following theorem proved in [9] charaterizes all digraphs which admit a homomorphism to a given cycle C which satisfies the property (B):

THEOREM 1 ([9]) *Suppose C is a cycle with property (B) and G is any digraph. Then $G \not\to C$ if and only if there exists an oriented path P such that $P \to G$ and $P \not\to C$.*

We are interested in characterizing those oriented cycles C that are multiplicative. If C is hom-equivalent to a proper subgraph G of C, then G must be an oriented path, and the results in [4, 20] describe whether or not the path G (and hence also the cycle C) is multiplicative. Furthermore, if C is a directed cycle, then the multiplicativity of C is determined by the length of C (those of prime power length are multiplicative, [4, 22], all others are not, [4, 12]). Therefore we may restrict our attention to oriented cycles C that are not directed cycles and that are not hom-equivalent to any oriented path. The crucial observation, due to H. Zhou, [18] (cf.also [21]), is the following result:

THEOREM 2 *Let C be an oriented cycle which is not a directed cycle and which is not hom-equivalent to any oriented path. If C is multiplicative, then C is a $\mathcal{C}$-cycle.*

Our main purpose in this paper is to prove the converse:

THEOREM 3 *Every $\mathcal{C}$-cycle is multiplicative.*

One common method of proving the multiplicativity of a digraph D depends on the following observation, cf. [4]:

LEMMA 2 *The digraph D is multiplicative if and only if there exists a set $\mathcal{O}$ of digraphs such that*

- *$G \not\to D$ for each $G \in \mathcal{O}$,*
- *for each H with $H \not\to D$ there is a $G \in \mathcal{O}$ such that $G \to H$, and*
- *for any two digraphs $G, G' \in \mathcal{O}$ there is a digraph $G^* \in \mathcal{O}$ such that $G^* \to G$ and $G^* \to G'$.*

The set $\mathcal{O}$ is called a *complete set of obstructions* for D.

We shall prove that each $\mathcal{C}$-cycle C is multiplicative by finding a complete set of obstructions $\mathcal{O}$ for C; in fact we shall show that $\mathcal{O} = \{P : P$ is an oriented path and $P \not\to C\}$ is such a set.

To begin **the proof of the main theorem** we fix a $\mathcal{C}$-cycle C as $c_0, c_1, \cdots, c_{n+1}, c_{n+2}, \cdots, c_m, c_0$ (where $n = N(C)$). We shall derive the multiplicativity of C from Lemma 2 by proving, in a sequence of lemmas, that the set of paths which are not homomorphic to C is a complete set of obstructions for C. It is clear that the first two properties of a complete set of obstructions are satisfied (by Theorem 1). Thus it only remains to verify the third property, namely the following result we call the **Common Preimage Theorem**:

THEOREM 4 *Let C be a $\mathcal{C}$-cycle as above. If oriented paths P and P' are not homomorphic to C, then there is an oriented path P^* which is homomorphic to both P and P', but not to C.*

We define $\mathcal{C}$-*paths* in a recursive manner analogous to the definition of $\mathcal{C}$-cycles, except we start with S_n instead of B_n. Also in analogy with $\mathcal{C}$-cycles we define $N(P) = n$ for any path P that arose starting from S_n. Thus $N(P)$ is the maximum length of a directed subpath of P. Note that a $\mathcal{C}$-path P with $N(P) = n$ is necessarily n-bounded.

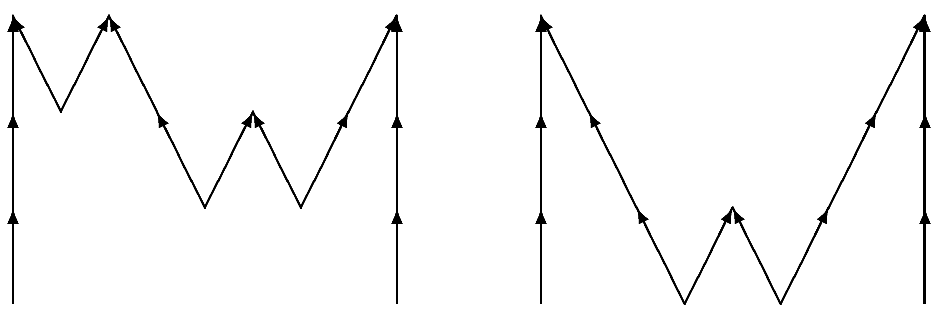

Fig. 2. Example C-paths

A fact frequently used in our proof is that homomorphisms of paths preserve length: If h is a homomorphism of an oriented path P (with initial vertex a and terminal vertex b) to an oriented path P' then $l(P) = l(P'[h(a), h(b)])$. In particular if the homomorphism h takes the initial vertex of P to the initial vertex of P', then the level of any vertex v in P is equal to the level of $h(v)$ in P'.

Suppose P is a minimal path $p_0, p_1, \ldots, p_k$ and $h : P \to C$ is a homomorphism, then the walk $h(p_0), h(p_1), \ldots, h(p_k)$ in C is also minimal, in the sense that it contains no proper subwalk of the same length. One can verify that any minimal walk of C of length $n+1$ must start at c_0 and end at c_{n+1}. Therefore if P is a minimal path of length $n+1$ then any homomorphism $h : P \to C$ has $h(p_0) = c_0$ and $h(p_k) = c_{n+1}$.

The proof of the Common Preimage Theorem: Let P, P' be oriented paths which are not homomorphic to C. We shall assume that P is the path $p_0, p_1, \ldots, p_t$ and P' is the path $p'_0, p'_1, \ldots, p'_{t'}$. It is clear that we may assume without loss of generality that any proper subpath of either P or P' is homomorphic to C. Then we have the following:

LEMMA 3 *Both P and P' are minimal oriented paths of length $n+2$.*

Proof. By symmetry, we only need to treat P, and we may assume that $l(P) \geq 0$. First we show that there is no subpath of P of length $n+1$ in $P[p_1, p_{t-1}]$. Therefore, we assume without loss of generality, that for some $i < j$, $p_i, p_{i+1}, \ldots, p_j$ is a minimal subpath of $P[p_1, p_{t-1}]$ of length $n+1$. Since $P[., p_j], P[p_i, .]$ are proper subpaths of P, there are homomorphisms $f : P[., p_j] \to C$ and $g : P[p_i, .] \to C$.

By the remark preceeding the proof of the Common Preimage Lemma we have $f(p_i) = g(p_i) = c_0$ and $f(p_j) = g(p_j) = c_{n+1}$. Define $h : P \to C$ as follows:

$$h(v) = \begin{cases} f(v) & \text{if } v \in P[p_0, p_j] \\ g(v) & \text{if } v \in P[p_j, p_t] \end{cases}$$

Then h is a homomorphism of P to C, contradicting the assumption that $P \not\to C$. Therefore $P[p_1, p_{t-1}]$ contains no path of length $n+1$.

Now both p_0p_1 and $p_{t-1}p_t$ are forward edges of P and $P[p_1, p_{t-1}]$ has length n, for otherwise P would contain no path of length $n+2$ and therefore would be homomorphic to the directed path $c_0, c_1, \ldots, c_{n+1}$ and hence to C. It is now easy to see that P is minimal.

We now define a $\mathcal{C}$-path S which corresponds to a walk in C in which we start at c_1, go all the way around C to c_0, then continue forward to c_{n+1}, and then back up to c_0. Formally, we introduce new vertices c_j^i (corresponding to the i-th visit to c_j) and define S to be the oriented path

$$c_1, c_2, \ldots, c_{n+1}, \ldots, c_m, c_0, c_1^2, \ldots, c_n^2, c_{n-1}^3, \ldots, c_1^3, c_0^2$$

where consecutive vertices c_j^i and $c_{j'}^{i'}$ of S are joined by a forward (respectively backward) edge if and only if $c_j c_{j'}$ is a forward (respectively backward) edge of C. It is easy to see that S is a $\mathcal{C}$-path, and that there is a canonical homomorphism $g : S \to C$ which takes each c_j^i to c_j.

Finally we introduce a construction which changes an arbitrary minimal path Y of length $n+2$ into an n-bounded path $\mathcal{F}(Y)$: Let Y be a minimal path of length $n+2$, say, $y_0, y_1, \cdots, y_j, \cdots, y_s, \cdots, y_k$, where y_j is the last vertex of level 1, and y_s is the first vertex after y_j of level $n+1$. (Such subscripts exist because Y has length $n+2$.) Then we define $\mathcal{F}(Y)$ to correspond to a walk of Y in which we start at y_1, go all the way to y_s, and then back to y_j. Formally, (introducing new vertices y_i^2 as above) $\mathcal{F}(Y)$ is the oriented path

$$y_1, y_2, \ldots, y_{s-1}, y_s, y_{s-1}^2, \ldots, y_j^2.$$

It is obvious from the construction that $\mathcal{F}(Y)$ is an n-bounded path.

LEMMA 4 *Let Y be a minimal path of length $n+2$. Then Y is homomorphic to C if and only if $\mathcal{F}(Y)$ can cross S.*

Proof. First we prove the "only if" part. Suppose that $\mathcal{F}(Y)$ can cross S. Let h be a homomorphism which carries $\mathcal{F}(Y)$ over S. Then we have $h(y_s) = c_n^2$, because y_s is the last vertex on $\mathcal{F}(Y)$ of level n and c_n^2 is the last vertex on S of level n. We define a homomorphism $f : P[p_1, p_{t-1}] \to S$ which equals h on $P[p_1, p_s]$ and maps $P[p_s, p_{t-1}]$ to the backward directed path $c_n^2, \ldots, c_0^2$ in the obvious way. The composition $g \circ f$ is a homomorphism of $P[p_1, p_{t-1}]$ to C, which can be extended to a homomorphism of the entire path P to C by mapping p_0 to c_0, and p_t to c_{n+1}. This contradicts the assumption that P is not homomorphic to C.

We now prove the "if" part. Suppose that h is a homomorphism of Y to C. Then we have $h(y_0) = c_0$ and $h(y_1) = c_1$ because c_0 is the initial point of a minimal subpath of Y of length $n+1$. Also the path $Y[y_j, y_k]$ is a minimal of length $n+1$. Therefore h maps this path to $c_0, c_1, \cdots, c_{n+1}$, and hence maps y_j to c_0, y_k to c_{n+1} and y_s to c_n. Since the $l(Y[y_0, y_j]) = 1$, we conclude that the walk $h(y_0), h(y_1), \cdots, h_(y_j), \cdots, h(y_s)$ of C starts at c_0, goes in the positive direction of C, passes through $c_0 = h(y_j)$, continues in the positive direction of C, and ends at c_n. Therefore h restricted to the path $y_1, y_2, \cdots, y_j, \cdots, y_s$ can be considered to be in fact a homomorphism to S, which sends y_1 to c_1 and y_s to c_n^2. It follows from the definitions of y_j, y_s that the subpath $y_s, y_{s-1}^2, \cdots, y_j^2$ of $\mathcal{F}(Y)$ is of length $-n$ and contains no path of length greater than n. Thus there is a homomorphism g of $y_s, y_{s-1}^2, \cdots, y_j^2$ to the backward directed path $c_{n+1}^2, c_n^2, \cdots, c_1^2$ of S such that $g(y_s) = c_{n+1}^2$ and $g(y_j^2) = c_1^2$. Then the mapping f, which equals h on $Y[y_1, y_s]$, and equals g on $y_s, y_s^2, \cdots, y_j^2$, is a homomorphism that carries $\mathcal{F}(Y)$ over S. This completes the proof of Lemma 4

Let $R = \mathcal{F}(P)$ and $R' = \mathcal{F}(P')$, and let p_j, p_s and $p'_{j'}, p'_{s'}$ be the vertices of P and P' corresponding to y_j, y_s in Y. By the above lemma, we see that neither R nor R' can cross S.

LEMMA 5 (The Common Crossing Lemma) *Let Q be a $\mathcal{C}$-path with $N(Q) = n$. If Z, Z' are two n-bounded oriented paths which cannot cross Q, then there exists an n-bounded oriented path Z^* which also cannot cross Q but can cross both Z and Z'.*

The lengthy proof of this lemma is left to the next section. Now we show how to use this lemma to prove the Common Preimage Theorem, and hence Theorem 3. We conclude from the Common Crossing Lemma that there exists an n-bounded path R^* which can cross both R and R', but cannot cross S. Suppose that R^* is the path $r_1^*, \ldots, r_b^*$, and let r_l^* be the last vertex of R^* of level n. Any homomorphism h which carries R^* over R must take r_l^* to p_s (and similarly for homomorphisms which carry R^* over R'). The restriction of h to $R^*[., r_l^*]$ may fail to carry $R^*[., r_l^*]$ over $R[p_1, p_s]$, as some images may lie in $R[p_s, .]$. However, because of the special form of the path R (since $R = \mathcal{F}(P)$, the path $R[p_s, .]$ is the same as the path $R[p_j, p_s]^T$), there must be a homomorphism which carries $R^*[., r_l^*]$ over $R[p_1, p_s]$.

Therefore the path $R^*[., r_l^*]$ can cross both $R[p_1, p_s] = P[p_1, p_s]$ and $R'[p'_1, p'_{s'}] = P'[p'_1, p'_{s'}]$. We now extend this path to a minimal path P^* of length $n+2$ so that P^* is homomorphic to both P and P' and such that $\mathcal{F}(P^*) = R^*$. This will imply that P^* is not homomorphic to C by Lemma 4, and complete the proof of Theorem 3.

To be specific, the path P^* is of the form $P^* = ab \circ R^*[., r_l^*] \circ R^\# \circ cd$ where ab and cd are two forward edges and $R^\#$ is a path of length zero which contains no subpath of length n and which admits homomorphisms to P and P' taking the initial vertex of $R^\#$ to p_s and $p'_{s'}$ respectively, and taking the terminal vertex of $R^\#$ to p_{t-1} and $p'_{t'-1}$ respectively. It is easy to see that if such a path $R^\#$ exists, the path P^* constructed as above satisfies our requirements.

The existence of such a path $R^{\#}$ can be derived from Lemma 1.

Consider the path $p_s, p_{s-1}, \ldots, p_{j+1}$ and the corresponding path in P'. Since these paths have length $1-n$ and contain no paths of length n, there is a path R_1 that can cross both. Consider the path $p_{j+1}, p_{j+2}, \ldots, p_{t-1}$, and the corresponding path in P'. These paths have length $n-1$ and contain no paths of length n. Hence there is a path R_2 that can cross both. It is easy to see $R^{\#} = R_1 \circ R_2$ contains no path of length n and it admits homomorphisms to P and P' taking the initial vertex of $R^{\#}$ to p_s and $p'_{s'}$ respectively, and taking the terminal vertex of $R^{\#}$ to p_{t-1} and $p'_{t'-1}$ respectively. As noted above, this completes the proof of Theorem 3.

3 The Proof of the Common Crossing Lemma

Suppose Q is a forward $\mathcal{C}$-path with $N(Q) = n$ and Z, Z' are n-bounded oriented paths that cannot cross Q. We will prove, by induction on n, that there is a path Z^* which also cannot cross Q but which can cross both Z and Z'.

We denote by q, z and z' the initial vertices of Q, Z and Z' respectively, and by s, w and w' the terminal vertices of Q, Z and Z' respectively.

Consider the case $N(Q) = n = 1$. Any 1-bounded path (including Q, Z and Z') is an even alternating path of forward and backward edges, and an even alternating path of length l can cross any even alternating path of length $m \leq l$. Thus the longer of Z, Z' can cross both Z and Z' and cannot cross Q. Therefore the lemma is true for any $\mathcal{C}$-path Q with $N(Q) = 1$. Suppose the lemma is true for any $\mathcal{C}$-path P with $N(P) = n-1$. We shall prove that it is true for Q, where $N(Q) = n$.

Let $I_1, I_2, \ldots, I_{2k}$ be the directed subpaths of Q with length n or $-n$, in the order encountered on Q. Note that I_i is a forward directed path when i is odd (i.e., $l(I_i) = n$) and a backward directed path when i is even (i.e., $l(I_i) = -n$). Let u_i and v_i be the initial vertex and terminal vertex of I_i respectively. (Note that u_i may be equal to v_{i-1}).

Since Z is n-bounded, it also has an even number of minimal paths of length n or $-n$. Let these minimal subpaths be $A_1, A_2, \ldots, A_{2m}$, in the order encountered on Z. It is easy to see that these minimal paths are edge disjoint, and A_i is of length n when i is odd and of length $-n$ when i is even. Let a_i and b_i be the initial and terminal vertices of A_i respectively. (Again a_i may be equal to b_{i-1}.) Similarly, let $A'_1, A'_2, \ldots, A'_{2m'}$ be the minimal subpaths of Z' of length n or $-n$, in the order encountered on Z', and let a'_i and b'_i be the initial and terminal vertices of A'_i respectively.

Suppose that X and Y are two oriented paths such that there is a homomorphism $h : X \to Y$ which takes the initial vertex of X to the initial vertex of Y. We define $f_Y(X)$ to be the latest vertex y of Y such that there is a homomorphism $h : X \to Y$ which takes the initial vertex of X to the initial vertex of Y and takes the terminal vertex of X to y.

Since Z is n-bounded there are homomorphisms $h : Z \to Q$ such that $h(z) = q$. Hence $f_Q(Z[., x])$ is well defined for any vertex x of Z. For any homomorphism h, each of the paths A_i is mapped to a certain minimal subpath of Q with length n or $-n$. Since $I_j (1 \leq j \leq 2k)$ are the only minimal subpaths of Q of length n or $-n$, we have $h(A_i) = I_j$ for some j. Furthermore the terminal vertex of A_i is mapped either to the terminal or to the initial vertex of I_j, i.e., $h(b_i) =$ either v_j or u_j. Therfore $f_Q(Z[., b_i]) = u_j$ or v_j for some j. If $h(A_i) = I_j$ and $h(b_i) = v_j$ then i and j have the same parity. If $h(A_i) = I_j$ and $h(b_i) = u_j$ then i and j have different parity. In case $f_Q(Z[., b_i]) = v_j = u_{j+1}$, we write $f_Q(Z[., b_i]) = u_{j+1}$ if there is a homomorphism $h : Z[., b_i] \to Q$ such that $h(z) = q$ and $h(A_i) = I_{j+1}$, and otherwise we write $f_Q(Z[., b_i]) = v_j$. Thus we can define a mapping $f : \{1, 2, \ldots, 2m\} \to \{1, 2, \ldots, 2k\}$ by letting $f(i) = j$ where $f_Q(Z[., b_i]) = v_j$ or u_j. Equivalently, we can define f as follows: $f(i) = max\{j :$ there is a homomorphism $h : Z[., b_i] \to Q$ such that $h(z) = q$ and $h(A_i) = I_j\}$. Since Z can not cross Q, we have that $f(2m) < 2k$.

Similarly we let $A'_1, A'_2, \ldots, A'_{2m'}$ be the minimal subpaths of Z' of length n or $-n$, in the order encountered in Z. We definee $f'(i) = j$ where $f_Q(Z'[., b'_i]) = v_j$ or u_j. Similarly we have $f(2m') < 2k$.

If there is a homomorphism $h : Z[., b_i] \to Q$ such that $h(z) = q$ and $h(A_i) = I_j$, then this homomorphism can be extended to $Z[., b_{i+1}]$ by sending all A_{i+1} to I_j. Therefore f is a nondecreasing mapping.

Claim. Let i and i' have the same parity and let $f(i) \leq t, f'(i') \leq t$ for some t, with $1 \leq t \leq 2k$. Then there is a path V^* which can cross both $Z[., b_i]$ and $Z'[., b'_{i'}]$, and such that $f_Q(V^*)$ precedes v_t.

Proof of the claim. We prove the claim by induction on the sum $i + i'$.

First we treat the case $i + i' = 2$, i.e., $i = i' = 1$. By Lemma 1 and the remarks following it, there is an oriented path V^* which can cross both $Z[., b_1]$ and $Z'[., b'_1]$, such that V^* contains exactly one minimal path of length

n, and such that V^* ends with this minimal path of length n. It is easy to see that $f_Q(Z[.,b_1]) = f_Q(Z'[.,b'_1]) = f_Q(V^*) = v_1$. Thus the lemma is true in this case.

To carry on with the induction, we consider two cases.

Case 1. $i > 1$ and $i' > 1$.

Case 1(a). The parity of i and i' is the same as the parity of t.

Without loss of generality we assume i, i', t are odd. Since $f(i-1) \leq f(i) \leq t$ and $f'(i'-1) \leq f'(i') \leq t$, by the induction hypothesis there is a path X^* which can cross both $Z[.,b_{i-1}]$ and $Z'[.,b'_{i-1}]$, and $f_Q(V^*)$ precedes v_t. Let x^* and z^* be the initial and terminal vertice of X^* respectively. Since X^* can cross $Z[.,b_{i-1}]$ and $i-1$ is even, we see that z^* has level zero. Hence for any homomorphism $h : V^* \to Q$ with $h(x^*) = q$ the level of $h(z^*)$ in Q is zero. Because u_t is the last vertex of Q which has level zero and precedes v_t, we see that $h(z^*)$ precedes u_t. By Lemma 1 and the remarks following it, there is a path U such that U contains only one minimal path A^* of length n, which ends at the terminal vertex y^* of U, and such that U can cross both $Z[b_{i-1}, b_i]$ and $Z'[b'_{i'-1}, b'_{i'}]$.

Let $V^* = X^* \circ U$. It is obvious that V^* can cross both $Z[.,b_i]$ and $Z'[.,b'_{i'}]$. We now need to show that $f_Q(V^*)$ precedes v_t. (Note that $f_Q(V^*)$ is well defined because V^* can cross $Z[.,b_i]$.) Suppose $h : V^* \to Q$ is a homomorphism such that $h(x^*) = q$. Then $h^*(A^*) = I_j$ for some j. Suppose $j \geq t+1$. Then since $h(z^*)$ precedes u_t, we see that $h(U)$ contains at least two minimal paths of length n, namely I_t and I^T_{t+1}. This is impossible because A^* is the only minimal path of lenght n contained in U. Therefore $j \leq t$, and hence $h(y^*)$ precedes v_t on Q.

Case 1(b). The parity of i and i' is different from the parity of t.

Without loss of generality we assume that t is odd and i and i' are even. If $a(i-1) \leq t-1$ and $a'(i'-1) \leq t-1$, then the same argument as in Case 1(a) applies. We shall distinguish two further subcases:

Case 1(b)(i). $f(i-1) = t$ and $f'(i'-1) = t$.

By the induction hypothesis there is a path X^* which can cross both $Z[.,b_{i-1}]$ and $Z'[.,b'_{i-1}]$ and $f_Q(X^*)$ precedes v_t. Denote by x^* and z^* the initial and terminal vertices of X.

Let a be the second point of I_t, b be the second to last point of I_{t+1}. Then $S = Q[a,b]$ is a $\mathcal{C}$-path with $N(S) = n-1$. (In case t is even and i, i' are odd, we consider S^R, the path obtained from S by reversing the direction of each of the edges of S. The argument will be the same except that instead of applying the induction hypothesis to S, T and T' (cf. below), we should apply the induction hypothesis to S^R, T^R and T'^R.)

Let x be the second point of A_{i-1}, let y be the second to last point of A_i, and let $T = Z[x, y]$. Let x' be the second point of $A'_{i'-1}$, let y' be the second to last point of $A'_{i'}$, and let $T' = Z'[x', y']$. It is easy to check that T and T' are $n-1$ bounded paths. We shall show that neither T nor T' can cross R. For otherwise let $g : T \to R$ be a homomorphism which carries T over R. Then we must have $g(b_{i-1}) = v_t$ because b_{i-1} is the first point of T with level $n-1$ and v_t is the first point of R with level $n-1$. Similarly we must have $g(a_i) = u_{t+1}$.

Since $i-1$ and t are odd and $f(i-1) = t$, we know that there is a homomorphism $r : Z[., b_{i-1}] \to Q$ such that $r(z) = q$ and $r(A_{i-1}) = I_t$, and therefore $r(b_{i-1}) = v_t$.

Obviously there is a homomorphism $k : Z[., a_i] \to I_{t+1}$ with $k(a_i) = u_{t+1}$. Now we define $h : Z \to Q$ as follows:

$$h(x) = \begin{cases} r(x) & \text{if } x \in Z[., b_{i-1}] \\ g(x) & \text{if } x \in T[b_{i-1}, a_i] \\ k(x) & \text{if } x \in Z[a_i, .] \end{cases}$$

Then h is a homomorphism such that $h(z) = q$ and $h(A_i) = I_{t+1}$, contradicting the assumption that $f(i) \le t$. Therefore T can not cross R.

Similarly T' can not cross R. Since $N(R) = n-1$, we conclude, from the induction hypothesis of the Common Crossing Lemma, that there is a path T^* which can cross both T and T' but can not cross R. Let u be the first vertex of T^* of level $n-1$ and v the last vertex of T^* of level $n-1$ (because T^* can cross T, such vertices exist). Since b_{i-1} is the first vertex of T of level $n-1$ and a_i is the last vertex of T of level $n-1$, we know that any homomorphism h which carries T^* over T takes u to b_{i-1} and takes v to a_i. Therefore r restricted to $T^*[u, v]$ is a homomorphism to T which takes u to b_{i-1} and v to a_i. Similarly there is a homomorphism r' of $T^*[u, v]$ to T' which takes u to $b'_{i'-1}$ and v to $a'_{i'}$. Also note that T^* can cross the $n-1$ bounded path T. Therefore T^* contains no path of length n and hence $T^*[u, v]$ contains no paths of length n.

Define $V^* = X^* \circ T^*[u, v] \circ A^*$, where A^* is a minimal path of length $-n$ which can cross both A_i and $A'_{i'}$. Let w^* and y^* be initial and the terminal vertices of A^*. By the discussion above, V^* can cross both $Z[., b_i]$ and $Z'[., b'_{i'}]$.

Now we will show that $f_Q(V^*)$ precedes v_t. Let $h^* : V^* \to Q$ be a homomorphism with $h^*(x^*) = q$. Then $h^*(A^*) = I_j$ for some j. Suppose that $j \geq t+1$. Since $h^*(z^*)$ precedes v_t, and $T^*[u,v]$ contains no path of length n, we must have $h^*(z^*) = v_t$, $h^*(w^*) = u_{t+1}$ and $h^*(A^*) = I_{t+1}$. Therefore h^* restricted to $T^*[u,v]$ is a homomorphism which takes u to v_t and takes v to u_{t+1}. It is easy to see that such a homomorphism can be extended to a homomorphism which carries T^* over R, contradicting the fact that T^* can not cross R.

Case 1(b)(ii). $f(i-1) = t$ and $f'(i'-1) \leq t-1$.

As in the proof of case 1(b)(i), let a be the second point of I_t, let b be the second to last point of I_{t+1}, and let $R = Q[a,b]$. Let x be the second point of A_{i-1}, let y be the second to last point of A_i and let $T = Z[x,y]$. As in Case 1(b)(i), we can prove that T can not cross R. In particular, this implies that $v_t \neq u_{t+1}$ and R has at least two points of level $n-1$, namely v_t and u_{t+1}.

Let E be a path isomorphic to $(A'_{i'})^T$, and let e be its terminal vertex. Let F be a path isomorphic to $A'_{i'}$, and let d be its terminal vertex. Define $Z'' = Z'[., b'_{i'}] \circ E \circ F$, an extension of $Z'[., b'_{i'}]$.

By the induction hypothesis, there is a path X^* which can cross both $Z[., b_{i-2}]$ and $Z[., b'_{i'}]$ and such that $f_Q(X^*)$ precedes v_t. Now we shall extend the path X^* to a path Y^* which can cross both $Z[., b_{i-1}]$ and $Z''[., e]$, and for which $f_Q(Y^*)$ precedes v_t. Since it is now the case that t and $i-1$ have the same parity, we can construct Y^* in a manner similar to the construction in Case 1(a).

Let x' be the second point of E and y' the second to last point of F. Then $T' = Z''[x', y']$ can not cross R because R has at least two point of level $n-1$ and T' has only one such point.

By the same argument as in Case 1(b)(i), we can use the induction hypothesis of Lemma 5 for R to construct an extention V^* of Y^* which can cross both $Z[., b_i]$ and $Z''[., d] = Z''$ and such that $f_Q(V^*)$ precedes v_t.

Now V^* can also cross $Z'[., b'_{i'}]$ because Z'' can cross $Z'[., b'_{i'}]$. This completes Case 1.

Case 2. $i > 1$ and $i' = 1$.

Since i and i' has the same parity, we know that i is odd and $i \geq 3$. As in Case 1(b)(ii), let E be a path isomorphic to $(A'_1)^T$, let F be a path isomorphic to A'_1, and consider the extension $Z'' = Z'[., b'_1] \circ E \circ F$. The remain der of the argument is similar to the Case 1(b)(ii). In the proof we need the following property: For any homomorphism $h'' : Z'' \to Q$ with

$h''(z) = q$, we have $h''(F) = I_j$ for some $j \leq t$.

This property is obviously true if $t \geq 3$. If $t \leq 2$, we derive the property by observing that since $f(3) \leq f(i) \leq t \leq 2$, we have $v_1 \neq u_2$ or $v_2 \neq u_3$. This kind of argument has already been used in the Case 1(b)(ii), and we leave the details to the reader.

Now we have completed the proof of the claim.

Applying the claim to A_{2m} and $A'_{2m'}$, we obtain a path V^*, with the terminal vertex y^*, which can cross both $Z[., b_{2m}]$ and $Z'[., b'_{2m'}]$, and such that for any homomorphism $h^* : V^* \to Q$ with $h^*(x^*) = q$ the vertex $h^*(y^*)$ precedes v_{2k-1}. As in the construction of $R^\#$ in the proof of theorem 3, we can construct a path U, with initial vertex u and terminal vertex v, which contains no path of length n and such that there are homomorphisms $h : U \to Z$ and $h' : U \to Z'$ with $h(u) = b_{2m}, h(v) = w, h'(u) = b'_{2m'}$ and $h'(v) = w'$.

Now let $Z^* = V^* \circ U$. It is easy to verify that Z^* can cross both Z and Z', and Z^* can not cross Q. This completes the proof of the Common Crossing lemma.

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