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Path homomorphisms

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Abstract

We investigate homomorphisms between finite oriented paths. We demonstrate the surprising richness of this perhaps simplest case of homomorphism between graphs by proving the density theorem for oriented paths. As a consequence every two dimensional countable poset is represented by finite paths and their homomorphisms, and every finite dimensional poset is represented by finite oriented trees and their homomorphisms. We then consider related problems of universal representability and extendability and on-line representability.

1. *Introduction*

We consider homomorphisms between oriented graphs: given two oriented graphs $G = (V,E)$ and $G' = (V',E')$, a *homomorphism* f of G to G' is any mapping $f\colon V \to V'$ which satisfies

$$(\bigstar)\quad f(x)f(y) \in E' \quad \text{for all} \quad xy \in E.$$

If there exists a homomorphism of G to G' then we say G is *homomorphic* to G', and write $G \to G'$. Otherwise we write $G \not\to G'$.

The graph homomorphisms were investigated in various contexts: graph colouring [**2**–**7**, **12**, **16**, **17**, **24**], graph products [**10**, **26**], automata theory and formal languages [**24**, **28**], various algebraical context (as a generalization of isomorphism: categories). Various of these questions can be conveniently expressed by considering the quasiorders (and partial orders) induced by the existence of homomorphism.

Here we consider the problem motivated by the existence of homomorphisms between perhaps the simplest type of graphs: oriented paths.

An *oriented path* P is any oriented graph (V,E) where $V = \{v_0, v_1, \ldots, v_n\}$ and for every $i = 1, 2, \ldots, n$ either $v_{i-1}v_i \in E$ or $v_i v_{i-1} \in E$ (but not both), and there are no other edges. Thus an oriented path is any orientation of an undirected path. We denote the initial vertex v_0 and the terminal vertex v_n of P by $i(P)$ and $t(P)$ respectively. Examples are in Fig. 1, and here all arcs are oriented upwards.

One can also express a given path P by a code $c(P) = a_1 a_2, \ldots, a_n$, where n is the number of edges in P, and $a_i = 0$ or 1 depending on whether the ith edge is a forward

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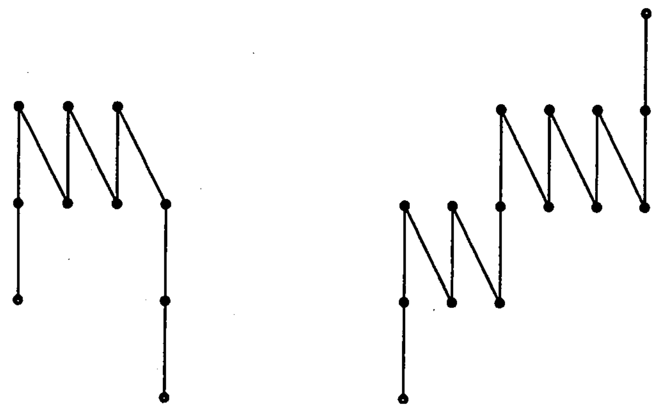

(a) P (b) P′

Fig. 1. *Example paths.*

or backward edge. For example the path P on Fig. 1 (a) has code $c(P) = 001010111$ and the path P' on Fig. 1 (b) has code $c(P') = 00101001010100$. Given a code $c = (a_1, a_2, \ldots, a_n)$ the code c^{-1} is the code $(a'_1, a'_2, \ldots, a'_n)$ given by $a'_i + a_{n+1-i} = 1$. The code $c(P)^{-1}$ corresponds to the flipping of the path P.

The *length* $l(P)$ of a path P is the number of edges in P. The *algebraic length* $al(P)$ of a path P is the number of 0s minus the number of 1s in the code of P. Thus the algebraic length of a path could be negative. The *net length* $nl(P)$ of a path is the absolute value of its algebraic length. The *height* $ht(P)$ of a path P is the maximum net length of a subpath P' of P. For a vertex p_i of $P = [p_0, p_1, \ldots, p_n]$, the *level* $l_P(p_i)$ of p_i is the algebraic length of the subpath $[p_0, p_1, \ldots, p_i]$ of P. An oriented path P of net length n is *minimal* if P contains no proper subpath of the same net length.

For two oriented paths P_1 and P_2 with codes $c(P_1)$ and $c(P_2)$ respectively, the concatenation $P_1 P_2$ of P_1 and P_2 is the path with code $c(P_1 P_2) = c(P_1)\, c(P_2)$. In other words, $P_1 P_2$ is the path obtained from the disjoint union of P_1 and P_2 by identifying the terminal vertex of P_1 with the initial vertex of P_2.

Homomorphisms between oriented paths were studied in [**7**, **20**, **13**], and one can quote the following theorems as examples of good understanding of this notion:

THEOREM 1·1 (Path duality [**20**]). *Let P be any oriented path. Then an oriented graph G is homomorphic to P if and only if any oriented path homomorphic to G is also homomorphic to P.*

This follows a pattern of homomorphism duality introduced in [**26**] and studied further in [**17**, **23**]. We shall make use of path duality below.

THEOREM 1·2 (Polynomial testing [**7**]). *There is an algorithm which decides for a given oriented graph G and oriented path P the existence of homomorphism $G \to P$ in time polynomial in the size of G and P.*

Considering the codes of oriented paths, an oriented path corresponds to a word on alphabet $\{0, 1\}$. Homomorphism of oriented paths can be interpreted as a kind of operation on words. One can see easily that the following holds:

THEOREM 1·3 (World homomorphism). *For paths P and P' the following two statements are equivalent:*

1. *P is homomorphic to P';*

2. *there exist words c, c' on alphabet $\{0,1\}$ such that $c(P)$ is a consecutive segment of c and c' is a consecutive segment of $c(P')$ and c' can be obtained from c by means of a sequence of substitutions of the form*

$$ww^{-1}w \to w.$$

Here w is any word on alphabet $\{0,1\}$.

Despite these results we show that the homomorphisms between finite oriented paths form a surprisingly rich structure. We illustrate this by the following easy result:

THEOREM 1·4. *Let $R = (X, \leqslant)$ be a finite poset. Then there exist finite oriented paths P_x, $x \in X$ such that*

$$(\bigstar) \quad x \leqslant y \quad \text{if and only if there exists a homomorphism } P_x \to P_y.$$

Proof. To every x associate a vector $x = (x_1, x_2, \ldots, x_d)$ such that $x \leqslant y$ if and only if $x_i \geqslant y_i$ for all $i = 1, 2, \ldots, d$. We may also assume that the coordinates x_i are positive integers bounded by n (d is of course at least the Dushnik Miller dimension of R). Denote by Z_m the code $0101\ldots010$ (m 0s and $m-1$ 1s. Then let P_x be the path with code $c(P_x) = 0Z_{x_1}Z_{x_2}\ldots Z_{x_d}0$.

It is easy to verify that P_x is homomorphic to P_y if and only if $x_i \geqslant y_i$ for each i. Therefore $x \leqslant y$ if and only if there exists a homomorphism of P_x to P_y. ▌

Note that all the oriented paths $P_x (x \in X)$ above have the same net length, and each P_x is a minimal path. This implies that any homomorphism of a path P_x to another path P_y takes the initial vertex of P_x to the initial vertex of P_y, and takes the terminal vertex of P_x to the terminal vertex of P_y.

The result above may briefly be expressed by saying that every finite poset R may be represented by finite paths and homomorphisms between them. Below we shall strengthen this statement in three directions.

1. *Density theorems.* We prove that the poset determined by all finite paths of height at least four and their homomorphisms is dense. Explicitly we have:

THEOREM 1·5. *Suppose that P and P' are paths such that $P \to P'$ and $P' \nrightarrow P$. Then there exist a path P'' with $P \to P'' \to P'$ but $P' \nrightarrow P''$ and $P'' \nrightarrow P$, provided that P' has height at least* 4.

This will be proved in Section 2 below and is the main result of this paper. Viewing the simplicity of path homomorphisms and its relationship to word homomorphisms, we consider this a surprising result, in contrast to the classical Higman result on well-quasi-ordering of finite words [**9**].

2. *Representation.* We prove in Section 3 that any two-dimensional countable poset can be represented by finite oriented paths and homomorphisms between them, and any finite dimensional poset can be represented by finite oriented trees and homomorphisms between them.

3. *Universal representability.* Density theorem may be viewed as 'on-line' representability (of chains). In Section 4 we show that every countable poset can be represented

by finite paths if and only if the class of all finite posets can be 'on-line' represented by finite oriented paths. As we do not know whether or not every countable poset can be represented by finite oriented paths, we propose here an intermediate notion of *universal representability*. A universal representation of a finite poset R is a representation such that for any finite poset S containing R there exists a representation of S whose restriction on R is the original representation of R. We prove that all finite chains are universally representable.

2. *Density theorem*

A graph C is a *core* if C is not homomorphic to any proper subgraph of C. The *core of a graph* G is a subgraph C of G such that C is a core and G is homomorphic to C. It is well known (cf. [**15**]) and easy to see that any finite graph has a unique core up to isomorphism. If an oriented path P is a core, then the first two edges of P must be of the same direction, i.e. either both edges are forward edges or both edges are backward edges. For otherwise the mapping of $V(P)$ to $V(P)$ which sends the first vertex to the third vertex and fixes every other vertex is a homomorphism of P to a proper subpath of P. Similarly the last two edges of P are also of the same direction.

LEMMA 2·1. *If an oriented path P is a core then the identity mapping of $V(P)$ to $V(P)$ is the only homomorphism of P to P.*

Proof. Suppose $P = [p_0, p_1, \ldots, p_n]$ is an oriented path which is a core. Let $f\colon V(P) \to V(P)$ be a homomorphism of P to P. Then f is one to one and the indices of $f(p_i)$ and $f(p_{i+1})$ differ exactly by one. Therefore if f is not the identity mapping then $f(p_i) = p_{n-i}$ for all i. If n is even then $f(p_{n/2}) = p_{n/2}$. It is easy to see that in this case the mapping g defined as $g(p_i) = p_i$ for $i \leqslant n/2$ and $g(p_i) = p_{n-i}$ for $i \geqslant n/2$ is a homomorphism of P to a proper subpath of P, contrary to the assumption that P is a core. If n is odd, say $n = 2k+1$, then $f(p_k) = p_{k+1}$ and $f(p_{k+1}) = p_k$. However, since P is an oriented path, exactly one of the two pairs $p_k p_{k+1}$ and $p_{k+1} p_k$ is an edge of P. Thus f is not a homomorphism. ∎

The main result of this paper is

THEOREM 2·1 (Density Theorem). *Suppose that P and P' are two oriented paths such that $P \to P'$ and $P' \nrightarrow P$. If the height of P' is at least four, then there is an oriented path P'' such that $P \to P'' \to P'$ and $P' \nrightarrow P'' \nrightarrow P$.*

Proof. Without loss of generality, we may assume that P and P' are cores. Let $f\colon P \to P'$ be a homomorphism. We also assume that $f(P)$ is a core. Thus by the remark at the beginning of this section, the first two edges of $f(P)$ are either both forward or both backward edges. The same is true for the last two edges of $f(P)$. Note that the first or last edge of $f(P)$ need not to be the image of the first or last edge of P.

We consider two cases.

Case 1. The mapping f is not onto.

In this case either the initial vertex or the terminal vertex of P' does not belong to $f(P)$. Without loss of generality, we may assume that the terminal vertex of P' does not belong to $f(P)$. Let v be the last vertex of P' which is contained in $f(P)$. The vertex v divides P' into two parts $P' = P'_1 P'_2$. Let n be the number of vertices of P'. Depending on whether the last edge of P'_1 is a forward edge or backward edge, we let Z be the oriented path with code $1010\ldots10$ or $0101\ldots01$ respectively, such that $|V(Z)| > |V(P)|$. Let $P'' = P'_1 Z P'_2$. We claim that $P \to P'' \to P'$ and $P' \not\to P'', P'' \not\to P$.

Since the subpath $f(P)$ of P' is contained in P'', we can consider f as a homomorphism of P to P''. To obtain a homomorphism of P'' to P', we simply choose the identity mapping on P'_1 and P'_2 and send all the edges of Z to the last edge of P'_1.

Now we show that $P'' \not\to P$. Otherwise let $g: P'' \to P$ be a homomorphism. The composition $g \circ f$ is a homomorphism of P to P, hence is equal to the identity mapping by Lemma 2·1. Thus the restriction of g to $f(P)$ is a one to one mapping. In particular g sends v, the terminal vertex of P'_1, to either the terminal vertex or the initial vertex of P. Since P starts with a directed path (either forward or backward) of length two, and also terminates with a directed path of length two, g must send all the edges of the zigzag path Z to a single edge of P (either the initial edge or the terminal edge). Therefore, the terminal vertex of P'_1 and the initial vertex of P'_2 are sent to the same vertex of P by the homomorphism g. This implies that g induces a homomorphism of P' to P, contrary to our assumption that $P' \not\to P$.

To show that $P' \not\to P''$, we assume to the contrary that there is a homomorphism $h: P' \to P''$. Since $h(P')$ is connected and $|V(P)| = n < |V(Z)|$, we see that either $h(P') \cap V(P'_1) = \varnothing$ or $h(P') \cap V(P'_2) = \varnothing$. If $h(P') \cap V(P'_2) = \varnothing$, then $h(P')$ is homomorphic to P'_1 since $P'_1 Z$ is homomorphic to P'_1. Thus P' is homomorphic to P'_1 which is a proper subpath of P', contrary to the assumption that P' is a core. If $h(P') \cap V(P'_1) = \varnothing$, then $h(P')$ is homomorphic to the proper subpath of P' which consists P'_2 and the last edge of P'_1. This is again a contradiction.

Case 2. The mapping f is onto.

In this case the mapping f cannot be one to one. For otherwise f^{-1} would be a homomorphism of P' to P, contrary to our assumption that $P' \not\to P$.

Thus there exist two distinct vertices u_1 and u_2 of P such that $f(u_1) = f(u_2)$. Actually, it is not difficult to see that there exist a pair of vertices u_1 and u_2 such that $f(u_1) = f(u_2)$ and u_1 and u_2 have a common neighbour. For the remaining part of the proof of Theorem 2·1, we assume that u_1 and u_2 are two vertices of P with a common neighbour v and $f(u_1) = f(u_2)$. Thus either both $vu_1$and vu_2 are edges of P or both $u_1 v$ and $u_2 v$ are edges of P. Identifying the two vertices u_1 and u_2, we obtain an oriented tree T. We denote the new vertex, which is the identification of u_1 and u_2, by u. The vertex u is adjacent to v and has degree three in T. Suppose the other two neighbours of u are w_1 and w_2, such that w_1 is a neighbour of u_1 and w_2 is a neighbour of u_2 in P.

It is obvious that $P \to T \to P'$. It is also easy to see that $T \not\to P$, for otherwise the composition of homomorphisms $P \to T \to P$ would give us a homomorphism of P to a proper subpath of P (as T has fewer vertices than P), contrary to the assumption that P is a core.

Lemma 2·2. *Let P, P' be oriented paths and let T be an oriented tree. Suppose that there exist homomorphisms $h: P \to T$ and $g: T \to P'$, and that $P' \not\to T \not\to P$. Then there exists an oriented path P'' such that $P \to P'' \to P'$ and $P' \not\to P'' \not\to P$. Moreover, if $g \circ h(i(P)) = i(P')$,* *then P'' can be chosen in such a way that there exists a homomorphism $h': P \to P''$ with $h'(i(P)) = i(P'')$ and a homomorphism $h'': P'' \to P'$ with $h''(i(P'')) = i(P')$.*

Proof. Since $T \not\to P$, by the Path Duality Theorem (Theorem 1·1), there exists a path P^* such that $P^* \to T$ and $P^* \not\to P$. Let $h^*: P^* \to T$ be a homomorphism of P^* to T. Without loss of generality we can assume that $h^*(i(P^*)) = h(t(P))$. (In fact by extending the path P^* we can make $h^*(i(P^*))$ any vertex of T.) Now we define an oriented path P'' to be the path obtained from the disjoint union of P and P^* by adding a sufficiently long (i.e. has more vertices than P') zigzag path $0101\ldots01$ or $1010\ldots10$ (depending on whether the terminal edge of P is a forward edge or backward edge) to connect the terminal vertex $t(P)$ of P to the initial vertex $i(P^*)$ of P^*). Note that P is the initial segment of P''. Therefore there is a homomorphism $h': P \to P''$ such that $h'(i(P)) = i(P'')$. The partial mapping $g': P'' \to P'$ which agrees with h on P (which is the initial segment of P'') and agrees with $g \circ h^*$ on P^* (which is the terminal segment of P'') can be easily extended to a homomorphism $h'': P'' \to P'$. On the other hand $P'' \not\to P$ because $P^* \not\to P$, and $P' \not\to P''$ because any homomorphism $P' \to P''$ would either give a homomorphism of P' to P or a homomorphism of P' to P^* (observe the zigzag path connecting P and P^* has more vertices than P'), contrary to our assumptions.

For the second half of the lemma, it suffices to observe that the homomorphism $h': P \to P''$ sends $i(P)$ to $i(P'')$, and the homomorphism $h'': P'' \to P'$ agrees with the homomorphism $g \circ h$ on the initial segment of P'', which is a copy of P. ∎

Applying Lemma 2·2, we see that in case P' is not homomorphic to T, then there exists P'' such that $P \to P'' \to P'$ and $P \not\to P'' \not\to P'$. We now consider the case that P' is homomorphic to T. In this case, we cannot obtain the path P'' from the tree T. We shall first investigate the structure of T, and then construct some other trees from which we can construct the path P''.

Let $h: P' \to T$ be a homomorphism. Then h must be one to one, for otherwise the composition $P' \to T \to P'$ would give us a homomorphism of P' to a proper subpath of P'.

The image $h(P')$ of P' is thus an isomorphic copy of P'. Since P' is a core and $T \to P'$, we have that $h(P')$ is the core of T.

We must have that $w_1, w_2 \in h(P')$, for otherwise $h(P')$ is a subpath of P, contrary to the assumption that $P' \not\to P$. Thus $u \in h(P')$ and $v \notin h(P')$ as $h(P')$ is a path of T isomorphic to P'.

Claim 1. The homomorphism h of P' to T is unique.

Otherwise suppose that $g: P' \to T$ is another homomorphism of P' to T. Then g must also be one to one, and $w_1, w_2 \in g(P')$. Now the images $g(P')$ and $h(P')$ must be different paths of T, for otherwise $g^{-1} \circ f$ would be an homomorphism of P' to P' which is not the identity mapping, contrary to Lemma 2·1. Since both paths $g(P')$ and $h(P')$ have the same length as P', we know that one of the end vertices of $g(P')$ is an interior vertex of $h(P')$. Consider a homomorphism $k: T \to g(P')$ (recall that $g(P')$ and $h(P')$ are isomorphic to P' and they are cores of T). By Lemma 2·1, k restricted to $g(P')$ is the identity mapping. Consider the restriction of k to $h(P')$. It is the identity mapping on the intersection of $h(P')$ and $g(P')$, which is not empty because $w_1, w_2 \in g(P') \cap h(P')$, and it flips the part of $h(P')$ that is not contained in $g(P')$ into

$g(P')$. Therefore, k is a homomorphism of $h(P')$ to $g(P')$ which is not one to one, contrary to the assumption that $h(P')$ and $g(P')$ are isomorphic cores of T. The claim is proved.

Let $h': T \to h(P')$ be a homomorphism. Then h' restricted to $h(P')$ is the identity mapping by Lemma 2·1. In particular we have $h'(u) = u$, $h'(w_1) = w_1$ and $h'(w_2) = w_2$. Thus $h'(v)$ is equal to either w_1 or w_2. Without loss of generality we may assume that $h'(v) = w_1$. Thus either both uv and uw_1 are edges of T or both vu and $w_1 u$ are edges of T.

The structure of the tree T, as well as that of P, is now quite clear. There is an isomorphic copy $h(P')$ of P' in the oriented tree T, and $u \in h(P')$. We name the vertices of this path as $h(P') = [x_0, x_1, \ldots, x_{m-1} = w_1, x_m = u, x_{m+1} = w_2, \ldots, x_s]$. The vertex u divides $h(P')$ into two parts, say $h(P') = AB$, $A = [x_0, x_1, \ldots, x_{m-1} = w_1, x_m = u]$ and $B = [x_m = u, x_{m+1} = w_2, \ldots, x_s]$. The path P contains the subpath $A[u_1, v, u_2]B$, and either all of $vu_1, vu_2, w_1 u_1$ are edges of P or all of $u_1 v, u_2 v, u_1 w_1$ are edges of P. Without loss of generality, we may assume that all of $vu_1, vu_2, w_1 u_1$ are edges of P.

We shall construct a tree T' from the oriented path P by adding a branch to either the vertex u_1 or to the vertex v so that T' is homomorphic to P', and P' is not homomorphic to T', and moreover T' is not homomorphic to P. We consider two subcases:

Case 2(*a*). $w_2 u_2$ is an edge of P.

We consider the two subpaths $A = [x_0, x_1, \ldots, x_{m-1}(=w_1)]$ and $B = [x_{m+1}(=w_2), \ldots, x_s]$ of $h(P')$ in T (the paths A and B are defined above). For each x_i we let $l(x_i)$ be the algebraic length of the subpath of P connecting v to x_i. Since P' has algebraic length at least 4, we see that there exist x_i such that $i \neq 0$, s and $l(x_i) \neq 0, 1$.

We choose $1 \leqslant i_0 \leqslant s-1$ such that $l(x_{i_0}) \neq 0, 1$ and such that $|i_0 - m| = \min\{|i-m| : 1 \leqslant i \leqslant s-1, l(x_i) \neq 0, 1\}$. If $i_0 > m$ then we let B' be a path which is isomorphic to the subpath $[w_2 = x_{m+1}, x_{m+2}, \ldots, x_{i_0}]$ of B. If $i_0 < m$ then we let B' be a path isomorphic to the path $[w_1 = x_{m-1}, x_{m-2}, \ldots, x_{i_0}]$. Without loss of generality we assume that $i_0 > m$ and let g be an isomorphism of $[w_2 = x_{m+1}, x_{m+2}, \ldots, x_{i_0}]$ to B'. Let T' be the tree obtained from the disjoint union of the paths P and B' by identifying the vertex v of P with the vertex $g(w_2)$ of B'. We now show that T' is homomorphic to P' but not homomorphic to P, and furthermore P' is not homomorphic to T'.

The mapping which is equal to f on the subpaths A and B of P and is equal to $f \circ g$ on B' is a homomorphism of T' to P'. Thus T' is homomorphic to P'.

To see that T' is not homomorphic to P, we assume to the contrary that there is a homomorphism h' of T' to P. Then h' restricted to P is a homomorphism of P to P. Since P is a core, we have that h' is the identity mapping on P by Lemma 2·1. Thus $h'(v) = v$. Suppose that $h'(g^{-1}(x_{i_0})) = x_j$ (recall that $g^{-1}(x_{i_0})$ is the terminal vertex of B'), and that X is the subpath of P connecting v to x_j, whose length is at most the length of the path B'. Therefore $|j-m| \leqslant i_0 - m - 2$. Thus by the choice of i_0, either X has algebraic length 0 or 1, or $j = 0$. However, since the algebraic length of B' is neither 0 nor 1, which implies that the algebraic length of X is neither 0 nor 1 (because homomorphisms preserve algebraic length), it must be the case that $j = 0$. This then implies that the subpath $[x_0, x_1, \ldots, x_{i_0-2}]$ of P is of height two. By the choice of i_0, we know that $[x_{i_0-2}, x_{i_0-1}, x_{i_0}]$ is a directed path of height two. Moreover we have $l(x_0) = l(x_{i_0}) = al(B')$, because homomorphisms of oriented paths preserve algebraic lengths. Therefore, there is a homomorphism of $[x_0, x_1, \ldots, x_{i_0-2}]$ to $[x_{i_0-2}, x_{i_0-1}, x_{i_0}]$, and hence a homomorphism of P to its proper subpath $[x_{i_0-2}, x_{i_0-1}, x_{i_0}, \ldots, x_s]$, contrary to the assumption that P is a core. Thus T' is not homomorphic to P.

Now we prove that P' is not homomorphic to T'. Otherwise let $g' : P' \to T'$ be a homomorphism. Then g' must be one to one, for otherwise the composition of g' and a homomorphism of T' to P' would give a homomorphism of P' to P' which is not one to one. Therefore, in particular $g'(P')$ is a path. Also $g'(P')$ must intersect B', for otherwise g' would be a homomorphism of P' to P, contrary to the assumption that $P' \nrightarrow P$. Thus $g'(P')$ is either disjoint from A or disjoint from B.

Let $k: T' \to T$ be the natural homomorphism of T' to T, i.e., k is equal to the identity mapping on A and B and is equal to g^{-1} on B'. The composition $g' \circ k$ is then a homomorphism of P' to T. It is easy to see that if $g'(P')$ is disjoint from A then $x_0 \notin g' \circ k(P')$, and if $g'(P')$ is disjoint from B then $x_s \notin g' \circ k(P')$. In any case $g' \circ k$ is not equal to the homomorphism h, contrary to Claim 1. Therefore $P' \nrightarrow T'$.

The path P'' required in Theorem 2·1 is then obtained by applying Lemma 2·2 to P, T' and P'.

Case 2(*b*). $u_2 w_2$ is an edge of P.

First we show that we can assume that w_2 is not an end vertex of $h(P')$, i.e. $s \geqslant m+2$. (In the dual case, i.e. in the case that $vu_1, vu_2, w_2 u_2$ and $u_1 w_1$ are edges of P, we need to assume that w_1 is not an end vertex of $h(P')$.) Otherwise suppose that $s = m+1$. Since $f(w_1) = f(v)$ and w_1 and v have the common neighbour u_1, we choose w_1, u_1, v to play the roles of u_1, v, u_2 respectively. By this choice, we obtain a different tree T^*. Explicitly, the tree T^* is obtained from the path P by identifying the vertices w_1 and v. We then have to discuss different cases with respect to this tree T^*. If it does not fall into Case 2(*b*), then the required path P'' can be constructed as before. If it does fall into Case 2(*b*), then the vertex x_{m-2} of $h(P')$ preceding w_1 will play the role of w_1. To make sure that we do not need to change back to let u_1, v, u_2 play their own roles, which will result in an endless changing of the roles of u_1, v, u_2 and that of w_1, u_2, v back and forth, we need to show that x_{m-2} is not the initial vertex of $h(P')$. This is obvious because P' has net length at least four and hence $h(P')$ has at least five vertices, which implies that $m \geqslant 3$ (recall that $x_{m+1} = w_2$ is the terminal vertex of $h(P')$). Thus we may assume that w_2 is not an end vertex of $h(P')$.

Let T' be the tree obtained from P by adding one new vertex a and an edge $u_1 a$. We shall show that T' is homomorphic to P', and P' is not homomorphic to T', and T' is not homomorphic to P.

The mapping f' which is equal to f on P and which sends a to $f((w_2))$ is a homomorphism of T' to P'. Thus T' is homomorphic to P'. To see that T' is not homomorphic to P, we assume to the contrary there is a homomorphism g' of T' to P. Then g' restricted to P is a homomorphism of P to P. Since P is a core, we have that g' is the identity mapping on P by Lemma 2·1. It is easy to see that this is impossible since $g'(a)$ cannot be defined.

Now we show that P' is not homomorphic to T'. Otherwise suppose that $h'' : P' \to T'$ is a homomorphism. Then h'' must be one to one, for otherwise $f' \circ h''$ would be a homomorphism of P' to a proper subpath of P', contrary to the assumption that P' is a core. Thus $h''(P')$ is a path of T'. The vertex a of T' must be in $h''(P')$, for otherwise

h'' is a homomorphism of P' to P. Therefore a is an end vertex of the path $h''(P')$. Observe that there is a homomorphism of T' to T which sends a to w_2. Therefore there is a homomorphism of P' to T which sends an end vertex of P' to w_2. However by the Claim, the homomorphism h is the unique homomorphism of P' to T, and by our assumption w_2 is not an end vertex of $h(P')$. This is a contradiction. This completes the proof of Theorem 2·1. ▌

For oriented paths of net length less than 4, the conclusion of this theorem is not true. Let P_1, P_2, P_3 be the directed paths with one, two and three vertices respectively. For each $i = 1, 2, 3$, P_i is the only oriented path which is a core and has net length $i-1$. It is easy to see that $P_1 \to P_2 \to P_3$, that there is no oriented path which is strictly between P_1 and P_2 and no oriented path which is strictly between P_2 and P_3. Among oriented paths of net length three, the only cores are the paths $Q_m = 001010\ldots10100 = 0Z_m0$ (recall that Z_m is the zigzag path with m forward edges and $m-1$ backward edges), cf. [**13**]. It is also easy to see that $Q_m \to Q_n$ if and only if $m \geqslant n$, and there is no oriented path which is strictly between Q_{m+1} and Q_m. Combining these observations with the Density Theorem, we see that these are the only 'jumps' in the poset of oriented paths and their homomorphisms.

3. *Representability*

Let $R = (X, \leqslant)$ be a poset. A *path representation* of R is a collection of finite oriented paths $\mathscr{P} = \{P_x : x \in R\}$ such that

($\bigstar$) $x \leqslant y$ if and only if there exists a homomorphism $P_x \to P_y$.

We may consider the set of all finite oriented paths and their homomorphisms as a poset $\mathscr{S}$ in which $P \leqslant P'$ if and only if $P \to P'$. Then a path representation of a poset R is actually an embedding of R into $\mathscr{S}$.

We have shown in Section 1 (cf. Theorem 1·4) that each finite poset has a path representation. By using the Density Theorem, it is easy to show that some countable posets also have path representations.

THEOREM 3·1. *Let Q be the linear order of rational numbers. Then there is a path representation $\mathscr{P}$ of Q. Moreover, it can be assumed that if two of the paths $P, P' \in \mathscr{P}$ are homomorphic then there is a homomorphism of P to P' which sends the initial vertex of P to the initial vertex of P'.*

Proof. The first part of the theorem is an obvious consequence of Theorem 2·1. The 'moreover' part of the theorem follows from the proof of Theorem 2·1, by using the 'moreover' part of Lemma 2·2. ▌

Given two paths P and P'. Let F and F' be obtained from P and P' by replacing each edge by the oriented path 001000110111. It is easy to see that P is homomorphic to P' if and only if F is homomorphic to F'. However, F and F' always have height four, and the initial vertices of F and F' always have level zero, no matter what the height of P and P' and what the levels of the initial vertices of P and P'. Thus a partially ordered set can be represented by finite oriented paths if and only if it can be represented by finite oriented paths of height four with initial vertices of level zero. In particular, we have

COROLLARY 3·1. *The rational numbers x can be represented by finite oriented paths P_x of height four with initial vertices of level zero so that $P_x \to P_y$ if and only if $x < y$. Furthermore in case $P_x \to P_y$, there exists a homomorphism of P_x to P_y which sends the initial vertex of P_x to the initial vertex of P_y.*

THEOREM 3·2. *Any countable poset of dimension 2 has a path representation.*

Proof. Let R be a countable poset of dimension 2, and let L_1, L_2 be two linear extensions of R that realize R. Since any countable linear order can be embedded in the rationals, we can apply Corollary 3·1 to L_1 and L_2, and obtain path representations $\mathscr{P}_1, \mathscr{P}_2$ of L_1, L_2 respectively, such that each path of $\mathscr{P}_i (i = 1, 2)$ has height four, and the initial vertex of each path has level zero. Furthermore, we can assume that for paths in $\mathscr{P}_1$, the homomorphisms between the paths send initial vertices to initial vertices; and for paths in $\mathscr{P}_2$, the homomorphisms between paths send terminal vertices to terminal vertices.

Let X be the oriented path 110111001000. For any element a of R, let P_a^1, P_a^2 be paths which represent a in $\mathscr{P}_1, \mathscr{P}_2$, respectively. Let $P_a = P_a^2 X P_a^1$, and let $\mathscr{P} = \{P_a : a \in Q\}$. We claim that $\mathscr{P}$ is a path representation of R. Note that each path P_a in $\mathscr{P}$ has height 8. The levels of each vertex of P_a^1, P_a^2 is between 4 and 8, and the levels of each vertex of X is between 0 and 4. The only vertices of X with level 4 are the two end vertices, which are also end vertices of P_a^1 and P_a^2 respectively. Therefore if h is a homomorphism of P_a to P_b then h must send the copy of X in P_a to the copy of X in P_b, because homomorphisms of paths with the same height preserve the levels. However, X is a core, and thus h restricted to X must be the identity mapping. This implies that $h(P_a^1)$ must be contained in P_b^1 and $h(P_a^2)$ must be contained in P_b^2. Therefore, $P_a^1 \to P_b^1$ and $P_a^2 \to P_b^2$, and hence $a < b$ in R. On the other hand it is easy to see that $P_a \to P_b$ if $P_a^1 \to P_b^1$ (which implies that there exists a homomorphism which sends the initial vertex of P_a^1 to the initial vertex of P_b^1) and $P_a^2 \to P_b^2$ (which implies that there exists a homomorphism which sends the terminal vertex of P_a^2 to the terminal vertex of P_b^2). Hence $\mathscr{P}$ is indeed a path representation of R. ▌

A *tree representation* of $R = (X, \leqslant)$ is a collection of finite oriented trees $\mathscr{T} = \{T_x : x \in R\}$ such that

$$(\bigstar) \quad x \leqslant y \text{ if and only if there exists a homomorphism } T_x \to T_y.$$

The same technique can be used to prove that any finite dimensional poset can be represented by finite oriented trees.

THEOREM 3·3. *Any countable poset of finite dimension has a tree representation.*

Proof. Suppose R is a countable poset of dimension n, and suppose further that $L_1, L_2, \ldots, L_n$ are n linear extensions of R which realize R. Applying Corollary 3·1 we obtain path rpresentations $\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_n$ of $L_1, L_2, \ldots, L_n$ respectively. All the paths have height four, initial vertices of level zero, and all homomorphisms send initial vertices to initial vertices.

For each $i = 1, 2, \ldots, n$, let X_i be the path $X_i = 00\ldots0100\ldots0$ which starts with $i+1$ forward edges, followed by 1 backward edge, and then followed by $n-i+2$ forward edges. Let T be the tree obtained from the disjoint union of $X_1, X_2, \ldots, X_n$ by identifying their initial vertices. For each element $a \in R$, let P_a^i be the path in $\mathscr{P}_i$ which represent a. Let T_a be the tree obtained from the disjoint union of T and $P_a^1, P_a^2, \ldots, P_a^n$ by identifying the initial vertex of P_a^i with the terminal vertex of X_i (which is a vertex of degree 1 in T) for each $i = 1, 2, \ldots, n$. A similar argument as in the proof of Corollary 3·2 will show that $\mathscr{T} = \{T_a : a \in R\}$ is a tree representation of R. ∎

Recall that $\mathscr{S}$ is the poset of all finite oriented paths and their homomorphisms. Since every finite poset has a path representation (cf. Theorem 1·4), i.e., every finite poset can be embedded into $\mathscr{S}$, we see that $\mathscr{S}$ has infinite dimension. Thus there are poset of infinite dimension which has a path representation. (The poset $\mathscr{S}$ is such a poset.) The following is another concrete example of an infinite dimensional poset which has a path representation.

Let X be a countable antichain. Denote by $\mathscr{F}$ the set of all finite subsets F of X. Let R be the poset $(X \cup \mathscr{F}, \leqslant)$ with the partial order $a \leqslant b$ if and only if $a \in X$ and $b \in \mathscr{F}$ and $a \in b$. Then R has infinite dimension, cf. [**27**]. We now construct a path representation of R.

Since the subposet of R induced by X, which we also denote by X, has dimension 2, there is a path representation $\mathscr{P} = \{P_x : x \in X\}$ of X by Theorem 3·2. The set $\mathscr{F}$ forms a countable antichain in R, and hence also has a path representation, say $\mathscr{Q} = \{Q_y : y \in \mathscr{F}\}$. For each $x \in X$, denote by P'_x the path obtained from P_x by replacing each edge by the path 0001100110000111001111; and for each $y \in \mathscr{F}$, denote by Q'_y the path obtained from Q_y by replacing each edge by the path 0001100001110011001111; and finally let Q''_y be the path obtained from Q'_y and $\{P'_x : x \in y\}$ by connecting them with the zigzag path 01 (in arbitrary order). It is not difficult to show that $\mathscr{P} = \{P'_x : x \in X\} \cup \{Q''_y : y \in \mathscr{F}\}$ is a path representation of R.

Open question. Is there a countable poset which does not have a path representation. How about tree representation?

4. *On-line representability and universal representability*

Some of the results of representation of posets in the previous section can be strengthened to 'on-line' representation. By on-line representability of a class of posets R, we mean that one can construct a representation of any poset R in this class under the circumstances that the elements of R are revealed one by one. The on-line representation of a class of posets can be considered as a game between two players A and B. Player A choose a poset R in the specified class, and reveal the elements of R one by one to player B. Whenever an element x of R is revealed to B, the relations among x and previously revealed elements are also revealed. Player B is required to construct an oriented path (or oriented tree in case of tree presentation) to represent x before the next element is revealed. Player B wins the game if he succeeds in constructing a representation of R. The class of posets is on-line representable if player B has a winning strategy.

It is easy to see that the Density Theorem implies that the class of countable linear ordered sets is on-line representable.

We do not know whether or not the class of all finite (or countable) posets is on-line representable. It turns out that this problem is equivalent to the representability of all countable posets.

THEOREM 4·1. *The following three statements are equivalent.* (1) *Every countable poset is path* (*tree*) *representable.* (2) *The class of all finite posets is path* (*tree*) *on-line representable.* (3) *The class of all countable posets is on-line path* (*tree*) *representable.*

Proof. It is obvious that (3) ⇒ (1). To see that (2) ⇒ (3), we note that if player B has a winning strategy for the class of all finite posets, then this strategy can be applied to construct an on-line representation of any countable poset R, because at each step the revealed part of R induces a finite poset. We now prove that (1) ⇒ (2): We construct a (homogeneous) countable poset P as follows:

Suppose a k-element subposet P_n of P has been constructed, and suppose there are m different $k+1$-element posets, say $Q_1, Q_2, \ldots, Q_m$, which contains P_n. Let P_{n+1} be obtained from P_n by adding m elements, say $v_1, v_2, \ldots, v_m$, such that $P_n \cup \{v_i\}$ is isomorphic to Q_i. Any two added elements are comparable only if they are forced to be comparable. We repeat the above process of construction countably many times, and obtain the countable poset P.

It is easy to see from the construction of P that if R is a subposet of P and R' is an extension of R, then there is a subposet R'' of P which contains R and is isomorphic to R' and the isomorphism from R' to R'' fixes each vertex of R. Therefore if P has a path representation, then this representation can be used to construct an on-line representation of any finite (or countable) poset. ▌

We have not been able to prove the on-line path representability even for posets of dimension 2.

We now introduce another representability of poset, the *universal representability*, which is weaker than the on-line path representability.

Let R be a finite poset, and let $\mathscr{P} = \{P_x : x \in R\}$ be a path representation of R. We say the representation is *universal* if for any finite poset Q containing R there is a path representation $\mathscr{P}' = \{P'_x : x \in Q\}$ with $P'_x = P_x$ for $x \in R$. We say R is universal representable if there is a universal representation of R.

THEOREM 4·2. *Each linear order has a universal path representation.*

Proof. Let $C = \{c_1 < c_2 < \ldots < c_n\}$ be a linear order. Let P_{c_i} be the directed path of length $i+4$. Then $\mathscr{P} = \{P_{c_i} : i = 1, 2, \ldots, n\}$ is a path representation of C. We now show that this representation is universal.

Let Q be a finite poset which contains C. Let $Q' = Q \backslash C$ be the poset induced by the elements of Q not in C. By Theorem 1·4 there is a path representation $\mathscr{P}'$ of Q', with element $x \in Q'$ represented by path $P_x \in \mathscr{P}$, such that for any $x < y$ in Q', there is a homomorphism of P_x to P_y which takes the initial vertex of P_x to the initial vertex of P_y, and takes the terminal vertex of P_x to the terminal vertex of P_y. For each $P_x \in \mathscr{P}'$, replace each edge of P_x by the oriented path $e = 110111001000$, we obtain another path F_x, which has height 4 and with initial and terminal vertices of level zero. Let $\mathscr{F} = (F_x : x \in Q'\}$. Then $\mathscr{F}$ is also a path representation of Q', and for $x < y$, there is a homomorphism of F_x to F_y which takes the initial vertex of F_x to the initial vertex of F_y, and takes the terminal vertex of F_x to the terminal vertex of F_y. For each $x \in Q'$, let c_{l_x} be the least element of C greater than x, and let c_{m_x} be the greatest element of C less than x. (In case no element of C is greater than x, let $l_x = n+1$, and in case no element of C is less than x, let $m_x = 0$). For each $x \in Q'$, we construct a path W_x as follows:

Let m be an even integer which is greater than the length of F_y for any $y \in Q'$. Let E_x be the path $0101\ldots0100\ldots0$ which starts with a zigzag path of length m, then followed by a directed path of length m_x+4. Let I_x be the path $001001001\ldots 0010101\ldots01$ which starts with two forward edges, then one backward edge, then two forward edges, then one backward edge, ..., until the net length of the path reaches $l_x - m_x \geqslant 1$, then followed by a zigzag path of length m. W_x is the concatenation $I_x F_x E_x$ of the three paths I_x, F_x, E_x.

We now show that $\{W_x : x \in Q'\} \cup \{P_{c_i} : i = 1, 2, \ldots, n\}$ is a path representation of Q. Thus we need to show that for any $x, y \in Q$, we have $x < y$ if and only if $W_x \to W_y$. If $x, y \in C$ then this is certainly true. Now suppose $x = c_i \in C$ and $y \in Q'$. If $x = c_i < y$ then $i \leqslant m_y$. This implies that $W_x \to W_y$ as W_y contains a directed path of length $m_y + 4$. If $x = c_i \nless y$ then $i > m_y$. This implies that $W_x \nrightarrow W_y$ as W_y does not contain a directed path of length $i+4$. If $x = c_i > y$ then $i \geqslant l_y$. This implies that $W_y \to W_x$, as W_x is a directed path of length $i+4$ and W_y has height $l_y + 4$.

It remains to consider the case that both x, y are not in C. If $x < y$ then $m_x \leqslant m_y$ and $l_x \geqslant l_y$. Thus we have $W_x \to W_y$, because $I_x \to I_y, F_x \to F_y, E_x \to E_y$, and furthermore the homomorphisms take the common vertex of I_x and F_x to the common vertex of I_y and F_y, and take the common vertex of F_x and E_x to the common vertex of F_y and E_y. Suppose x and y are noncomparable. Then $l_x > m_y$ and $l_y > m_x$. Suppose there is a homomorphism h of W_x to W_y. Then $h(W_x)$ can not be contained in $F_y \cup E_y$, as W_x has height $l_x + 4$ and $F_y \cup E_y$ has height $m_y + 4$. Also $h(W_x)$ cannot be contained in $I_y \cup F_y$ because W_x contains a directed path of length 4, and $I_y \cup F_y$ does not. By considering the two long zigzag paths in W_x and W_y, we conclude that $h(F_x) = F_y$. Thus $F_x \to F_y$, contrary to our assumption that x and y are not comparable (which implies that $F_x \nrightarrow F_y$). ▌

We do not know whether or not there are other finite posets which have universal path representation. It would be interesting to know whether or not the two element antichain has a universal path representation.

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