A Characterization of (X, kX)-Intersection Graphs

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Abstract

For two fixed graphs X and Y, the (X, Y)-*intersection graph* of a graph G is a graph whose vertices are induced subgraphs of G isomorphic to Y and where two vertices are adjacent if their intersection in G contains an induced subgraph isomorphic to X. A *conformal* k-*graph* is a simple hypergraph whose hyperedges are exactly k-cliques in the 2-section of the hypergraph.

Let kX denote the disjoint union of k copies of X. We show that for any integer $k \geq 2$ and connected graph X with no bipartite blocks, the family of (X, kX)-intersection graphs coincides with the family of line graphs of conformal k-graphs. On the other hand, we obtain a Ramsey type result on vertex splitting and use it to prove that for any connected bipartite graph X with at least two vertices, the family of (X, kX)-intersection graphs is strictly contained in the family of line graphs of conformal k-graphs.

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1 Introduction

To represent the intersection of various induced subgraphs in a graph, Cai, Corneil and Proskurowski [2] introduced the following notion of (X,Y)-intersection graphs. Let (X,Y) be a pair of fixed graphs. For a graph G, any induced subgraph in G isomorphic to a given graph G' is an *induced G'-subgraph*. The *(X,Y)-intersection graph* of a graph G, denoted $I_{X,Y}(G)$, is a graph where

1. each vertex corresponds to a distinct induced Y-subgraph in G, and
2. two vertices are adjacent iff the intersection of their corresponding induced Y-subgraphs contains an induced X-subgraph.

Various classes of graphs in the literature are special (X,Y)-intersection graphs. The *line graph* $L(G)$ of G is precisely the (K_1,K_2)-intersection graph of G, the *edge intersection graph of triangles* $H_1(G)$ [11] of G is exactly the (K_2,K_3)-intersection graph of G, and the *k-line graph* [8] (also known as *K_k-intersection graph* [5] and *kth interchange graph* [6]) of G is the the same as the (K_{k-1},K_k)-intersection graph of G. Furthermore, a *strict G'-factor* [7] of G corresponds to an independent set in the (K_1,G')-intersection graph of G.

As with line graphs, a natural question concerning (X,Y)-intersection graphs is to characterize all graphs that are (X,Y)-intersection graphs. It was shown in [2] that (X,Y)-intersection graphs are line graphs of k-uniform hypergraphs if Y contains exactly k induced X-subgraphs, and line graphs of simple k-uniform hypergraphs if, in addition, these k induced X-subgraphs contain all vertices of Y. This raises the issue of determining (X,Y) for which the family of (X,Y)-intersection graphs coincides with the family of line graphs of (simple) k-uniform hypergraphs.

The case $k=2$ was studied in [1, 2]. The issue there was to characterize (X,Y) for which the family of (X,Y)-intersection graphs equals the family of line graphs or the family of line graphs of multigraphs. It was shown in [2] that for such (X,Y) the two induced X-subgraphs inside Y must be connected in a highly symmetric manner. Furthermore, if these two induced X-subgraphs contain all vertices of Y, the family of (X,Y)-intersection graphs admits a forbidden induced subgraph characterization only when it coincides with the family of line graphs. For $k\geq 3$, there are (X,Y) pairs for which the family of (X,Y)-intersection graphs coincides with the family of line graphs of k-uniform hypergraphs. However, it has been shown recently in [3] that no (X,Y) pair makes the family of (X,Y)-

intersection graphs equal to the family of line graphs of simple k-uniform hypergraphs, which is quite different from the case $k = 2$.

In this paper, we study (X, kX)-intersection graphs, where kX denotes the disjoint union of k copies of X. As observed in [2], for any (X, Y), the family of (X, Y)-intersection graphs is exactly the same as the family of $(\overline{X}, \overline{Y})$-intersection graphs, where $\overline{X}$ and $\overline{Y}$, respectively, are the complement graphs of X and Y. Therefore the family of line graphs is also the same as the family of $(K_1, 2K_1)$-intersection graphs, and the family of intersection graphs of k-cliques equals the family of (K_1, kK_1)-intersection graphs. Note that kX contains k X-subgraphs iff X is connected, and the family of (X, kX)-intersection graphs is contained in the family of line graphs of simple k-uniform hypergraphs when X is connected.

The case $k = 2$ for (X, kX)-intersection graphs has been studied by Cai [1]. He showed that the family of $(X, 2X)$-intersection graphs equals the family of line graphs whenever X is a connected graph with no bipartite blocks, and that for any complete bipartite graph X with at least two vertices, the family of $(X, 2X)$-intersection graphs is a strict subfamily of the family of line graphs. In this paper, we generalize his results to (X, kX)-intersection graphs for any $k \geq 2$. We show that for any integer $k \geq 2$ and connected graph X with no bipartite blocks, the family of (X, kX)-intersection graphs coincides with the family of line graphs of conformal k-graphs. On the other hand, we obtain a Ramsey type result on vertex splitting and use it to prove that for any connected bipartite graph X with at least two vertices, the family of (X, kX)-intersection graphs is strictly contained in the family of line graphs of conformal k-graphs. This settles two conjectures of Cai [1] in affirmative.

We define terms and fix notation in Section 2. In Section 3, we obtain a Ramsey type result on vertex splitting and use it to derive a necessary condition for $(X, 2X)$ to make the family of $(X, 2X)$-intersection graphs coincide with the family of line graphs. In Section 4, we prove a result on X-disjoint root graphs, which will be used in Section 5 to establish a close relation between $(X, 2X)$-intersection graphs and (X, kX)-intersection graphs, and to prove the main results of the paper.

2 Definitions and notation

In this paper, all graphs are simple undirected graphs. A *hypergraph* $H = (V, \mathcal{E})$ consists of a finite set V of vertices and a family $\mathcal{E}$ of hyperedges, where each hyperedge is a nonempty subset of V and the union of all hyperedges equals V. A hypergraph H is *simple* if no hyperedge is contained in another hyperedge, and k-*uniform* if every hyperedge has k vertices. The 2-*section* $[H]_2$ of a hypergraph $H = (V, \mathcal{E})$ is a graph on V such that two vertices are adjacent iff they are both contained in a hyperedge of H. A *conformal* k-*graph* H is a simple hypergraph whose hyperedges are exactly k-cliques of the 2-section $[H]_2$ of H. Note that conformal 2-graphs are precisely graphs with no isolated vertices. The *line graph* $L(H)$ *of a hypergraph* H is a graph whose vertices are hyperedges of H and where two vertices are adjacent iff they have a nonempty intersection.

The following notion was introduced in [2] to facilitate the study of (X, Y)-intersection graphs. For a pair (X, Y) of fixed graphs, the (X, Y)-*containment hypergraph* of a graph G, denoted $C_{X,Y}(G)$, is a hypergraph in which

1. each vertex corresponds to a distinct induced X-subgraph in G that is contained in some induced Y-subgraph of G,
2. each hyperedge corresponds to a distinct induced Y-subgraph in G, and
3. a vertex is contained in a hyperedge iff the induced X-subgraph corresponding to the vertex is contained in the induced Y-subgraph corresponding to the hyperedge.

Clearly, $C_{X,Y}(G)$ is a k-uniform hypergraph if Y contains exactly k induced X-subgraphs. Furthermore, $I_{X,Y}(G) = L(C_{X,Y}(G))$.

Remark. The above definition of (X, Y)-containment hypergraphs is slightly different from the one in [2]. In the definition of [2], each vertex of the hypergraph corresponds to an X-subgraph of G (not necessarily contained in a Y-subgraph). Hence, an (X, Y)-containment hypergraph in [2] may contain isolated vertices, whereas such a hypergraph in this paper contains no isolated vertices. Also note that the (K_1, P_4)-containment hypergraph of G is the same as the P_4-*structure* [4, 10] of G, which is an important concept in the study of perfect graphs.

For a k-uniform hypergraph H, any graph G satisfying $C_{X,Y}(G) \cong H$ is a *root graph* of H. If, in addition, the induced X-subgraphs in G corresponding to vertices of H are mutually disjoint, then G is an X-*disjoint root graph* of H. An induced X-subgraph in a root graph is *isolated* if it is not contained in any induced Y-subgraph.

For any family $\mathcal{H}$ of hypergraphs (graphs), $L(\mathcal{H})$ denotes the family $\{L(H) : H \in \mathcal{H}\}$. Let $\mathcal{C}_k$ denote the family of conformal k-graphs and $\mathcal{G}$

the family of graphs. Note $\mathcal{C}_2 \neq \mathcal{G}$ but $L(\mathcal{C}_2) = L(\mathcal{G})$. For a pair (X, Y), $\mathcal{I}_{X,Y}$ denotes the family of (X, Y)-intersection graphs, and $\mathcal{C}_{X,Y}$ the family of (X, Y)-containment hypergraphs. For a family $\mathcal{F}$ of graphs, (X, Y) is an *$\mathcal{F}$-generator* (or a *generator for $\mathcal{F}$*) if $\mathcal{I}_{X,Y} = \mathcal{F}$. We use K_n and K_n^k, respectively, to denote the complete graph and complete k-uniform hypergraph on n vertices. For two disjoint graphs H and G, $H + G$ denotes the *join* of H and G, i.e., the graph obtained from H and G by adding edges $\{uv : u \in V(H) \text{ and } v \in V(G)\}$.

3 Vertex splitting

Let $t \geq 2$ be an integer. A *vertex t-split* of a graph G is any graph obtained from G by splitting each vertex of G into at most t vertices, i.e., replacing each vertex v by a set $s(v)$, $1 \leq |s(v)| \leq t$, of new vertices, called the *split-image* of v, and each edge uv by an edge joining a vertex in $s(u)$ with a vertex in $s(v)$. For a vertex t-split G' of G, an induced subgraph H' of G' is *inherited* if the subgraph of G induced by the corresponding vertices of H' in G is isomorphic to H'. In other words, an inherited induced H'-subgraph in G' is an induced H'-subgraph in G that was not destroyed in the splitting process. See Figure 1 for an example.

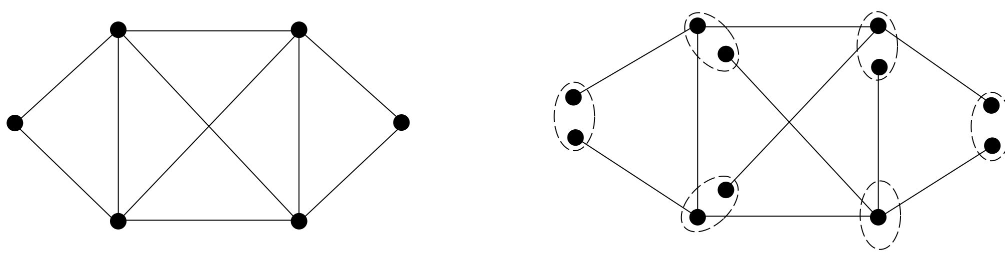

Figure 1: A vertex 2-split with many induced P_4-subgraphs, but only the top and bottom ones are inherited.

Vertex splitting has a close connection with $(X, 2X)$-type generators for line graphs. By exploring the relation between vertex splitting and X-disjoint root graphs, Cai [1] obtained the following necessary condition for $(X, 2X)$ to be a generator for line graphs. Note $\overline{2X} = \overline{X} + \overline{X}$ and recall $\mathcal{I}_{X,Y} = \mathcal{I}_{\overline{X},\overline{Y}}$.

Theorem 3.1 [Cai [1]] *Let X be a connected bipartite graph with $t \geq 2$ vertices. If there is a bipartite graph of which any vertex t-split contains* *an inherited induced X-subgraph, then neither $(X, 2X)$ nor $(\overline{X}, \overline{X} + \overline{X})$ is a generator for line graphs.*

Cai [1] also conjectured that if X is a connected bipartite graph with at least two vertices, then $(X, 2X)$ is not a generator for line graphs; and that for any bipartite graph X and any integer $t \geq 2$, there exists a bipartite graph of which any vertex t-split contains an inherited induced X-subgraph.

Here we settle the above two conjectures in affirmative by relating vertex splitting to edge colouring and using the following theorem in Ramsey theory.

Theorem 3.2 [Nešetřil and Rödl [9]] *For every bipartite graph B and every positive integer n, there is a bipartite graph B' such that for any n-colouring of the edges of B', B' contains a monochromatic induced B-subgraph.*

Theorem 3.3 *For every bipartite graph B and every integer $t \geq 2$, there exists a bipartite graph B' of which any vertex t-split contains an inherited induced B-subgraph.*

Proof. Let B' be the bipartite graph in Theorem 3.2 with $n = t^2$, and G' be an arbitrary vertex t-split of B'. Let (U, W) be a bipartition of B'. For each vertex v of B', let $v_1, v_2, \ldots, v_{t'}$, where $t' \leq t$, be the split-image of v in G'. For each edge uw of B', where $u \in U$ and $w \in W$, colour it with colour (i, j) if the corresponding edge of uw in G' is $u_i w_j$. This gives us an n-colouring of the edges of B'. By Theorem 3.2, B' contains a monochromatic induced B-subgraph, which corresponds to an inherited induced B-subgraph in G'. ∎

It follows from Theorem 3.1 and Theorem 3.3 that $(X, 2X)$ is not a generator for line graphs whenever X is a connected bipartite graph with at least two vertices. We will generalize this to (X, kX)-intersection graphs in Section 5.

4 Disjoint root graphs

The concept of X-disjoint root graphs plays an important role in the study of $(X, 2X)$-type generators for line graphs. It was proved in [1] that in order for $(X, 2X)$ to be a generator for line graphs, certain graphs must have X-disjoint root graphs. In this section, we generalize this result to (X, kX)-type generators for $L(\mathcal{C}_k)$, which will be used in the next section to establish a relation between $(X, 2X)$-type generators for line graphs and (X, kX)-type generators for $L(\mathcal{C}_k)$.

To begin with, we present a result on line graphs of conformal k-graphs. In general, two nonisomorphic conformal k-graphs may have isomorphic line graphs. However, some conformal k-graph H can be uniquely determined by its line graph $L(H)$, i.e., if H' is a k-uniform hypergraph with $L(H') \cong L(H)$, then $H' \cong H$. A simple example is the complete k-uniform hypergraph K_n^k with $n \geq 2k^2$, which follows from a general result in [3]. Here we construct a family of conformal k-graphs in which each hypergraph is uniquely determined by its line graph. This family of conformal k-graphs will be used later in proving a useful theorem on X-disjoint root graphs.

Let G be a graph on n vertices $v_1, v_2, \cdots, v_n$, and $G_1, G_2, \cdots, G_n$ be n pairwise disjoint graphs. The composition $G[G_1, G_2, \cdots, G_n]$ is the graph obtained from G by replacing vertex v_i with graph G_i and adding all edges between G_i and G_j whenever v_i is adjacent to v_j in G. To be precise, $G[G_1, G_2, \cdots, G_n]$ has vertex set $\bigcup_{i=1}^{n}\{(v_i, u) : u \in V(G_i)\}$ where two vertices $(v_i, u), (v_j, u')$ are adjacent iff either $v_i v_j \in E(G)$ or $i = j$ and $uu' \in E(G_i)$.

Lemma 4.1 *Let G be an arbitrary graph on n vertices $v_1, v_2, \cdots, v_n$, and let H be the conformal k-graph whose 2-section is isomorphic to the composition $G[K_{t_1}, K_{t_2}, \cdots, K_{t_n}]$, where each $t_i \geq 2k^2$. For any k-uniform hypergraph H', if $L(H') \cong L(H)$ then $H' \cong H$.*

Proof. Let G' denote $G[K_{t_1}, K_{t_2}, \cdots, K_{t_n}]$. First, since each edge of G' is contained in a k-clique, these is indeed a unique conformal k-graph H with $[H]_2 \cong G'$.

By the definition of conformal k-graphs, the complete k-uniform hypergraph $K_{t_i}^k$ is an induced subhypergraph of H. Therefore, each $L(K_{t_i}^k)$ is an induced subgraph of $L(H)$. Since $L(H') \cong L(H)$, H' contains a subhypergraph A_i such that $L(A_i) \cong L(K_{t_i}^k)$. Therefore $A_i \cong K_{t_i}^k$ for each i as A_i is a k-uniform hypergraph and $K_{t_i}^k$ has at least $2k^2$ vertices [3]. This implies that each A_i is an induced subhypergraph of H'. Since $K_{t_1}^k, K_{t_2}^k, \cdots, K_{t_n}^k$ are pairwise disjoint, $A_1, A_2, \cdots, A_n$ are also pairwise disjoint. It follows that $A_1 \cup \cdots \cup A_n$ is isomorphic to $K_{t_1}^k \cup \cdots \cup K_{t_n}^k$.

Since the 2-section of H is isomorphic to G', each induced subhypergraph $H[V(K_{t_i}) \cup V(K_{t_j})]$, where $i \neq j$, equals either $K_{t_i+t_j}^k$ or $K_{t_i}^k \cup K_{t_j}^k$. Therefore for any $i \neq j$, $H'[V(A_i) \cup V(A_j)] \cong K_{t_i+t_j}^k$ iff $H[V(K_{t_i}) \cup V(K_{t_j})] \cong K_{t_i+t_j}^k$ and $H'[V(A_i) \cup V(A_j)] \cong K_{t_i}^k \cup K_{t_j}^k$ iff $H[V(K_{t_i}) \cup V(K_{t_j})] \cong K_{t_i}^k \cup K_{t_j}^k$.

Therefore $H' \cong H$ since H' contains the same number of hyperedges as H. ∎

We now consider (X, kX)-intersection graphs. We show that for any connected X, an (X, kX)-containment hypergraph is always a conformal k-graph. Therefore, an (X, kX)-intersection graph is isomorphic to the line graph of some conformal k-graph. This motivates us to determine (X, kX) for which the family of (X, kX)-intersection graphs exactly equals the family of line graphs of conformal k-graphs.

Lemma 4.2 *For any integer $k \geq 2$ and connected graph X, $\mathcal{C}_{X,kX} \subseteq \mathcal{C}_k$.*

Proof. Let G be an arbitrary graph. Then the (X, kX)-containment hypergraph H of G is a simple k-uniform hypergraph since kX contains exactly k induced X-subgraphs when X is connected. Observe that each edge in the 2-section $[H]_2$ of H corresponds to two disjoint induced X-subgraphs in G and there is no edge between these two X-subgraphs. Let K be an arbitrary k-clique in $[H]_2$, and S be the set of induced X-subgraphs in G corresponding to vertices in K. Then any two induced X-subgraphs in S are disjoint and not connected by edges. This implies that vertices in S induce a kX-subgraph in G, which corresponds to a hyperedge in H. Therefore H is a conformal k-graph. ∎

Corollary 4.3 *For any integer $k \geq 2$ and connected graph X, $\mathcal{I}_{X,kX} \subseteq L(\mathcal{C}_k)$.*

To prove that (X, kX) is an $L(\mathcal{C}_k)$-generator, we need only construct an X-disjoint root graph G for an arbitrary conformal k-graph H. On the other hand, to prove that (X, kX) is not an $L(\mathcal{C}_k)$-generator, we need to show that the line graph of some conformal k-graph H is not an (X, kX)-intersection graph. This appears to be very difficult because induced X-subgraphs in possible root graphs of H can interwine in a complicated manner. Here we show that we need only consider X-disjoint root graphs.

Note (X, kX) being an $L(\mathcal{C}_k)$-generator only guarantees that for any conformal k-graph H, there is a graph R satisfying $I_{X,kX}(R) \cong L(H)$. However, because of the lack of the one-to-one correspondence between conformal k-graphs and their line graphs, it is not even clear if H has a root graph. To tackle this, we construct from H a conformal k-graph H' of Lemma 4.1, and use a root graph of H' to produce an X-disjoint root graph of H.

Theorem 4.4 *For any integer $k \geq 2$ and connected graph X, (X, kX) is an $L(\mathcal{C}_k)$-generator iff every conformal k-graph has an X-disjoint root graph that contains no isolated induced X-subgraphs.*

Proof. The sufficiency of the theorem follows from Corollary 4.3 and the fact that $I_{X,kX}(G) = L(C_{X,kX}(G))$. We now prove the necessity. Let H be an arbitrary conformal k-graph on n vertices $v_1, v_2, \ldots, v_n$ and G be the 2-section of H. Let m be the number of vertices in X. Set $t_n = 2k^2 + (mn)^2$ and let t_i, $1 \leq i < n$, be a finite integer satisfying $t_i > (mn)^2 + m\sum_{j=i+1}^{n} t_j$. (The reason for setting t_i in the way will become clear later in the proof.) Denote by G' the composition $G[G_1, G_2, \cdots, G_n]$ with G_i being the complete graph on t_i vertices, and let H' be the conformal k-graph corresponding to G'.

Since (X, kX) is an $L(\mathcal{C}_k)$-generator, there is a graph R' whose (X, kX)-intersection graph $I_{X,kX}(R')$ is isomorphic to the line graph $L(H')$ of H'. It follows from Lemma 4.1 that the (X, kX)-containment hypergraph $C_{X,kX}(R')$ of R' is isomorphic to H'. We now use R' to construct an X-disjoint root graph R of H.

Each vertex of H' corresponds to an induced X-subgraph in R'. Let S_i denote the set of induced X-subgraphs in R' that correspond to the vertices in the complete graph G_i. Since vertices of any two induced X-subgraphs in S_i induce a $2X$-subgraph in R', we deduce the following two facts:

1. Inside each S_i, every pair of induced X-subgraphs are disjoint and not connected by edges.
2. For any $i \neq j$, an induced X-subgraph in S_i can share vertices with at most m induced X-subgraphs in S_j.

To construct an X-disjoint root graph R of H, we use the following algorithm to choose one induced X-subgraph X_i from each S_i to form a set $S^* = \{X_i : 1 \leq i \leq n\}$ of disjoint induced X-subgraphs such that no induced X-subgraph in $\bigcup_{i=1}^{n} S_i - S^*$ intersects more than one induced X-subgraph in S^*.

For each value of i from 1 to n in increasing order perform the following two steps. Initially, all induced X-subgraphs are unmarked.

Step 1. Choose from S_i an unmarked induced X-subgraph X_i that is disjoint from all induced X-subgraphs in $\bigcup_{j=i+1}^{n} S_j$.

Step 2. Let S' be the set of induced X-subgraphs (marked or unmarked) in $\bigcup_{j=1}^{i-1} S_j$ that share vertices with X_i. For each induced X-subgraph in $\bigcup_{j=i+1}^{n} S_j$, mark it if it shares vertices with an induced X-subgraph in S'.

It is clear that if $X_1, \cdots, X_i$ are selected in the first i-iterations, these induced X-subgraphs are mutually disjoint and disjoint from all induced X-subgraphs in $\bigcup_{j=i+1}^{n} S_i$. Furthermore, no induced X-subgraphs inside $\bigcup_{j=1}^{i} S_j - \{X_1, \cdots, X_i\}$ intersect more than one induced X-subgraph in $X_1, \cdots, X_i$. Therefore after the n-th iteration, the set S^* of selected induced X-subgraphs has the required property. It remains to be shown that indeed one induced X-subgraph is chosen at Step 1 of each iteration. Recall that S_1 contains $t_1 \geq (mn)^2 + m \sum_{j=2}^{n} t_j$ induced X-subgraphs and none of them are marked. Since each induced X-subgraph in $\bigcup_{j=2}^{n} S_j$ intersects at most m induced X-subgraphs in S_1, there is an induced X-subgraph in S_1 that is disjoint from all induced X-subgraphs in $\bigcup_{j=2}^{n} S_j$, and hence all induced X-subgraphs in $\bigcup_{j=1}^{n} S_j - X_1$.

Assume that $i-1$ induced X-subgraphs $X_1, \cdots, X_{i-1}$ have been selected. Consider the situation right before the execution of Step 1 of the i-th iteration. Each selected induced X-subgraph X_j, $1 \leq j < i$, intersects at most m induced X-subgraphs in each $S_{j'}$, $j' < j$. Hence the total number of induced X-subgraphs in $S_1, \cdots, S_{j-1}$ that share vertices with X_j is at most $m(j-1)$. Each of these $m(j-1)$ induced X-subgraphs share vertices with at most m induced X-subgraphs in S_i. Therefore at most $m^2(j-1)$ unmarked induced X-subgraphs in S_i became marked in Step 2 right after X_j was selected. This implies that the total number of marked induced X-subgraphs in S_i is at most $m^2(i-1)(i-2)/2$, which is less than $(mn)^2$. Therefore S_i contains more than $t_i - (mn)^2 > m \sum_{j=i+1}^{n} t_j$ unmarked induced X-subgraphs. Since $\bigcup_{j=i+1}^{n} S_j$ contains $\sum_{j=i+1}^{n} t_j$ induced X-subgraphs, and each induced X-subgraph intersects at most m induced X-subgraphs in S_i, S_i has at least one unmarked induced X-subgraph that is disjoint from all induced X-subgraphs in $\bigcup_{j=i+1}^{n} S_j$. Therefore the algorithm indeed constructs a set S^* of n disjoint induced X-subgraphs with the required property.

Let R be the subgraph of R' induced by all vertices of induced X-subgraphs of S^*. By the construction of G' and the choice of S^*, it is clear that, for any $i \neq j$, $V(X_i) \cup V(X_j)$ induces a $2X$-subgraph in R iff $v_i v_j$ is an edge in G. Therefore the vertices of any k X-subgraphs from S^* induce a kX-subgraph in R iff the corresponding vertices of these k X-subgraphs in G form a k-clique. Furthermore, by the construction of S^*, no induced X-subgraph in $\bigcup_{i=1}^{n} S_i - S^*$ can share vertices with more than one induced X-subgraph in S^*, implying that R contains no induced X-

subgraph in $\bigcup_{i=1}^{n} S_i - S^*$. Therefore $C_{X,kX}(R) \cong H$ and we have obtained an X-disjoint root graph of H.

Now we show that actually H has an X-disjoint root graph with no isolated induced X-subgraphs. Let H^* be the conformal k-graph whose 2-section is isomorphic to $[H]_2 + K_{k-1}$. Then, as we have just proved, H^* has an X-disjoint root graph R^*. Let V^* denote the vertices of X-subgraphs in R^* that correspond to vertices of H, and let $R = R^*[V^*]$. Then $C_{X,kX}(R) \cong H$. Suppose that R contains an isolated induced X-subgraph. Then it forms an induced kX-subgraph with the $k-1$ disjoint induced X-subgraphs corresponding to K_{k-1}, contradicting to the fact that $C_{X,kX}(R^*) \cong H^*$. Therefore H has an X-disjoint root graph with no isolated induced X-subgraphs. ∎

5 Generators for line graphs of conformal k-graphs

Having the tools from the previous sections, we now characterize (X, kX)-type generators for line graphs of conformal k-graphs. First, we use Theorem 4.4 to establish the following relation between $L(\mathcal{G})$-generators and $L(\mathcal{C}_k)$-generators, which enables us to concentrate on $(X, 2X)$ when studying (X, kX)-type generators for $L(\mathcal{C}_k)$.

Theorem 5.1 *For any integer $k \geq 3$, (X, kX) is an $L(\mathcal{C}_k)$-generator iff $(X, 2X)$ is an $L(\mathcal{G})$-generator.*

Proof. Suppose that $(X, 2X)$ is an $L(\mathcal{G})$-generator and let H be an arbitrary conformal k-graph. By Theorem 4.4, the 2-section $[H]_2$ of H has an X-disjoint root graph R, i.e., $C_{X,2X}(R) \cong [H]_2$. For each k-clique of $[H]_2$, its corresponding induced X-subgraphs in R are mutually disjoint and not connected by edges. Since each k-clique of $[H]_2$ corresponds to a hyperedge of H, $C_{X,kX}(R) \cong H$ and hence $I_{X,kX}(R) \cong L(H)$. It follows from Corollary 4.3 that (X, kX) is an $L(\mathcal{C}_k)$-generator.

Conversely, suppose that (X, kX) is an $L(\mathcal{C}_k)$-generator and let G be an arbitrary graph. Let $G' = G + K_{k-1}$. Since every edge of G' is contained in some k-clique, there is a conformal k-graph H' corresponding to G', i.e., $[H']_2 = G'$. By Theorem 4.4, H' has an X-disjoint root graph R' with no isolated induced X-subgraphs. For each vertex v of G, let X_v be its corresponding induced X-subgraph in R'. Let R denote the subgraph of R' induced by $\bigcup_{v \in V(G)} V(X_v)$. Then the set of all induced X-subgraphs in R is precisely $\{X_v : v \in V(G)\}$ since R' contains no isolated induced X-subgraphs and induced X-subgraphs in R' are mutually disjoint.

Let u and v be two arbitrary vertices of G. We claim that uv is an edge of G iff $R[V(X_u) \cup V(X_v)] \cong 2X$. If uv is an edge of G, then uv is contained in a k-clique of G' and thus u and v are contained in a hyperedge of H'. Therefore no edges connecting X_u with X_v in R' and hence in R, implying $R[V(X_u) \cup V(X_v)] \cong 2X$.

If u is not adjacent to v, then no hyperedge in H' contains both u and v since $[H']_2 = G'$ and G is an induced subgraph of G'. Suppose that X_u and X_v are not connected by edges in R'. Let S denote the set of induced X-subgraphs in R' that correspond to the vertices of the complete graph K_{k-1} of G'. Since $V(K_{k-1}) \cup \{u\}$ and $V(K_{k-1}) \cup \{v\}$ are hyperedges in H', any two X-subgraphs in $S \cup \{X_u, X_v\}$ are mutually disjoint and not connected by edges in R'. Then any $k-2$ vertices in K_{k-1} together with u and v would be a hyperedge of H', a contradiction. Therefore X_u and X_v are connected by some edges in R' and hence in R, implying $R[V(X_u) \cup V(X_v)]$ is not isomorphic to $2X$.

From the above argument, we have $C_{X,2X}(R) \cong G$ and hence $I_{X,2X}(R) \cong L(G)$. It follows from Corollary 4.3 that $(X, 2X)$ is an $L(\mathcal{G})$-generator. ∎

It was proved in [1] that $(X, 2X)$ is an $L(\mathcal{G})$-generator whenever X is a connected graph with no bipartite blocks. Therefore we can derive from Theorem 5.1 the following sufficient condition for (X, kX) to be an $L(\mathcal{C}_k)$-generator. Recall $\mathcal{I}_{X,Y} = \mathcal{I}_{\overline{X},\overline{Y}}$

Theorem 5.2 *For any integer $k \geq 2$ and connected graph X with no bipartite blocks, both (X, kX) and $(\overline{X}, \overline{kX})$ are $L(\mathcal{C}_k)$-generators.*

On the other hand, combining Theorem 3.1, Theorem 3.3, and Theorem 5.1, we obtain the following necessary condition for (X, kX) to be an $L(\mathcal{C}_k)$-generator.

Theorem 5.3 *For any integer $k \geq 2$ and connected bipartite graph X with at least two vertices, neither (X, kX) nor $(\overline{X}, \overline{kX})$ is an $L(\mathcal{C}_k)$-generator.*

Therefore (X, kX)-type $L(\mathcal{C}_k)$-generators are fully characterized when X is 2-connected.

Corollary 5.4 *Let X be a 2-connected graph and $k \geq 2$ an integer. Then (X, kX) (likewise, $(\overline{X}, \overline{kX})$) is an $L(\mathcal{C}_k)$-generator iff X is not bipartite.*

We leave the full characterization of (X, kX)-type $L(\mathcal{C}_k)$-generators as an open problem for the reader to ponder.

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