On the chromatic number of the products of hypergraphs

Xuding Zhu

March 6, 2003

Abstract

This paper discusses the chromatic number of the products of $n+1$-chromatic hypergraphs. The following two results are proved:

Suppose G and H are $n+1$-chromatic hypergraphs such that each of G and H contains a complete sub-hypergraph of order n and each of G and H contains a vertex critical $n+1$-chromatic sub-hypergraph which has non-empty intersection with the corresponding complete sub-hypergraph of order n. Then the product $G \times H$ is of chromatic number $n+1$.

Suppose G is an $n+1$-chromatic hypergraph such that each vertex of G is contained in a complete sub-hypergraph of order n. Then for any $n+1$-chromatic hypergraph H, $G \times H$ is an $n+1$-chromatic hypergraph.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set, and let $\mathcal{E} = \{e_i : i \in I\}$ be a family of subsets of X. The couple $H = (X, \mathcal{E})$ is called a *hypergraph* if $e_i \neq \emptyset (\forall i \in I)$ and $\bigcup_{i \in I} e_i = X$. X is called the vertex set of H, denoted by $V(H)$ and $\mathcal{E}$ is called the edge set of H, denoted by $E(H)$.

An edge $e \in \mathcal{E}$ is a *loop* if $|e| = 1$. In the following we discuss loopless hypergraphs.

For $X = \{x_1, x_2, \ldots, x_n\}$, if $\mathcal{E} = \{e \subseteq X : |e| \geq 2\}$, then the hypergraph $H = (X, \mathcal{E})$ is a complete hypergraph of order n.

A hypergraph H is said to be *n-colorable* if there is a mapping $\Delta : V(H) \mapsto \{1, 2, \ldots, n\}$ such that $\forall e \in \mathcal{E}, \forall i \in \{1, 2, \ldots, n\}, e \nsubseteq \Delta^{-1}(i)$, i.e. there is no monochromatic edge. Such a mapping is called a *proper* n-coloring of H and the set {1, 2, ..., n } is the set of colors. The chromatic number of H, denoted by $\chi(H)$, is the least integer n such that H is n-colorable.

Suppose $H = (X, \mathcal{E})$ and $G = (X', \mathcal{E}')$ are hypergraphs. The product $G \times H$ is a hypergraph with vertex set $V(G \times H) = V(G) \times V(H) = \{(g, h) : g \in V(G), h \in V(H)\}$ and a subset $e = \{(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)\}$ of $V(G \times H)$ is an edge of $G \times H$ if and only if $\{x_1, x_2, \ldots, x_k\}$ is an edge of G and $\{y_1, y_2, \ldots, y_k\}$ is an edge of H, where $x_1, x_2, \ldots, x_k$ and $y_1, y_2, \ldots, y_k$ need not to be distinct.

Suppose G is n-colorable and $\Delta : V(G) \mapsto \{1, 2, \ldots, n\}$ is a proper n-coloring of G. Then for any hypergraph H, define $\Delta' : V(G \times H) \mapsto \{1, 2, \ldots, n\}$ by $\Delta'((g, h)) = \Delta(g)$. $\Delta'^{-1}(j)$ contains no edge of $G \times H$ for any $j \in \{1, 2, \ldots, n\}$. Hence Δ' is a proper n-coloring of $G \times H$. Therefore $\chi(G \times H) \leq min\{\chi(G), \chi(H)\}$

Similar to Hedetniemi's conjecture about the chromatic number of the product of graphs, we ask the same question for the product of hypergraphs. Is it true that $\chi(G \times H) = min\{\chi(G), \chi(H)\}$?

Conjecture 1 $\chi(G \times H) = min\{\chi(G), \chi(H)\}$.

The same conjecture about graphs was proposed by Hedetniemi[6] in 1966 and only some special cases are proved.

El-Zahar and N. Sauer[4] proved that the chromatic number of the product of two 4-chromatic graphs is 4. Burr, Erdös and Lovász[2] proved that if $min\{\chi(G), \chi(H)\} = \chi(G) = n + 1$ and each vertex of G is contained in a complete subgraph of order n, then $\chi(G \times H) = n + 1$. Duffus, Sands and Woodrow[3] and Welzl[8] proved that if $\chi(G) = \chi(H) = n + 1$ and each of

G and H are connected and contains a complete subgraph of order n, then $\chi(G \times H) = n + 1$.

We prove in the following some special cases of conjecture 1.

DEFINITION 1 *Suppose H is a hypergraph. The n-coloring hypergraph of H, denoted by $\mathcal{C}_n(H)$, has as its vertex set all mappings $f : V(H) \mapsto \{1, 2, \ldots, n\}$. A subset $\{f_1, f_2, \ldots, f_k\}$ is an edge of $\mathcal{C}_n(H)$ if and only if $|\{f_{\alpha(i)}(x_i) : i \in I$ for some index set $I\}| \geq 2$ whenever $\{x_i : i \in I\}$ is an edge of H (x_i need not to be distinct) and $\{f_{\alpha(i)} : i \in I\} = \{f_1, f_2, \ldots, f_k\}$.*

By the definition, we know that the n-coloring graph of H, $\mathcal{C}_n(H)$, has a loop if and only if H is n-colorable (each loop corresponds to a proper n-coloring of H).

Conjecture 2 *Suppose H is an $n+1$-chromatic hypergraph. Then the n-coloring hypergraph of H, $\mathcal{C}_n(H)$, is n-colorable.*

THEOREM 1 *Conjecture 1 and conjecture 2 are equivalent.*

Proof. (1)$\Rightarrow$ (2): Suppose (2) fails and $\mathcal{C}_n(H)$ is not n-colorable for some $n+1$-chromatic hypergraph H. To show that (1) fails, it is enough to show that $H \times \mathcal{C}_n(H)$ is n-colorable (then H and $\mathcal{C}_n(H)$ witnesses that (1) fails). We claim that the mapping $\Delta : V(H \times \mathcal{C}_n(H)) \mapsto \{1, 2, \ldots, n\}$ defined as $\Delta(h, f) = f(h)$ is a proper n-coloring of $H \times \mathcal{C}_n(H)$. If $\{(h_1, f_1), (h_2, f_2), \ldots, (h_k, f_k)\}$ is an edge of $H \times \mathcal{C}_n(H)$, then $\{h_1, h_2, \ldots, h_k\}$ is an edge of H and $\{f_1, f_2, \ldots, f_k\}$ is an edge of $\mathcal{C}_n(H)$. By the definition of $\mathcal{C}_n(H)$, $|\{f_1(h_1), f_2(h_2), \ldots, f_k(h_k)\}| \geq 2$. Therefore $|\{\Delta((h_1, f_1)), \Delta((h_2, f_2)), \ldots, \Delta((h_k, f_k))\}| \geq 2$, i.e. this edge (hence every edge) of $H \times \mathcal{C}_n(H)$ is not a monochromatic edge under the coloring Δ and therefore Δ is a proper n-coloring of $H \times \mathcal{C}_n(H)$.

(2)$\Rightarrow$ (1): Suppose (1) fails and G, H are $n+1$-chromatic hypergraphs such that $G \times H$ is n-colorable. Now we show that $\mathcal{C}_n(H)$ is not n-colorable hence (2) fails.

Otherwise suppose $\mathcal{C}_n(H)$ is n-colorable and $\phi : V(\mathcal{C}_n(H)) \mapsto \{1,2,\ldots,n\}$ is a proper n-coloring of $\mathcal{C}_n(H)$. Let $\Delta : V(G \times H) \mapsto \{1,2,\ldots,n\}$ be a proper n-coloring of $G \times H$. Define $\Delta' : V(G) \mapsto \{1,2,\ldots,n\}$ by $\Delta'(g) = \phi(f_g)$ where $f_g : V(H) \mapsto \{1,2,\ldots,n\}$ is the mapping defined as $f_g(h) = \Delta(g,h)$. We claim that Δ' is a proper n-coloring of G. This then contradicts the assumption that G is not n-colorable.

Otherwise suppose $\{g_1, g_2, \ldots, g_k\}$ is an edge of G and $\Delta'(g_1) = \Delta'(g_2) = \ldots = \Delta'(g_k)$. then $\phi(f_{g_1}) = \phi(f_{g_2}) = \ldots = \phi(f_{g_k})$. Since ϕ is a proper n-coloring of $\mathcal{C}_n(H)$, $\{f_{g_1}, f_{g_2}, \ldots, f_{g_k}\}$ is not an edge of $\mathcal{C}_n(H)$. By definition, this means that there exists an edge $\{h_1, h_2, \ldots, h_t\}$ of H such that $|\{f_{\alpha(i)}(h_{\beta(i)}) : i \in I\}| = 1$ where $\{\alpha(i) : i \in I\} = \{g_1, g_2, \ldots, g_k\}$ and $\{\beta(i) : i \in I\} = \{1,2,\ldots,t\}$. Suppose $\alpha(i) = g_{\gamma(i)} (\forall i \in I)$. Then $\{(g_{\gamma(i)}, h_{\beta(i)}) : i \in I\}$ is an edge of $G \times H$ by definition. But $|\{\Delta((g_{\gamma(i)}, h_{\beta(i)})) : i \in I\}| = 1$, i.e. this edge is a monochromatic edge of $G \times H$ under the coloring Δ, contradicting the assumption that Δ is a proper n-coloring of $G \times H$.

LEMMA 1 *Suppose H is a vertex critical $n+1$-chromatic hypergraph, i.e. for any vertex $x \in V(H)$, the hypergraph $H - x$ is n-colorable (where the hypergraph $H - x$ has vertex set $V(H) \setminus \{x\}$ and edge set $\{e \in E(H) : x \notin e\}$). The n-coloring graph of H, $\mathcal{C}_n(H)$, contains exactly one complete sub-hypergraph of order n, namely, the sub-hypergraph induced by the constant mappings.*

Proof. Obviously the sub-hypergraph of $\mathcal{C}_n(H)$ induced by the constant mappings is a complete hypergraph of order n. Suppose that $f_1, f_2, \ldots, f_n \in V(\mathcal{C}_n(H))$ induces a complete sub-hypergraph of order n of $\mathcal{C}_n(H)$. We need to show that $f_1, f_2, \ldots, f_n$ are just the n constant mappings.

Claim 1. $\forall x_1 \in V(H), \forall i \in \{1,2,\ldots,n\}, \exists$ an edge $e \in E(H)$ containing x_1, say $e = \{x_1, x_2, \ldots, x_k\}$, such that $f_i(x_1) = f_i(x_2) = \ldots = f_i(x_k)$.

If this is not true, then $\exists x_1 \in V(H), \exists i \in \{1,2,\ldots,n\}$ such that for every edge e of H containing x_1 we have $|\{f_i(v) : v \in e\}| \geq 2$. By assumption, $H - x_1$ is n-colorable. Let $\phi : V(H - x_1) \mapsto \{1,2,\ldots,n\}$ be a proper n-coloring of $H - x_1$. Define $\Delta : V(H) \mapsto \{1,2,\ldots,n\}$ as follows:

$\Delta(x_1) = f_i(x_1)$,

$\Delta(y) = f_j(y)$ if $y \in V(H - x_1)$ and $\phi(y) = j$.

Now we show that Δ is a proper n-coloring of H, which gives a contradiction and finishes the proof of claim 1.

Suppose $\{y_1, y_2, \ldots, y_t\}$ is an edge of H. We need to show that $|\{\Delta(y_j) : 1 \leq j \leq t\}| \geq 2$.

Case 1. $x_1 \notin \{y_1, y_2, \ldots, y_t\}$. Since ϕ is a proper n-coloring of $H - x_1$, we have $|\{\phi(y_j) : 1 \leq j \leq t\}| \geq 2$. Then $\{f_{\phi(y_j)} : 1 \leq j \leq t\}$ is an edge of $\mathcal{C}_n(H)$, because $f_1, f_2, \ldots, f_n \in V(\mathcal{C}_n(H))$ induces a complete sub-hypergraph of order n of $\mathcal{C}_n(H)$. By definition $|\{f_{\phi(y_j)}(y_j) : 1 \leq j \leq t\}| \geq 2$. Hence $|\{\Delta(y_j) : 1 \leq j \leq t\}| = |\{f_{\phi(y_j)}(y_j) : 1 \leq j \leq t\}| \geq 2$.

Case 2. $x_1 \in \{y_1, y_2, \ldots, y_t\}$. we divide this case into two subcases.

Case 2a. $\phi(y_2) = \phi(y_3) = \ldots = \phi(y_t) = i$. By assumption, $|\{f_i(y_j) : 1 \leq j \leq t\}| \geq 2$. Hence $|\{\Delta(y_j) : 1 \leq j \leq t\}| = |\{f_i(y_j) : 1 \leq j \leq t\}| \geq 2$.

Case 2b. $\exists j \in \{2, 3, \ldots, t\}$ such that $\phi(y_j) \neq i$. In this case $|\{f_{\phi(y_j)} : 2 \leq j \leq t\} \cup \{y_i\}| \geq 2$. Hence $\{f_{\phi(y_j)} : 2 \leq j \leq t\} \cup \{y_i\}$ is an edge of $\mathcal{C}_n(H)$. Therefore $|\{f_{\phi(y_j)}(y_j) : 2 \leq j \leq t\} \cup \{f_i(x_1)\}| \geq 2$. Claim 1 is now proved.

Claim 2. For each $x \in V(H), f_i(x) \neq f_j(x)$ if $i \neq j$.

If false, say $f_1(x_1) = f_2(x_1)$ for some $x_1 \in V(H)$. By claim 1, there is an edge $e \in E(H)$ containing x_1 such that $|\{f_1(v) : v \in e\}| = 1$. Suppose $e = \{x_1, x_2, \ldots, x_k\}$ and $f_1(x_1) = f_1(x_2) = \ldots = f_1(x_k) = i$. Then $f_2(x_1) = f_1(x_2) = \ldots = f_1(x_k) = i$. Therefore $\{f_1, f_2\}$ is not an edge of $\mathcal{C}_n(H)$ by definition. This contradicts the assumption that $f_1, f_2, \ldots, f_n$ induces a complete sub-hypergraph of order n.

Claim 3. For each $i \in \{1, 2, \ldots, n\}, f_i$ is constant on every edge of H.

If false, suppose $e = \{x_1, x_2, \ldots, x_k\}$ is an edge of H and f_i is not constant on e for some $i \in \{1, 2, \ldots, n\}$. Let $j = f_i(x_1)$. Since $\forall x \in V(H), f_1(x), f_2(x), \ldots, f_n(x)$ are pairwise distinct by claim 2, $f_1(x), f_2(x), \ldots,$

$f_n(x)$ use up all the n colors. So for each $t \in \{2, 3, \ldots, k\}$, $\exists \alpha(t) \in \{1, 2, \ldots, n\}$ such that $f_{\alpha(t)}(x_t) = j$. Since $f_i(x_t) \neq j$ for some t (otherwise f_i is constant on e), we have $\alpha(t) \neq i$ for some t. Hence $\{f_i, f_{\alpha(2)}, \ldots, f_{\alpha(k)}\}$ is an edge of $\mathcal{C}_n(H)$. But $|\{f_i(x_1), f_{\alpha(2)}(x_2), \ldots, f_{\alpha(k)}(x_k)\}| = 1$ while $\{x_1, x_2, \ldots, x_k\}$ is an edge of H. This is a contradiction. Claim 3 is proved.

Since H is vertex critical, it must be connected. f_i is constant on every edge of H, hence it is constant on $V(H)$.

The proof of this lemma in fact shows that if H is a connected $n+1$-chromatic hypergraph such that each vertex of H is contained in a vertex critical $n+1$-chromatic sub-hypergraph of H, then $\mathcal{C}_n(H)$ contains exactly one complete sub-hypergraph of order n.

THEOREM 2 *Suppose G and H are $n+1$-chromatic hypergraphs. G contains a complete sub-hypergraph of order n induced by $\{g_1, g_2, \ldots, g_n\}$ and H contains a complete sub-hypergraph of order n induced by $\{h_1, h_2, \ldots, h_n\}$. Furthermore G contains a vertex critical $n+1$-chromatic sub-hypergraph G' such that $V(G') \cap \{h_1, h_2, \ldots, h_n\} \neq \emptyset$ and H contains a vertex critical $n+1$-chromatic sub-hypergraph H' such that $V(H') \cap \{h_1, h_2, \ldots, h_n\} \neq \emptyset$. Then $G \times H$ is $n+1$-chromatic.*

Proof. Suppose the theorem is not true. Let $\Delta : V(G \times H) \mapsto \{1, 2, \ldots, n\}$ be a proper n-coloring of $G \times H$. First we show that $\phi_i : V(H) \mapsto \{1, 2, \ldots, n\}$ defined as $\phi_i(h) = \Delta(g_i, h), i = 1, 2, \ldots, n$, induces a complete sub-hypergraph of $\mathcal{C}_n(H)$ of order n.

This is equivalent to show that for each edge $\{x_1, x_2, \ldots, x_k\}$ of H, if $|\{\alpha(j) : 1 \leq j \leq k\}| \geq 2$, then $|\{\phi_{\alpha(j)}(x_j) : 1 \leq j \leq k\}| \geq 2$.

If $|\{\alpha(j) : 1 \leq j \leq k\}| \geq 2$, then $\{g_{\alpha(j)} : 1 \leq j \leq k\}$ is an edge of G. Therefore $\{(g_{\alpha(j)}, x_j) : 1 \leq j \leq k\}$ is an edge of $G \times H$. Thus $|\{\Delta((g_{\alpha(j)}, x_j)) : 1 \leq j \leq k\}| \geq 2$ and $|\{\phi_{\alpha(j)}(x_j) : 1 \leq j \leq k\}| \geq 2$.

Similarly, $\psi_i : V(G) \mapsto \{1, 2, \ldots, n\}$ defined as $\psi_i(g) = \Delta((g, h_i)), i = 1, 2, \ldots, n$, induces a complete sub-hypergraph of $\mathcal{C}_n(G)$.

Now $\phi_i (i = 1, 2, \ldots, n)$ restricted to H' is a complete sub-hypergraph of $\mathcal{C}_n(H')$ of order n. Hence ϕ_i are constant mappings on $V(H')$ by lemma 2.

Suppose $h_1 \in V(H') \cap \{h_1, h_2, \ldots, h_n\}$. Then $\phi_i(h_1)(i = 1, 2, \ldots, n)$ are pairwise distinct and use up all the n colors.

We claim that $\phi_i(h_j) = \phi_i(h_1)$ for all $1 \leq i \leq n$ and $2 \leq j \leq n$.

If not, say $\phi_i(h_j) \neq \phi_i(h_1)$ for some $1 \leq i \leq n$ and $2 \leq j \leq n$, then $\phi_t(h_1) = \phi_i(h_j)$ for some $t \neq i$. Since $\{\phi_t, \phi_j\}$ is an edge of $\mathcal{C}_n(H')$ and $\{h_1, h_j\}$ is an edge of H', we must have $|\{\phi_t(h_1)), \phi_i(h_j)\}| \geq 2$. This is a contradiction. Therefore $\phi_i(i = 1, 2, \ldots, n)$ are constant on $\{h_1, h_2, \ldots, h_n\}$.

Similarly $\psi_i(i = 1, 2, \ldots, n)$ are constant on $\{g_1, g_2, \ldots, g_n\}$. But this is an obvious contradiction because $\phi_i(h_j) = \Delta(g_i, h_j) = \psi_j(g_i)$ and Δ is a proper n-coloring of $G \times H$.

THEOREM 3 *Suppose G is an $n+1$-chromatic hypergraph such that each vertex of G is contained in a complete sub-hypergraph of order n. Then for any $n+1$-chromatic hypergraph H, $G \times H$ is an $n+1$-chromatic hypergraph.*

Proof. Suppose G and H are hypergraphs satisfy the above conditions. Let H' be any vertex critical $n+1$-chromatic sub-hypergraph of H. It is enough to show that $G \times H'$ is not n-colorable.

Otherwise let $\Delta : V(G \times H') \mapsto \{1, 2, \ldots, n\}$ be a proper n-coloring of $G \times H'$. For each $g \in V(G)$, let $\phi_g : V(H') \mapsto \{1, 2, \ldots, n\}$ be defined as $\phi_g(h) = \Delta(g, h)$. Similar to the proof of theorem 2, we can show that ϕ_g is contained in a complete sub-hypergraph of $\mathcal{C}_n(H')$ of order n. Therefore ϕ_g is constant on $V(H')$ for each $g \in V(G)$.

Now define $\psi : V(G) \mapsto \{1, 2, \ldots, n\}$ as follows:

> $\forall g \in V(G), \psi(g) = \phi_g(h)$ for some $h \in V(H')$ (since ϕ_g is constant on $V(H')$, ψ is well-defined).

It is easy to check that ψ is a proper n-coloring of G, contradicting the assumption that G is $n+1$-chromatic.

Suppose $H = (X, \mathcal{E})$ is a hypergraph. Let $H^* = (X, \mathcal{E}')$ where $\mathcal{E}' = \{e \in \mathcal{E} : e$ is a minimal element of $\mathcal{E}$, i.e. e is not a proper subset of any other member of $\mathcal{E}\}$. Then $\chi(H) = \chi(H^*)$. Let $k = max\{|e| : e \in E(H^*)$ or $e \in E(G^*)\}$, let $k' = max\{|e| : e \in E((G \times H)^*)\}$, then $k \leq k'$. Especially if $k = 2$, then $k' = k = 2$. If we define the relative complete hypergraph of order n to be a hypergraph K with $|V(K)| = n$ and $E(K)$ contains all those subsets of $V(K)$ of cardinality between 2 and $k' (= max\{|e| : e \in E((G \times H)^*)\})$, all the above results are still true and this discussion will include the case of graphs as a special case.

Acknowledgement

I would like to thank professor N.Sauer for many valuable discussions.

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