An approach to Hedetniemi's conjecture

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Abstract

For a fixed integer $n \in \omega$, a graph G of chromatic number greater than n is called persistent if for all $n+1$-chromatic graphs H, the products $G \times H$ are $n+1$-chromatic graphs. Whether all graphs of chromatic number greater than n are persistent is a long standing open problem due to Hedetniemi. We call a graph G strongly persistent if G is persistent and the Hajos sum of G with any other persistent graph H is still persistent. This paper extends the class of known persistent graphs by proving the following result: If G is constructed from copies of K_{n+1} by Hajos sums, adding vertices and edges and at most one contraction of non-adjacent vertices then G is strongly persistent.

1 Introduction

Let G be a simple graph. We denote by $V(G)$ and $E(G)$ the vertex set and edge set of G respectively. Instead of writing $(x, y) \in E(G)$, we may write $xy \in E(G)$ or $x \sim y$. A proper n-coloring of G is a mapping ϕ :

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$V(G) \mapsto \{1, 2, \ldots, n\}$ such that whenever $xy \in E(G)$ we have $\phi(x) \neq \phi(y)$. The chromatic number $\chi(G)$ of G is the least integer n such that there is a proper n-coloring of G.

If G and H are simple graphs, the product $G \times H$ of G and H has vertex set $V(G \times H) = \{(g, h) : g \in V(G), h \in V(H)\}$ and edge set $E(G \times H) = \{(g, h)(g', h') : gg' \in E(G), hh' \in E(H)\}$. Given a proper n-coloring ϕ of G, there is a natural induced proper n-coloring ψ of $G \times H$, namely $\psi((g, h)) = \phi(g)$. Therefore we have $\chi(G \times H) \leq min\{\chi(G), \chi(H)\}$. The "product conjecture" due to Hedetniemi [12] asserts that the equality holds for all graphs G and H. Or equivalently, for all n, $\chi(G) = \chi(H) = n + 1$ implies that $\chi(G \times H) = n + 1$.

This conjecture has enjoyed considerable interest, [5, 6, 7, 9, 11, 14, 15, 16, 18, 19, 21, 22], the principal result being that it holds for $n = 3$, [7]. Some other special cases of the conjecture have also been proved to be true, [5, 6, 18, 19].

Among various approaches to this conjecture, D. Duffus, B. Sands and R. Woodrow [6] tried to prove it by using a theorem of Hajos. Suppose that G and H are simple graphs, $ab \in E(G)$ and $uv \in E(H)$. The Hajos sum of G and H with respect to ab and uv, denoted by $G \oplus_{ab-uv} H$, is the graph obtained from the disjoint union of G and H by contracting a and u into a single vertex, deleting the edges ab and uv and adding the edge bv. Although the edges ab and uv are unordered pairs, the order in which we write the Hajos sum is important. For instance, $G \oplus_{ab-uv} H$ and $G \oplus_{ba-uv} H$ are different graphs. We will write $G \oplus H$ instead of $G \oplus_{ab-uv} H$ in case the edges ab and uv need not be specified.

If G and H are graphs of chromatic number greater than n, then it is easy to see that $G \oplus H$ is also of chromatic number greater than n. Adding vertices and edges to a graph G or contracting two non-adjacent vertices of G will not decrease its chromatic number. Hajos proved that all graphs with chromatic number at least $n + 1$ can be constructed from copies of K_{n+1} by these operations, [10].

Fix an integer n, a graph G of chromatic number greater than n is called *persistent* if for all $n+1$-chromatic graphs H we have $\chi(G \times H) = n+1$. Notice that when we discuss whether a graph G is persistent, we should fix the integer n first. Throughout this paper, n is a fixed integer unless otherwise specified. Hedetniemi's conjecture, for a given fixed n, is equivalent to asserting that all graphs G with $\chi(G) > n$ are persistent. Since every graph G with $\chi(G) > n$ can be constructed from copies of K_{n+1} by adding vertices and edges, contracting non-adjacent vertices and performing Hajos sums, it is sufficient to prove that every such constructed graph is persistent in order to establish Hedetniemi's conjecture.

D. Duffus, B. Sands and R. Woodrow [6] proved that if a graph G is constructed from copies of K_{n+1} by performing the three operations in such a way that all contractions are performed after all the Hajos sums then G is persistent.

We call a graph G *strongly persistent* if G is persistent and the Hajos sum of G with any other persistent graph H is still persistent. In this paper we prove that if G is constructed from copies of K_{n+1} by Hajos sums, adding vertices and edges and at most one contraction then G is strongly persistent. By observing that contractions of non-adjacent vertices preserve persistence, we can see that this extends the class of known persistent graphs.

A homomorphism ϕ of G to H is an edge preserving mapping of $V(G)$ to $V(H)$. We write $G \to H$ if there exists a homomorphism of G to H. Otherwise we write $G \not\to H$. A graph G is n-colorable if and only if $G \to K_n$. We usually write $G \to K_n$ instead of writing G is n-colorable. Also we do not distinguish a proper n-coloring from a homomorphism of G to K_n, and we always assume that the vertex set of the n-complete graph K_n is $\{1, 2, \ldots, n\}$.

Given graphs G and H, the graph H^G has as vertex set $V(H^G) = \{f : f$ is a mapping of $V(G)$ into $V(H)\}$ and edge set $E(H^G) = \{(f,g) : \forall xy \in E(G), f(x)g(y) \in E(H)\}$. Obviously H^G has loops if and only if G is homomorphic to H. The graph K_n^G is called the n-coloring graph of G. When K_n^G has loops, we denote by $\mathcal{L}(K_n^G)$ the graph obtained from K_n^G by deleting the vertices of K_n^G with a loop on it.

2 The main theorem

In order to establish that all graphs G with $\chi(G) > n$ are persistent, it is sufficient to prove that K_{n+1} is persistent and that the three types of operations, adding vertices and edges, taking Hajos sums and contracting non-adjacent vertices, preserve persistence. While it is easy to verify that K_{n+1} is persistent and that the operations of adding vertices and edges and contracting non-adjacent verices preserve persistence, we do not know whether or not the operation of taking Hajos sums preserves persistence.

D. Duffus, B. Sands and R. Woodrow [6] introduced a property of graphs G with $\chi(G) > n$ which is stronger than persistence and which is satisfied by K_{n+1} and preserved by Hajos sums. By investigating this stronger property, they were able to prove that those $n+1$-chromatic graphs constructed from copies of K_{n+1} by first taking Hajos sums and then performing contractions are persistent.

In this paper we introduce some other properties of graphs G with $\chi(G) > n$. The discussion of these properties enables us to prove the following theorem which extends the class of known persistent graphs.

THEOREM 1 *If G is obtained from copies of K_{n+1} by means of adding vertices and edges, taking Hajos sums and at most one contraction of non-adjacent vertices, then G is strongly persistent. Furthermore the Hajos sum of two strongly persistent graphs is a strongly persistent graph.*

The properties of graphs G with $\chi(G) > n$ introduced in this paper are motivated by the following result of El-Zahar and N. Sauer:

THEOREM 2 *[7] A graph G is persistent if and only if $K_n^G \to K_n$.*

This theorem suggests that we should look into the structure of the coloring graph K_n^G in order to prove the persistence of G.

Let G be a graph of chromatic number greater than n and let a be a vertex of G. We denote by $G-a$ the subgraph of G induced by $V(G)\setminus\{a\}$. The vertex a is said to have property (*) if

- $(1)^*$: The graph $\mathcal{L}(K_n^{G-a})$ is n-colorable, i.e., $\mathcal{L}(K_n^{G-a})\to K_n$.

- $(2)^*$: If ff and fg are edges of K_n^{G-a} then $f=g$.

The graph G is said to have property (*) if every vertex of G has property (*).

For an edge ab of G, we denote by $G-ab$ the subgraph of G obtained from G by removing the edge ab. An edge ab of G is said to have property (**) if

- $(1)^{**}$: The graph $\mathcal{L}(K_n^{G-ab})$ is n-colorable, i.e., $\mathcal{L}(K_n^{G-ab})\to K_n$.

- $(2)^{**}$: If ff and fg are edges of K_n^{G-ab} then $f=g$.

The graph G is said to have property (**) if every edge of G has property (**).

Now we proceed to prove Theorem 1 by first establishing a sequence of lemmas.

LEMMA 3 *If G is a graph of chromatic number greater than n and a is a vertex of G with property (*), then for any other vertex b of G with $ab\in E(G)$, the edge ab has property (**).*

Proof. First we prove that $\mathcal{L}(K_n^{G-ab})\to K_n$. Observe that if (f,g) is an edge of K_n^{G-ab} then $(f|G-a, g|G-a)$ is an edge of K_n^{G-a}, where $f|G-a$ is the restriction of f to $G-a$.

Let S be the subgraph of $\mathcal{L}(K_n^{G-ab})$ induced by the set $\{f \in \mathcal{L}(K_n^{G-ab}) : f|G-a \in \mathcal{L}(K_n^{G-a})\}$. Then the mapping which takes f to $f|G-a$ is a homomorphism of S to $\mathcal{L}(K_n^{G-a})$. Therefore $S \to K_n$ since $\mathcal{L}(K_n^{G-a}) \to K_n$.

We claim that $\mathcal{L}(K_n^{G-ab}) \setminus S$ is a set of isolated vertices, which would then imply that $\mathcal{L}(K_n^{G-ab}) \to K_n$.

First we show that there is no edge between elments of $\mathcal{L}(K_n^{G-ab}) \setminus S$. Otherwise suppose that $f, g \in \mathcal{L}(K_n^{G-ab}) \setminus S$ and that $f \sim g$. Then $f|G-a \sim g|G-a$ and $f|G-a \sim f|G-a$ in K_n^{G-a}. Therefore $f|G-a = g|G-a$ since $G-a$ satisfies $(2)^*$. Because f is not a homomorphism of $G-ab$ to K_n, there is a neighbor c of a in $G-ab$ such that $f(a) = f(c)$. Thus $f(a) = f(c) = g(c)$, in contradiction to $f \sim g$.

A similar argument shows that there is no edge between S and $\mathcal{L}(K_n^{G-ab}) \setminus \mathcal{S}$. Therefore the claim is proved and $G-ab$ satisfies $(1)^{**}$.

To prove that $G-ab$ satisfies $(2)^{**}$, we let f, g be vertices of K_n^{G-ab} such that $f \sim f$ and $f \sim g$. Then $f|G-a \sim f|G-a$ and $f|G-a \sim g|G-a$ in K_n^{G-a}. Therefore $f|G-a = g|G-a$ because $G-a$ satisfies $(2)^*$. Since $\chi(G) > n$ and $f \sim f$ in K_n^{G-ab}, we have $f(a) = f(b)$ for otherwise f would be a proper n-coloring of G. Also $f \sim g$ in K_n^{G-ab} implies that $f(a) = g(a)$ for otherwise we would have a proper n-coloring ϕ of G defined as $\phi(a) = g(a)$ and $\phi(t) = f(t)$ for $t \neq a$. Therefore $f = g$ and hence $G-ab$ satisfies $(2)^{**}$.

LEMMA 4 *The complete graph K_{n+1} has property (*).*

Proof. For any vertex a of K_{n+1}, $K_{n+1} - a$ is a copy of K_n. We first prove that $\mathcal{L}(K_n^{K_n}) \to K_n$ by explicitly giving a homomorphism ϕ of $\mathcal{L}(K_n^{K_n})$ to K_n.

For each $f \in V(\mathcal{L}(K_n^{K_n}))$, since f is not a homomorphism, there exist $x, y \in K_n$ such that $f(x) = f(y)$. We set $\phi(f) = f(x) = f(y)$. If $\phi(f) = \phi(g) = f(x) = f(y) = g(x') = g(y')$, then $x \sim x'$ or $x \sim y'$. Hence $(f, g) \notin E(\mathcal{L}(K_n^{K_n}))$. Therefore ϕ is a homomorphism.

Now we prove that $K_{n+1} - a$ satisfies $(2)^*$. Suppose that $f \sim f$ and $f \sim g$

in $K_n^{K_n}$. Since f is a homomorphism of K_n to K_n, it must be a one-to-one mapping. Hence if $f(a) = i$ and $j \neq i$, then there is a neighbor b of a such that $f(b) = j$. Therefore $g(a) \neq j$ for any $j \neq i$. Thus $g(a) = i = f(a)$ and $f = g$.

LEMMA 5 *If both G and H have property (*), then the Hajos sum $G \oplus H$ also has property (*).*

Proof. We need to show that every vertex a of $G \oplus H$ has property (*). Suppose $G \oplus H$ is the graph as shown in Fig. 1.

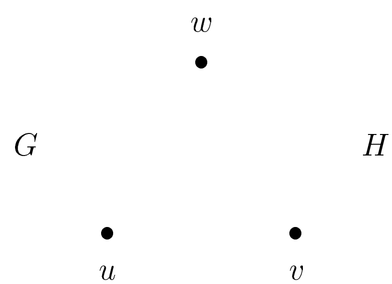

Fig. 1. $G \oplus H$

Case 1. $a \neq w$. Without loss of generality, we assume that $a \in V(H)$. Now $\mathcal{L}(K_n^{G \oplus H - a})$ is the union of the following subsets:

- $S_1 = \{f \in \mathcal{L}(K_n^{G \oplus H - a}) : f|G - w \in \mathcal{L}(K_n^{G-w})\}$;

- $S_2 = \{f \in \mathcal{L}(K_n^{G \oplus H - a}) : f \notin S_1$ and $f|G - u \in \mathcal{L}(K_n^{G-u})\}$;

- $S_3 = \{f \in \mathcal{L}(K_n^{G \oplus H - a}) : f \notin S_1 \cup S_2$ and $f|H - a \in \mathcal{L}(K_n^{H-a})\}$;

- $S_4 = \mathcal{L}(K_n^{G \oplus H - a}) \setminus S_1 \cup S_2 \cup S_3$.

Claim 1: Let S_1, S_2, S_3 and S_4 be the subsets of $\mathcal{L}(K_n^{G\oplus H-a})$ defined as above. Then we have

- (1): There is no edge between S_i and S_j if $i \neq j$.

- (2): The subgraph S_2 of $\mathcal{L}(K_n^{G\oplus H-a})$ contains no edge.

- (3): There is a homomorphism of S_3 to $\mathcal{L}(K_n^{H-a})$.

- (4): The subgraph S_4 of $\mathcal{L}(K_n^{G\oplus H-a})$ contains no edge.

Proof of Claim 1: (1). Suppose that $f \in S_1$, $g \notin S_1$ and $f \sim g$. Then $g|G-w \sim g|G-w$ and $g|G-w \sim f|G-w$ in K_n^{G-w}. Since $G-w$ satisfies $(2)^*$, we have $f|G-w = g|G-w$. This contradicts the assumption that $f \in S_1$ and $g \notin S_1$.

Similarly we can prove that there is no edge between S_2 and $S_3 \cup S_4$ and that there is no edge between S_3 and S_4.

(2). Suppose to the contrary that there are two elements f, g of S_2 such that $f \sim g$. Then $f|G-w \sim f|G-w$ and $f|G-w \sim g|G-w$. Therefore $g|G-w = f|G-w$ because $G-w$ satisfies $(2)^*$.

Since $f|G-w$ is a homomorphism and $f|G-u$ is not, there is a neighbor b of w in $G-u$ such that $f(w) = f(b)$. Thus $g(b) = f(b) = f(w)$, in contradiction to $f \sim g$. Therefore there is no edge between elements of S_2.

(3). For each $f \in S_3$, we must have $f(u) = f(w)$. Otherwise f would be a proper n-coloring of G which has chromatic $\chi(G)$ greater than n. Therefore if $f, g \in S_3$ and $f \sim g$ then $f(v) \neq g(u) = g(w)$ and $f(w) = f(u) \neq g(v)$. This implies that $f|H-a \sim g|H-a$. Thus the mapping which sends f to $f|H-a$ is a homomorphism of the subgraph S_3 of $\mathcal{L}(K_n^{G\oplus H-a})$ to $\mathcal{L}(K_n^{H-a})$.

(4). If $f, g \in S_4$ and $f \sim g$, then it is easy to see that we should have $f(u) = f(v) = g(u) = g(v)$. This is a contradiction because $uv \in E(G \oplus H)$.

Therefore there is no edge between elements of S_4. This completes the proof of claim 1.

Summing up the discussion above, we have $S_1 \to \mathcal{L}(K_n^{G-w}) \to K_n$, $S_2 \to K_n$, $S_3 \to \mathcal{L}(K_n^{H-a}) \to K_n$ and $S_4 \to K_n$. Furthermore S_1, S_2, S_3 and S_4 are isolated from each other. Therefore $\mathcal{L}(K_n^{G\oplus H-a}) \to K_n$ and $G \oplus H - a$ satisfies $(1)^*$.

Now we prove that $G \oplus H - a$ also satisfies $(2)^*$. Suppose that $f \sim g$ and $f \sim f$ in $\mathcal{L}(K_n^{G\oplus H-a})$. Then $f|G-w \sim g|G-w$ and $f|G-w \sim f|G-w$ imply that $f|G-w = g|G-w$. Also $f|G-u \sim g|G-u$ and $f|G-u \sim f|G-u$ imply that $f|G-u = g|G-u$, and $f(w) = g(w) = f(u) = g(u)$ implies that $f(w) \neq g(v), f(w) \neq f(v)$ and that $f(v) \neq g(w)$. Thus $f|H-a \sim f|H-a$ and $g|H-a \sim f|H-a$, which imply that $f|H-a = g|H-a$. Therefore $f = g$ and hence $G \oplus H - a$ satisfies $(2)^*$ and a has property (*).

Case 2. $a = w$. Then $\mathcal{L}(K_n^{G\oplus H-w})$ is the union of the following subsets:

- $S_1 = \{f \in \mathcal{L}(K_n^{G\oplus H-w}) : f|G-w \in \mathcal{L}(K_n^{G-w})\}$;

- $S_2 = \{f \in \mathcal{L}(K_n^{G\oplus H-w}) : f \notin S_1$ and $f|H-w \in \mathcal{L}(K_n^{H-w})\}$;

- $S_3 = \mathcal{L}(K_n^{G\oplus H-w}) \setminus S_1 \cup S_2$.

As in the proof of Claim 1, we can show that there is no edge between S_i and S_j if $i \neq j$, and $S_i \to K_n$ for $i = 1, 2, 3$. Therefore $\mathcal{L}(K_n^{G\oplus H-a}) \to K_n$ and $G \oplus H - a$ satisfies $(1)^*$.

To prove that $G \oplus H - a$ satisfies $(2)^*$, let $f \sim g$ and $f \sim f$ in $\mathcal{L}(K_n^{G\oplus H-a})$. Then $f|G-w \sim g|G-w$ and $f|G-w \sim f|G-w$ imply that $f|G-w = g|G-w$, and $f|H-w \sim f|H-w$ and $g|H-w \sim f|H-w$ imply that $f|H-w = g|H-w$. Therefore $f = g$ and hence $G \oplus H - a$ satisfies $(2)^*$ and a has property (*).

COROLLARY 6 *If G is constructed from copies of K_{n+1} by Hajos sums then G has property (*).*

LEMMA 7 *If G has property (*) and G' is obtained from G by contracting a set T of independent vertices of G into a single vertex v_T, then G' has property (**).*

Proof. First we show that if $a \in V(G')$ and $a \neq v_T$, then a has property (*). Define a mapping $\phi : \mathcal{L}(K_n^{G'-a}) \mapsto \mathcal{L}(K_n^{G-a})$ as follows:

For each $f \in \mathcal{L}(K_n^{G'-a}), \phi(f) \in \mathcal{L}(K_n^{G-a})$ is the mapping defined by $\phi(f)(y) = f(y)$ if $y \notin T$ and $\phi(f)(y) = f(v_T)$ if $y \in T$.

It is easy to see that ϕ is a homomorphism. Therefore $\mathcal{L}(K_n^{G'-a}) \to \mathcal{L}(K_n^{G-a}) \to K_n$. So $G' - a$ satisfies $(1)^*$.

If $f \sim f$ and $f \sim g$ in $\mathcal{L}(K_n^{G'-a})$, then $\phi(f) \sim \phi(f)$ and $\phi(f) \sim \phi(g)$ in $\mathcal{L}(K_n^{G-a})$. Therefore $\phi(f) = \phi(g)$ and hence $f = g$. Thus $G' - a$ satisfies $(2)^*$ and a has property (*).

Now every edge of G' has at least one vertex which is not v_T. By lemma 3, every edge of G' has property (**). Therefore G' has property (**).

LEMMA 8 *If both G and H have property (**), then the Hajos sum $G \oplus H$ also has property (**).*

Proof. Suppose $G \oplus H$ is the graph as shown in Fig. 1. We need to show that every edge ab of $G \oplus H$ has property (**).

Case 1. $ab \neq uv$. Without loss of generality, we assume that $ab \in E(H)$. Then $\mathcal{L}(K_n^{G \oplus H - ab})$ is the union of the following subsets:

- $S_1 = \{f \in \mathcal{L}(K_n^{G \oplus H - ab}) : f|G \in \mathcal{L}(K_n^{G-wu})\}$;

- $S_2 = \{f \in \mathcal{L}(K_n^{G \oplus H - ab}) : f \notin S_1 \text{ and } f|H \in \mathcal{L}(K_n^{H-ab})\}$;

- $S_3 = \mathcal{L}(K_n^{G \oplus H - ab}) \setminus S_1 \cup S_2$.

As in the proof of Claim 1, we can show that there is no edge between S_i and S_j if $i \neq j$, and $S_i \to K_n$ for $i = 1, 2, 3$. Therefore $\mathcal{L}(K_n^{G \oplus H - ab}) \to K_n$ and $G \oplus H - ab$ satisfies $(1)^{**}$.

To prove that $G \oplus H - ab$ also satisfies $(2)^{*}$, let $f \sim g$ and $f \sim f$ in $\mathcal{L}(K_n^{G \oplus H - ab})$. Then $f|G \sim g|G$ and $f|G \sim f|G$ in K_n^{G-wu} imply that $f|G = g|G$. Since f is a homomorphism of $G - wu$ to K_n and $\chi(G) > n$, we have $f(w) = f(u) = g(w) = g(u)$. This implies that $f|H \sim f|H$ and $f|H \sim g|H$ in K_n^{H-ab}. Therefore $f|H = g|H$ and hence $f = g$.

Case 2. $ab = uv$. Then $\mathcal{L}(K_n^{G \oplus H - uv})$ is the union of the following subsets:

- $S_1 = \{f \in \mathcal{L}(K_n^{G \oplus H - uv}) : f|G \in \mathcal{L}(K_n^{G-wu})\}$;
- $S_2 = \{f \in \mathcal{L}(K_n^{G \oplus H - uv}) : f \notin S_1 \text{ and } f|H \in \mathcal{L}(K_n^{H-wv})\}$;
- $S_3 = \mathcal{L}(K_n^{G \oplus H - ab}) \setminus S_1 \cup S_2$.

As in the proof of Claim 1, we can show that there is no edge between S_i and S_j if $i \neq j$ and that $S_i \to K_n$ for $i = 1, 2, 3$. Therefore $\mathcal{L}(K_n^{G \oplus H - uv}) \to K_n$ and $G \oplus H - uv$ satisfies $(1)^{**}$.

If $f \sim g$ and $f \sim f$ in $\mathcal{L}(K_n^{G \oplus H - uv})$, then $f|G \sim g|G$ and $f|G \sim f|G$ in K_n^{G-wu} imply that $f|G = g|G$. Also $f|H \sim f|H$ and $f|H \sim g|H$ in K_n^{H-wv} imply that $f|H = g|H$. Therefore $f = g$ and $G \oplus H - uv$ satisfies $(2)^{**}$.

COROLLARY 9 *If G is constructed from copies of K_{n+1} by Hajos sums and at most one contraction of an independent set, then G has property (**).*

LEMMA 10 *If G has property (**), then G is persistent and for any persistent graph H, the Hajos sum $G \oplus H$ is also persistent. Or equivalently, if G has property (**), then G is strongly persistent.*

Proof. Since G has property (**), $\mathcal{L}(K_n^{G-ab}) \to K_n$ for any given edge ab of G. Furthermore any element $f \in V(K_n^{G-ab}) \setminus V(\mathcal{L}(K_n^{G-ab}))$ is an isolated loop. The graph K_n^G has the same vertex set as K_n^{G-ab} and less edges. So K_n^G can be viewed as a subgraph of K_n^{G-ab}. When we consider $f \in V(K_n^{G-ab}) \setminus V(\mathcal{L}(K_n^{G-ab})$ as a vertex of K_n^G, the loop disappears because $\chi(G) > n$. Therefore $K_n^G \to K_n$ and hence G is persistent by Theorem 2.

To prove that $G \oplus H$ is persistent, let $G \oplus H$ be the graph as shown in Fig. 1. Again we only need to prove that $K_n^{G\oplus H} \to K_n$ by Theorem 2. $K_n^{G\oplus H}$ is the union of $S_1 = \{f \in K_n^{G\oplus H} : f|G \in \mathcal{L}(K_n^{G-wu})\}$ and $S_2 = K_n^{G\oplus H} \setminus S_1$.

Obviously $S_1 \to \mathcal{L}(K_n^{G-wu}) \to K_n$ because $G-wu$ satisfies $(1)^{**}$. Suppose that $f \in S_2$. Then $f|G$ is a homomoephism of $G-wu$ to K_n, i.e., $f|G \sim f|G$ in K_n^{G-wu}. Therefore f is not adjacent to any elements of S_1 because $G-wu$ satisfies $(2)^{**}$. Also we have $f(w) = f(u)$ because otherwise f would be a proper n-coloring of G. Therefore $f \sim g$ and $f, g \in S_2$ imply that $f|H \sim g|H$ in K_n^H. Hence $S_2 \to K_n^H \to K_n$ because H is persistent. So $K_n^{G\oplus H} \to K_n$ and $G \oplus H$ is persistent.

LEMMA 11 *If a graph G is strongly persistent and G' is a graph obtained from G by adding vertices and edges, then G' is also strongly persistent.*

The proof of this lemma is straight forward and omitted.

Suppose that a graph G' is obtained from G by first adding vertices and edges and then taking a Hajos sum or contraction. It is easy to see that either G' can be obtained from G by first taking a Hajos sum or contraction and then adding vertices and edges, or G' can be obtained from G by simply adding vertices and edges.

Now we can finish the proof of Theorem 1 as follows:

Suppose G is obtained from copies of K_{n+1} by Hajos sums, adding vertices and edges and at most one contraction. Then by the above argument, G can be obtained from copies of K_{n+1} by first taking Hajos sums and at most one contraction, then adding vertices and edges. By corollary 9 and lemmas 10 and 11, such a graph is strongly persistent.

Suppose both G and H are strongly persistent. We shall prove that $G \oplus H$ is also strongly persistent. Let Q be a persistent graph, we need to show that $(G \oplus H) \oplus Q$ is persistent. This is true because $(G \oplus H) \oplus Q$ either equals $G \oplus (H \oplus Q)$ or equals $H \oplus (G \oplus Q)$ by choosing the corresponding edges appropriately. To be precise, let $(G \oplus H) \oplus Q = (G \oplus_{ab-cd} H) \oplus_{xy-uv} Q$. If $\{x, y\} \neq \{b, d\}$, then $(G \oplus_{ab-cd} H) \oplus_{xy-uv} Q = G \oplus_{ab-cd} (H \oplus_{xy-uv} Q)$. If $\{x, y\} = \{b, d\}$, we assume that $x = b$ and $y = d$. Then $(G \oplus_{ab-cd} H) \oplus_{xy-uv} Q = (G \oplus_{ba-uv} Q) \oplus_{av-cd} H$.

It was observed in [6] that contractions preserve persistence. By Theorem 1 and this observation, we have

THEOREM 12 *If G is a graph constructed from copies of K_{n+1} by the three types of operations in such a way that whenever a Hajos sum is performed, one of the factor graphs can be constructed from copies of K_{n+1} by Hajos sums, adding vertices and edges and at most one contraction, then G is persistent.*

3 Unique colorability

A graph G is uniquely n-colorable if there is a unique partition of $V(G)$ into n independent sets.

THEOREM 13 *[8] If G is a connected graph with $\chi(G) > n$, then $K_n \times G$ is uniquely n-colorable.*

D. Duffus, B. Sands and R.Woodrow [6] noticed a relation between uniquely colorable graphs and Hedetniemi's conjecture. As generalizations of Theorem 13, they proposed the following two conjectures concerning unique colorabilty:

- (A_n) : For all uniquely colorable graphs G and H, each proper n-coloring of $G \times H$ is induced by G or by H, i.e., if $\phi : V(G \times H) \mapsto$

$\{1, 2, \ldots, n\}$ is a proper n-coloring of $G \times H$, then either $\phi((g, h)) = \phi((g, h'))$ for all $g \in V(G)$ or $\phi((g, h)) = \phi((g', h))$ for all $h \in V(H)$.

- (B_n) : For all uniquely n-colorable graphs G and all connected graphs H with $\chi(H) > n$, $G \times H$ is uniquely n-colorable.

Let (C_{n+1}) be the statement that $\chi(G) = \chi(H) = n + 1$ implies that $\chi(G \times H) = n + 1$. It was proved in [6] that $(A_n) \Rightarrow (B_n) \Rightarrow (C_{n+1})$. By proving a special case of (A_n), they established the following special case of Hedetniemi's conjecture:

THEOREM 14 *[6] If G and H are connected $n+1$-chromatic graphs such that $K_n \rightarrow G$ and $K_n \rightarrow H$, then $\chi(G \times H) = n + 1$.*

We propose a conjecture concerning the coloring graph of uniquely colorable graphs which is between (B_n) and (C_{n+1}).

- (D_n) : For all uniquely n-colorable graphs G, $\mathcal{L}(K_n^G)$ is n-colorable.

THEOREM 15 *The conjecture (D_n) is between conjecture (B_n) and conjecture (C_{n+1}), i.e., $(B_n) \Rightarrow (D_n) \Rightarrow (C_{n+1})$.*

Proof. Suppose (B_n) is true and (D_n) is false. Let G be a uniquely n-colorable graph with $\mathcal{L}(K_n^G)$ not n-colorable and let H be a connected component of $\mathcal{L}(K_n^G)$ such that $\chi(H) > n$. By (B_n), $G \times H$ is uniquely n-colorable. However the n-coloring $\phi : V(G \times H) \mapsto V(K_n)$ defined by

$$\phi((x, f)) = f(x)$$

is a proper n-coloring of $G \times H$ which is different from the proper n-coloring induced by the proper n-coloring of G. For otherwise $f(x) = g(x)$ for all $f, g \in V(H)$ and all $x \in V(G)$, i.e., H contains a single vertex. This contradicts the assumption that $\chi(H) > n$. Thus $(B_n) \Rightarrow (C_{n+1})$.

Suppose that (D_n) is true. For any connected graph H with $\chi(H) > n$, the graph $K_n \times H$ is uniquely n-colorable by Theorem 13. Therefore $\mathcal{L}(K_n^{K_n \times H}) \to K_n$.

Since $\mathcal{L}(K_n^{K_n}) \to K_n$ and $\chi(H) > n$, we know that $(\mathcal{L}(K_n^{K_n})^H$ has no loops, for otherwise we would have $H \to \mathcal{L}(K_n^{K_n}) \to K_n$. Therefore we have $(\mathcal{L}(K_n^{K_n}))^H \to \mathcal{L}(K_n^{K_n \times H})$. (The mapping $\psi : V((\mathcal{L}(K_n^{K_n}))^H) \mapsto V(\mathcal{L}(K_n^{K_n \times H}))$ defined by $\psi(f)(i,h) = f(h)(i)$ is the required homomorphism).

The constant maps in $\mathcal{L}(K_n^{K_n})$ induce a complete graph of order n. Therefore $K_n \to \mathcal{L}(K_n^{K_n})$ and hence $K_n^H \to (\mathcal{L}(K_n^{K_n}))^H \to \mathcal{L}(K_n^{K_n \times H}) \to K_n$. By Theorem 2, H is persistent. So (D_n) implies (C_{n+1}).

THEOREM 16 *Conjecture (A_n) holds if and only if for all uniquely n-colorable graphs G, we have $\mathcal{L}(K_n^G) \to K_n$ and $\mathcal{L}(K_n^G)$ contains no uniquely n-colorable subgraphs other than the complete graph of order n induced by the constant maps.*

Proof. "only if": Since $(A_n) \Rightarrow (B_n) \Rightarrow (D_n)$, we only have to show that when (A_n) holds, the graph $\mathcal{L}(K_n^G)$ contains no uniquely n-colorable subgraphs other than the complete subgraph of order n induced by the constant maps.

Suppose Y is a uniquely n-colorable subgraph of $\mathcal{L}(K_n^G)$. Then by (A_n), the proper n-coloring $\psi : V(G \times Y) \mapsto V(K_n)$ defined by $\psi(g,f) = f(g)$ is either induced by the unique n-coloring of G or induced by the unique n-coloring of Y. If ψ is induced by the coloring of G, then for all $f, f' \in V(Y)$, we have $f(g) = f'(g)$ for all $g \in V(G)$, i.e., Y contains only one element, contradicting the assumption that Y is uniquely n-colorable if $n \geq 2$. (When $n = 1$, K_n^G contains only a single constant mapping). If ψ is induced by the coloring of Y, then for all $g, g' \in V(G)$, we have $f(g) = f(g')$ for all $f \in V(Y)$, i.e., Y contains only constant maps.

"if": Let G and H be uniquely n-colorable graphs and let ψ be a homomorphism of $G \times H$ to K_n. Then the mapping $\phi : V(H) \mapsto K_n^G$ defined by $\phi(h)(g) = \psi(g, h)$ is a homomorphism of H to K_n^G. Now either we have $\phi(H) \subset \mathcal{L}(K_n^G)$ or $\phi(y)$ is a homomorphism of G to K_n for some $y \in V(H)$.

If $\phi(H) \subset \mathcal{L}(K_n^G)$, then $\phi(H)$ is a uniquely n-colorable subgraph of $\mathcal{L}(K_n^G)$. Hence $\phi(H)$ is the complete graph of order n induced by the constant maps. So for all $h \in V(H)$, we have $\phi(h)(g) = \phi(h)(g')$ for all $g, g' \in V(G)$. Therefore $\psi(g, h) = \psi(g'h)$ for all $g, g' \in V(G)$ and $h \in V(H)$, which implies that ψ is induced by a proper n-coloring of H.

If $\phi(y)$ is a homomorphism of G to K_n for some $y \in V(H)$, then $\phi(y)$ is an isolated loop in K_n^G because G is uniquely n-colorable. Therefore $\phi(H) = \{\phi(y)\}$ and hence ψ is induced by a proper n-coloring of G.

4 Some comments

In order to prove Theorem 1, we first proved that K_{n+1} has property (*), and then showed that this property is preserved by Hajos sums. However if we perform a contraction, this property may be lost. For example, the Hajos sum of two triangles is a pentagon, which has property (*). If we contract two non-adjacent vertices of this pentagon, the resulting graph has no longer property (*). However if we perform one contraction , although property (*) may be lost, the resulting graphs still have property (**), which is weaker than property (*) but strong enough for our purpose. Property (**) is also preserved by Hajos sums. Again it is not preserved by contractions.

Since property (**) is too strong, it is natural to look for some weaker ones. It is not difficult to prove (similar to the proof of lemma 8) that the following property is preserved by Hajos sums and it implies Hedetniemi's conjecture.

We say a graph G with $\chi(G) > n$ has property (**)' if for every edge ab of G, we have:

- $(1)'$: The graph $\mathcal{L}(K_n^{G-ab})$ is n-colorable.

- $(2)'$: If f is a loop in K_n^{G-ab} and g is connected to f, then $f(a) = g(a) = f(b) = g(b)$.

Although this property is weaker than property (**), it is still stronger than Hedetniemi's conjecture. The following is an argument to show that $(1)'$ implies (D_n), while (D_n) has already been proved to imply Hedetniemi's conjecture.

Let H be a uniquely n-colorable graph. If H is the complete graph K_n, then (D_n) holds trivialy for H. If H is not complete, then add an edge ab to two non-adjacent vertices a, b of H which are in the same color class in the unique n-color partition. Then the resulting graph H' is of chromatic number $n+1$. Hence every edge of H' has property (**)'. In particular we have $\mathcal{L}(K_n^{H'-ab}) = \mathcal{L}(K_n^H) \to K_n$. So (D_n) holds.

Property (**)' is also stronger than the following property (C^*_{n+1}) introduced by D. Duffus, B. Sands and R. Woodrow in [6]:

- (C^*_{n+1}): An $n+1$-chromatic graph G ia said to have property (C^*_{n+1}) if for all edges ab of G, for all connected $n+1$-chromatic graphs H, for all proper n-colorings ϕ of $(G - ab) \times H$, and for all $h \in V(H)$, $\phi(a, h) = \phi(b, h)$.

THEOREM 17 *If G is an $n+1$-chromatic graph and has property (**)', then G satisfies condition (C^*_{n+1}).*

Proof. Suppose that G has property (**)', ab is an edge of G, H is a connected $n+1$-chromatic graph and that ϕ is a proper n-coloring of $(G - ab) \times H$. Define $\psi : V(H) \mapsto V(K_n^{G-ab})$ as $\psi(h) = \phi_h$, where $\phi_h(x) = \phi(x, h)$. It is easy to verify that ψ is a homomorphism of H to K_n^{G-ab}. Since $\mathcal{L}(K_n^{G-ab}) \to K_n$, we know that $\psi(H)$ is not contained in $\mathcal{L}(K_n^{G-ab})$, for otherwise we would have $H \to K_n$. Therefore $\psi(h_0)$ is a loop in K_n^{G-ab} for some $h_0 \in V(H)$. Since for all $h \in V(H), \psi(h)$ is connected to $\psi(h_0)$, we have $\psi(h)(a) = \psi(h_0)(a) = \psi(h_0)(b) = \psi(h)(b)$, i.e., $\phi(h, a) = \phi(h, b)$ for all $h \in V(H)$.

El-Zahar constructed a 6-chromatic graph which is persistent but fails to satisfy (C_6^*), and one can add vertices and edges to this graph to obtain n-chromatic graphs violating (C_n^*) for all $n \geq 6$, [6]. In light of such an example, we propose another property which is still weaker:

- (C_{n+1}^{**}) : An $n+1$-chromatic graph G is said to have property (C_{n+1}^{**}) if for all edges ab of G, for all vertex critical $n+1$-chromatic graphs H, for all proper n-colorings ϕ of $(G - ab) \times H$, and for all $h \in V(H)$, $\phi(a, h) = \phi(b, h)$.

This property can be proved to be preserved by Hajos sums, and it implies Hedetniemi's conjecture. Also the example mentioned above has this property.

In the process of forming Hajos sums and contractions, the good properties of K_{n+1} are lost gradually. In order to establish Hedetniemi's conjecture, what we need to prove is that K_{n+1} is good enough so that no matter how many (finite) operations are performed, the resulting graph is still persistent.

Let $\mathcal{N}_1 = \{G : \chi(G) > n \text{ and } G \text{ is persistent }\}$, and for $k \geq 1$, let $\mathcal{N}_{2k} = \{G \in \mathcal{N}_{2k-1} : \forall G' \in \mathcal{N}_{2k-1}, G \oplus G' \in \mathcal{N}_{2k-1}\}, \mathcal{N}_{2k+1} = \{G \in \mathcal{N}_{2k} :$ any contraction of G is still in $\mathcal{N}_{2k}$ $\}$. Hedetniemi's conjecture is equivalent to asserting that $K_{n+1} \in \mathcal{N}_i$ for all integer i. By noting that the Hajos sum is associative in a certain sense, the result of D. Duffus, B. Sands and R. Woodrow in [6] is that $K_{n+1} \in \mathcal{N}_2$. The main result of this paper is still weaker than the assertion that $K_{n+1} \in \mathcal{N}_4$.

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