Colouring graphs with bounded generalized colouring number

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Abstract

Given a graph G and a positive integer p, $\chi_p(G)$ is the minimum number of colours needed to colour the vertices of G so that for any $i \leq p$, any subgraph H of G of tree-depth i gets at least i colours. This paper proves an upper bound for $\chi_p(G)$ in terms of the k-colouring number $\mathrm{col}_k(G)$ of G for $k = 2^{p-2}$. Conversely, for each integer k, we also prove an upper bound for $\mathrm{col}_k(G)$ in terms of $\chi_{k+2}(G)$. As a consequence, for a class $\mathcal{K}$ of graphs, the following two statements are equivalent:

(a) For every positive integer p, $\chi_p(G)$ is bounded by a constant for all $G \in \mathcal{K}$.

(b) For every positive integer k, $\mathrm{col}_k(G)$ is bounded by a constant for all $G \in \mathcal{K}$.

It was proved by Nešetřil and Ossona de Mendez that (a) is equivalent to the following:

(c) For every positive integer q, $\nabla_q(G)$ (the grad, i.e., the greatest reduced average density, of G with rank q) is bounded by a constant for all $G \in \mathcal{K}$.

This implies that (b) and (c) are also equivalent. We shall give a direct proof of this equivalence, by introducing $\nabla_{q-(1/2)}(G)$ and by showing that there is a function F_k such that $\nabla_{(k-1)/2}(G) \leq (\mathrm{col}_k(G))^k \leq F_k(\nabla_{(k-1)/2}(G))$. This gives an alternate proof of the equivalence of (a) and (c).

1 Introduction

The *colouring number* $\mathrm{col}(G)$ of a graph G is the minimum integer k such that there is a linear ordering L of the vertices of G for which each vertex v has *back-degree* at

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most $k-1$, i.e., v has at most $k-1$ neighbours u with $u <_L v$. It is well-known that for any graph G, the chromatic number $\chi(G)$ satisfies $\chi(G) \leq \mathrm{col}(G)$. Indeed, given a linear ordering L of G for which each vertex has back-degree at most $k-1$, then with k colours, one can simply colour the vertices in the order of L and with arbitrary legal colour at each step. Although the upper bound $\chi(G) \leq \mathrm{col}(G)$ is not very good in general, this provides a common tool in proving many classes of graphs having bounded chromatic number.

Many variations of the chromatic number of graphs have been studied extensively in the literature. We list a few examples here. An *acyclic colouring* f of a graph G colours the vertices of G in such a way that each colour class is an independent set, and the union of every two colour classes induces a forest. The *acyclic chromatic number* $\chi_a(G)$ of a graph G is the least number of colours needed in an acyclic colouring of G. A *star colouring* f of a graph G colours the vertices in such a way that each colour class is an independent set and the union of every two colour classes induces a star forest. The *star chromatic number* $\chi_s(G)$ of G is the least number of colours needed in a star colouring of G. A *low tree-width colouring* of a graph G is a generalization of acyclic colouring. Given an integer k, one wants to colour the vertices of G in such a way that the union of any $i \leq k$ colour classes induces a graph of tree-width at most $i-1$. Let $\chi(k, G)$ be the minimum number of colours needed in such a colouring. Then $\chi(2, G)$ is the acyclic chromatic number of G.

For the many variants of the chromatic number, are there corresponding generalizations of the colouring number? To be precise, are there linear orderings of $V(G)$ which can be used to derive meaningful upper bounds on these parameters? There are indeed some generalizations of the colouring number of a graph that are studied in the literature. These include the *arrangability* used in [1] in the study of Ramsey numbers of graphs, the *admissability* used in [4] in the study of the game chromatic number of graphs, and the *rank* used in [3] also in the study of game chromatic number of graphs. However, for the purpose stated above, it seems that a natural generalization is the k-colouring number of a graph introduced by Kierstead and Yang [5], which we define now.

For a graph G, let $\Pi(G)$ be the set of linear orderings of the vertices of G. For $L \in \Pi(G)$, the orientation $G_L = (V, E_L)$ of G with respect to L is obtained from G by orienting an edge xy of G from x to y if $y <_L x$. For a vertex u of G, let $N_G(u)$ be the set of neighbours of u in G, and let $N^+_{G_L}(u)$ the set of out-neighbours of u in G_L. Thus the colouring number of G can be defined as

$$\mathrm{col}(G) = 1 + \min_{L \in \Pi(G)} \max_{u \in V(G)} |N^+_{G_L}(u)|.$$

Definition 1.1 *Suppose $L \in \Pi(G)$ and x, y are vertices of G. We say x is* weakly k-accessible from y *if $x <_L y$ and there is an x-y-path P of length at most k (i.e, with at most k edges) and for any $z \in P$, $x <_L z$. If in addition every internal vertex z of P satisfies that $y <_L z$ then we say x is* k-accessible from y.

Let $Q_k(G_L, y)$ be the set of vertices that are weakly k-accessible from y and let $R_k(G_L, y)$ be the set of vertices that are k-accessible from y.

Definition 1.2 *The* k-colouring number $\mathrm{col}_k(G)$ *and the* weak k-colouring number $\mathrm{wcol}_k(G)$ *of* G *are defined as follows:*

$$\mathrm{col}_k(G) = 1 + \min_{L \in \Pi(G)} \max_{v \in V(G)} |R_k(G_L, v)|,$$

$$\mathrm{wcol}_k(G) = 1 + \min_{L \in \Pi(G)} \max_{v \in V(G)} |Q_k(G_L, v)|.$$

Kierstead and Yang introduced the k-colouring number of a graph mainly for the purpose of studying colouring games and marking games on graphs. They also proved that for any graph G, the acyclic chromatic number $\chi_a(G)$ of G is bounded from above by $\mathrm{col}_2(G)$. In this paper, we prove that the k-colouring number provides upper bounds for those variants of the chromatic number mentioned above, and for 'thousands' more other variants of chromatic number. However, we do not need to work with the variants of chromatic number one by one. Using a nice notion introduced by Nešetřil and Ossona de Mendez in [8], one can treat these variants altogether. Nešetřil and Ossona de Mendez observed that the colourings used in the definitions of many variants of chromatic number put lower bounds on the number of colours received by certain subgraphs. For example, in an acyclic colouring of a graph, it is required that each edge receives at least 2 colours (be a proper colouring) and each cycle receives at least 3 colours. In the star colouring, it is required that each edge receives at least 2 colours and each path on 4 vertices receives at least 3 colours. Generalizing this idea, Nešetřil and Ossona de Mendez in [8] introduced a new graph parameter $\chi(f, G)$ defined as follows: Let f be a *graph function*, which assigns to each graph H a positive integer $f(H) \leq |V(H)|$. We colour a graph G with N colours such that any subgraph H' of G isomorphic to H gets at least $f(H)$ colours. If $N = |V(G)|$, then such a colouring exists. The smallest integer N for which such a colouring exists is denoted by $\chi(f, G)$. This definition takes many variations of chromatic number as its special cases. For example, if $f_1(C_n) = 2$ for any n, and $f_1(H) = 1$ otherwise, then $\chi(f_1, G)$ is the point-arboricity of G (i.e., the smallest size of vertex partition whose parts induces forests). If $f_2(K_2) = 2$ and $f_2(H) = 1$ otherwise, then $\chi(f_2, G)$ is the same as $\chi(G)$. If $f_3(K_2) = 2$, $f_3(C_n) = 3$ for any $n \geq 3$, and $f_3(H) = 1$ otherwise, then $\chi(f_3, G)$ is the acyclic chromatic number of G. If $f_4(K_2) = 2, f_4(P_4) = 3$ (where P_4 is the path on 4 vertices) and $f_4(H) = 1$ otherwise, then $\chi(f_4, G)$ is the star-chromatic number of G. If $f_5(K_{1,d}) = 2$ and $f_5(H) = 1$ otherwise, then $\chi(f_5, G)$ is the d-relaxed chromatic number of G.

Given any particular graph function f, we can ask the question whether $\chi(f, G)$ is bounded by a function of the k-colouring number of G for some k. However, instead of considering a particular graph function f, we can also ask what properties need to be satisfied by a graph function f, so that $\chi(f, G)$ is bounded by a function of the k-colouring number of G. This is how Nešetřil and Ossona de Mendez treated such problems. In [8], Nešetřil and Ossona de Mendez studied the problem for which graph functions f, the parameter $\chi(f, G)$ is bounded by a constant on any proper minor closed class of graphs, i.e., a class of graphs which is closed under minor, and which excludes at least one finite graph. This question was answered there by introducing the notion of tree-depth of a graph. It was proved in [8] that $\chi(f, G)$ is bounded by a constant for each proper minor closed class of graphs if and only if there is a constant p such that $f(H) \leq \min\{p, \mathrm{td}(H)\}$, where $\mathrm{td}(H)$ is the tree-depth of H (see definition in Section 2).

As proper minor closed class of graphs have bounded k-colouring number, for the problem mentioned above, it is necessary to require that the graph functions f satisfy the condition that $f(H) \leq \min\{p, \mathrm{td}(H)\}$ for some constant p. The question of interest is whether or not for such graph functions f, the parameter $\chi(f, G)$ is bounded by a function of the k-colouring number of G for some k. The answer is affirmative, and surprisingly (or naturally) the proof is easy. It is obvious that if f, f' are graph functions such that $f'(H) \leq f(H)$ for every H, then $\chi(f', G) \leq \chi(f, G)$. Thus for the question above, we only need to consider graph functions f_p defined as $f_p(H) = \min\{p, \mathrm{td}(H)\}$. We denote $\chi(f_p, G)$ by $\chi_p(G)$. Equivalently, $\chi_p(G)$ is defined to be the minimum number N for which there is an N-colouring of $V(G)$ so that for each $i \leq p$, any subgraph of tree-depth i get at least i colours. We shall prove that for any integer p, $\chi_p(G)$ is bounded from above by $(\mathrm{col}_{2^{p-2}}(G))^{2^{p-2}}$. Conversely, we also prove that for each k, $\mathrm{col}_k(G)$ is bounded from above by a function of $\chi_{k+2}(G)$. Thus for any class $\mathcal{K}$ of graphs, we have the equivalence of statements (a) and (b) as stated in the abstract.

(a) For every integer p, there is an integer $\phi(p)$ such that $\chi_p(G) \leq \phi(p)$ for all $G \in \mathcal{K}$.

(b) For every positive integer p, there is an integer $\psi(k)$ such that for every $G \in \mathcal{K}$, $\mathrm{col}_k(G) \leq \psi(k)$.

The question that for which class $\mathcal{K}$ of graphs, the parameter $\chi_p(G)$ is bounded on $\mathcal{K}$ has been studied and answered by Nešetřil and Ossona de Mendez in [9] already. They introduced the grad of a graph G with rank k (see definition in Section 3), denoted by $\nabla_k(G)$, and proved that (a) is equivalent to the following statement.

(c) For every positive integer p, there is an integer $\kappa(p)$ such that for every $G \in \mathcal{K}$, $\nabla_p(G) \leq \kappa(p)$, where $\nabla_p(G)$ is the grad of G with rank p.

This implies that (b) and (c) are equivalent. In this paper, we shall provide a direct proof of the equivalence of (b) and (c). This provides an alternate proof of the equivalence of (a) and (c). The proofs in this paper are short and easy, and provide an alternate point of view to look at these interesting classes of graphs, and the interesting parameter $\chi(f, G)$.

2 Bounding $\chi_p(G)$ and $\mathrm{col}_k(G)$

A *rooted tree* is a tree with a vertex designated as its root. The *level* of a vertex v in a rooted tree T is the number of vertices on the path connecting the root and v. In particular, the root is the only vertex with level 1. The *height* of a rooted tree is the maximum level of a vertex. A *rooted forest* is the disjoint union of rooted trees, and the height of a rooted forest F is the maximum of the height of the rooted trees contained in F. A rooted forest defines a partial order on its vertices: $x \leq_F y$ if x is an ancestor of y. The comparability graph of this partial order is the *closure* of F and is denoted by $\mathrm{clos}(F)$.

Definition 2.1 *For a graph G, the* tree-depth *of G is the minimum height of a rooted forest F such that G is isomorphic to a subgraph of* $\mathrm{clos}(F)$.

Tree-depth of a graph is related to many other graph parameters. In particular, for a graph on n vertices, it is known [9, 10, 11] that $\mathrm{tw}(G)+1 \leq \mathrm{td}(G) \leq \mathrm{tw}(G)\log_2 n$. Recall that a *$k$-tree* is a graph whose vertices can be ordered as $x_1, x_2, \cdots, x_n$ in such a way that $\{x_1, x_2, \cdots, x_k\}$ induces a clique and for any $i \geq k+1$, the set $\{x_j : j < i, x_j \sim x_i\}$ induces a clique of order k. The *tree-width* of a graph G, denoted by $\mathrm{tw}(G)$, is the minimum k such that G is a subgraph of a k-tree. Readers are referred to [8] for basic properties of tree-depth of graphs. We just list the following property of tree-depth of a graph G which can be proved easily by induction on $|V(G)|$.

Lemma 2.2 *[8] Suppose G is a graph with components $G_1, \cdots, G_t$. Then*

$$\mathrm{td}(G) = \begin{cases} 1, & \text{if } |V(G)| = 1, \\ 1 + \min_{v \in V(G)} \mathrm{td}(G - v), & \text{if } |V(G)| \geq 2 \text{ and } t = 1, \\ \max\{\mathrm{td}(G_1), \cdots, \mathrm{td}(G_t)\}, & \text{if } t \geq 2, \end{cases}$$

Another concept related to the tree-depth of a graph is the centered colouring of a graph.

Definition 2.3 *A colouring of a graph G is called a* centered colouring *if for any connected subgraph H, some colour $c(H)$ appears exactly once in H.*

Note that centered colouring is necessarily proper. It is proved in [8] that for any graph G, the minimum number of colours in a centered colouring of G is equal to $\mathrm{td}(G)$. For our purpose, we are interesting in colouring of graphs G so that any subgraph H of G receives at least $\min\{p, \mathrm{td}(H)\}$ colours for some integer p. Such colourings are related to the p-centered colouring of G.

Definition 2.4 *For a positive integer p, a p-*centered colouring *of a graph G is a vertex colouring of G such that for any connected subgraph H of G, either some colour $c(H)$ appears exactly once in H, or H receives at least p colours.*

The following lemma can be proved easily by using Lemma 2.2.

Lemma 2.5 *[8] Let G be a graph, and c be a p-centered colouring of G. Then any subgraph H of G with $\mathrm{td}(H) \leq p$ gets at least $\mathrm{td}(H)$ colours.*

Now we derive an upper bound for $\chi_p(G)$ in terms of $\mathrm{col}_k(G)$.

Theorem 2.6 *If G is a graph with $\mathrm{wcol}_{2^{p-2}}(G) \le m$, then G has a p-centered colouring using at most m colours.*

Proof. Let $L = v_1 v_2 \cdots v_n$ be an ordering of the vertices of G with $\max_{v \in V(G)} |Q_{2^{p-2}}(G_L, v)| \le m-1$. Colour the vertices of G greedily, using the order L, so that the colour assigned to v is distinct from colours assigned to vertices in $Q_{2^{p-2}}(G_L, v)$. As $|Q_{2^{p-2}}(G_L, v)| \le m-1$, m colours suffice. We claim that such a colouring is a p-centered colouring. Let H be a connected subgraph of G. Let v be the minimum vertex of H with respect to L. If the colour $c(v)$ of v appears exactly once in H, then we are done.

Assume $c(v)$ is used more than once in H. We shall prove that H uses at least p colours. Let $u \ne v$ be a vertex of H with $c(u) = c(v)$, and let $P_0 = (v = v_0, v_1, v_2, \cdots, v_q = u)$ be a path in H connecting v and u. We must have $q > 2^{p-2}$, for otherwise v is weakly 2^{p-2}-accessible from u, i.e., $v \in Q_{2^{p-2}}(G_L, u)$, and we should have $c(u) \ne c(v)$.

Let $u_0 = v$, and let $P_1 = (v_1, \cdots, v_{2^{p-2}})$ be the subpath of P_0. Observe that no vertex of P_1 uses colour $c(u_0)$ and P_1 contains 2^{p-2} vertices. Assume $0 \le j \le p-2$, and a vertex u_j of P_j, and a subpath P_{j+1} of P_j are chosen such that the following hold:

- No vertex of P_{j+1} uses the colour of $c(u_j)$.
- P_{j+1} contains at least 2^{p-j-2} vertices.

Let u_{j+1} be the minimum vertex of P_{j+1} with respect to L. If $j \le p-3$, then let P_{j+2} be the largest component of $P_{j+1} - u_{j+1}$. Then u_{j+1} is weakly 2^{p-2}-accessible from each vertex of P_{j+2}, and hence no vertex of P_{j+2} uses the colour $c(u_{j+1})$. Moreover P_{j+2} is a path containing at least 2^{p-j-3} vertices. So we can repeat the process until $j = p-2$, and we obtain vertices $u_0, u_1, \cdots, u_{p-1}$. By the choices of these vertices, their colours are distinct. So H uses at least p colours. □

Lemma 2.7 below was proved in [5].

Lemma 2.7 *For any graph G, for any linear ordering L of G,*

$$\max_{v \in V(G)} |Q_k(G_L, v)| \le \max_{v \in V(G)} |R_k(G_L, v)|^k.$$

Consequently, we have

$$\mathrm{col}_k(G) \le \mathrm{wcol}_k(G) \le (\mathrm{col}_k(G))^k.$$

Combining Lemma 2.7, Theorem 2.6 and Lemma 2.5, we have the following corollary.

Corollary 2.8 *For any graph G, for any integer p,*

$$\chi_p(G) \leq (\mathrm{col}_{2^{p-2}}(G))^{2^{p-2}}.$$

□

The following corollary was proved by DeVos et al. in [2].

Corollary 2.9 *[2] For every graph K and integer $j \geq 2$, there is an integer $i = i(K, j)$ such that every graph with no K-minor has a vertex partition into i parts such that any $j' \leq j$ parts form a graph with tree-width at most $j' - 1$.*

Proof. It was proved in [5] that every topologically closed class of graphs has bounded k-colouring numbers. The same proof actually shows that for any graph K, the class of graphs with no K-minor has bounded k-colouring numbers for any k. In particular, there is an integer m such that $\mathrm{col}_{2^{j-2}}(G) \leq m$ for any graph G with no K-minor. By Corollary 2.8, $V(G)$ can be partitioned into m^m parts such that for any $j' \leq j$, any j' parts form a graph with tree-depth at most j', and hence with tree-width at most $j' - 1$. □

Next we prove that for any integer p, $\mathrm{col}_p(G)$ is bounded from above by a function of $\chi_{p+2}(G)$.

Theorem 2.10 *Suppose p is a positive integer, G is a graph with $\chi_{p+2}(G) = m$. Then $\mathrm{col}_p(G) \leq 1 + (2\binom{m}{p+1}p)^2$.*

Proof. Let c be an m-colouring of $V(G)$ such that for any $i \leq p+2$, any subgraph H of G of tree-depth i get at least i colours. Let $[m] = \{1, 2, \cdots, m\}$ be the colour set, and let $\binom{[m]}{p+1}$ be the family of $(p+1)$-subsets of $[m]$. For each $I \in \binom{[m]}{p+1}$, $c^{-1}(I)$ induces a subgraph G_I of tree-depth $\leq p+1$. Let F_I be the rooted forest of height $p+1$ such that G_I is a subgraph of $\mathrm{clos}(F_I)$. Let

$$H = \bigcup_{I \in \binom{[m]}{p+1}} \mathrm{clos}(F_I).$$

Orient the edges of $\mathrm{clos}(F_I)$ so that xy is oriented from x to y if y is an ancestor of x. Thus $\mathrm{clos}(F_I)$ has maximum out-degree at most p. The union of the orientations of all the $\mathrm{clos}(F_I)$'s induce an orientation of H in which each vertex has out-degree at most $\binom{m}{p+1}p$ (if an edge e of H appears in more than one of the $\mathrm{clos}(F_I)$'s, then we arbitrarily choose one of the $\mathrm{clos}(F_I)$ to determine the orientation of e). This implies that any subgraph H' of H has average degree at most $2\binom{m}{p+1}p$. Therefore H has colouring number at most $2\binom{m}{p+1}p + 1$. Let $M = 2\binom{m}{p+1}p$ and let L be a linear ordering of the vertices of H so that for each vertex v, $|R_1(H_L, v)| \leq M$.

Now we prove that for each vertex v, $|R_p(G_L, v)| \leq M^2$. Let v be a fixed vertex of G. For each $u \in R_p(G_L, v)$, select a v-u-path P_{uv} of length at most p such that all the interior vertices z of P_{uv} satisfies $v <_L z$. By definition of $R_p(G_L, v)$ such a path exists. As $|V(P_{uv})| \leq p+1$, there is an $I \in \binom{[m]}{p+1}$ such that $c(P_{uv}) \subseteq I$. For each $u \in R_p(G_L, v)$, fix $I_u \in \binom{[m]}{p+1}$ so that $c(P_{uv}) \subseteq I_u$. Let $z_u \in P_{uv}$ be the vertex of P_{uv} which has minimum level in F_{I_u}. If $z_u = v$, then v is an ancestor of u in F_{I_u} and hence $uv \in \mathrm{clos}(F_{I_u})$ implying that $uv \in H$. So we have

$$|\{u \in R_p(G_L, v) : z_u = v\}| \leq |R_1(H_L, v)| \leq M.$$

Similarly, if $z_u = u$, we have $\{u \in R_p(G_L, v) : z_u = u\} \subseteq R_1(H_L, v)$. Assume $z_u \neq v, u$. Then z_u is an ancestor of v and u in F_{I_u}. As v has at most p ancestors, and $u \in R_1(H_L, z_u)$, we conclude that

$$|\{u \in R_p(G_L, v) : z_u \neq v\}| \leq \binom{m}{p+1} pM = M^2/2.$$

Hence

$$|R_p(G_L, v)| \leq M + M^2/2 < M^2.$$

□

A class $\mathcal{K}$ of graphs is said to have *low tree-depth colouring* if for any positive integer p, $\chi_p(G)$ is bounded from above by a constant $\phi(p)$ for all $G \in \mathcal{K}$. We say $\mathcal{K}$ have *bounded generalized colouring number* if for any positive integer p, $\mathrm{col}_p(G)$ is bounded from above by a constant $\psi(p)$ for all $G \in \mathcal{K}$. Combining Corollary 2.8 and Theorem 2.10, we have the following corollary.

Corollary 2.11 *A class $\mathcal{K}$ of graphs has low tree-depth colouring if and only if it has bounded generalized colouring number.* □

3 Graphs with bounded expansion

Let G be a graph and let $\mathcal{P} = \{V_1, V_2, \cdots, V_n\}$ be a family of disjoint subsets of $V(G)$ such that each V_i induces a connected subgraph of G. We denote by $G/\mathcal{P}$ the graph with vertex set $\{1, 2, \cdots, n\}$ such that ij is an edge if and only if there is an edge in G connecting a vertex of V_i and a vertex of V_j. A graph G is a *minor* of a graph G' if there is a family $\mathcal{P}$ of disjoint subsets of $V(G')$ such that G is isomorphic to a subgraph of $G'/\mathcal{P}$. A class $\mathcal{K}$ of graphs is *minor closed* if for every $G' \in \mathcal{K}$ and for every minor G of G', we have $G \in \mathcal{K}$. A class $\mathcal{K}$ of graphs is *proper minor closed* if it is minor closed and excludes at least one finite graph (and hence excludes K_n for some integer n). Proper minor closed classes of graphs are of fundamental importance, as the chromatic number of graphs and many of its variations are bounded for proper minor closed classes of graphs, even though such parameters are usually unbounded for other very restrictive classes of graphs, for example, for graphs of large girth. Besides proper minor closed classes of graphs, the chromatic number and many of its variants are bounded for classes of graphs of bounded maximum degree, although the later is not a subclass of any proper minor closed class of graphs. The classes of graphs of bounded expansion, introduced recently by Nešetřil and Ossona de Mendez [9], takes both proper minor closed classes of graphs and classes of graphs of bounded maximum degree as subclasses.

For a connected graph G, the *radius* of G is defined as

$$\rho(G) = \min_{v \in V(G)} \max_{x \in V(G)} d(x, v).$$

If $v \in V(G)$ and $\max_{x\in V(G)} d(x,v) = \rho(G)$, then v is a *center* of G. Suppose $\mathcal{P} = \{V_1, V_2, \cdots, V_n\}$ is a partition of $V(G)$ such that each $G[V_i]$ is a connected graph. We shall denote by v_i a fixed center of $G[V_i]$. Let

$$\rho(\mathcal{P}) = \max\{\rho(G[V_i]) : i = 1, 2, \cdots, n\}.$$

Suppose r is a positive integer. The *grad of* G *with rank* r is defined as

$$\nabla_r(G) = \max\left\{\frac{|E(G/\mathcal{P})|}{|\mathcal{P}|} : \rho(\mathcal{P}) \le r\right\}.$$

The *grad of* G *with rank* $r - (1/2)$ is defined as

$$\nabla_{r-(1/2)}(G) = \max\left\{\frac{|E'(G/\mathcal{P})|}{|\mathcal{P}|} : \rho(\mathcal{P}) \le r\right\},$$

where $V_iV_j \in E'(G/\mathcal{P})$ if there is a vertex $x \in V_i$ adjacent to a vertex $y \in V_j$ and $d_{V_i}(v_i, x) + d_{V_j}(v_j, y) \le 2r - 1$.

Definition 3.1 *A class $\mathcal{K}$ of graphs is said to have bounded expansion if there is a function $\phi : Z \to R$ such that for any graph $G \in \mathcal{K}$ and for any positive integer r, $\nabla_r(G) \le \phi(r)$.*

It is well-known [7, 12] that if K_n is not a minor of G, then the average degree of G is bounded by $cn\sqrt{\log n}$. Hence if $\mathcal{K}$ is a proper minor closed class of graphs then $\mathcal{K}$ has bounded expansion as $\nabla_r(G) \le cn\sqrt{\log n}/2$ for all r. If G is a graphs with maximum degree at most d, then it is obvious that $G/\mathcal{P}$ has maximum degree at most d^{r+1} if $\rho(\mathcal{P}) \le r$. Thus if $\mathcal{K}$ is a family of graphs of bounded maximum degree, then it is a class of graphs of bounded expansion with $\phi(r) = d^{r+1}$. So the classes of graphs of bounded expansion is a common generalization of the two very different types of graph classes: proper minor closed class and class of graphs of bounded maximum degree. Nešetřil and Ossona de Mendez [9] proved the following result.

Theorem 3.2 *[9] A class $\mathcal{K}$ of graphs has bounded expansion if and only if it has low tree-depth colouring.*

It follows from Theorem 3.2 and Corollary 2.11 that a class of graphs has bounded expansion if and only if it has bounded generalized colouring number. In this section, we present a direct proof of this fact, and present explicit upper bounds for grad of G of rank r in terms of its k-colouring number for some integer k; and explicit upper bounds for the k-colouring number of a graph G in terms of its grad of rank r for some r.

Lemma 3.3 *For any graph G, for any integer k,*

$$\nabla_{(k-1)/2}(G) + 1 \leq \mathrm{wcol}_k(G) \leq (\mathrm{col}_k(G))^k.$$

Proof. Consider a linear order L on the vertex set of G such that $\max_{v \in V(G)} |Q_k(G_L, v)| = \mathrm{wcol}_k(G) - 1$. Let $\mathcal{P} = \{V_1, V_2, \cdots, V_n\}$ be a witness of $\nabla_{(k-1)/2}(G)$. Let H be the subgraph obtained from G by replacing each $G[V_i]$ by a tree T_i such that for each $x \in V_i$, $d_{T_i}(v_i, x) = d_{G[V_i]}(v_i, x)$, and that there is between V_i and V_j one edge xy with $d_{T_i}(v_i, x) + d_{T_j}(v_j, y) \leq k-1$, if such an edge exists. Then we have

$$\frac{|E(H/\mathcal{P})|}{|\mathcal{P}|} = \nabla_{(k-1)/2}(H) = \nabla_{(k-1)/2}(G).$$

For V_i, V_j adjacent in $H/\mathcal{P}$, orient the edge V_iV_j from V_i to V_j (in $H/\mathcal{P}$) if the minimum vertex $m_{i,j}$ of the (unique) v_i-v_j-path in $H[V_i \cup V_j]$ (with respect to the linear order L) belongs to V_i. In such a case, $m_{i,j}$ is weakly k-accessible from v_j. Moreover, if $i \neq i'$ then $m_{i,j} \neq m_{i',j}$ as X_i and $X_{i'}$ are disjoint sets. It follows that X_j has indegree at most $|Q_k(G_L, v_j)|$ in $H/\mathcal{P}$, and hence $\nabla_{(k-1)/2}(G) \leq \mathrm{wcol}_k(G) - 1$. □

Lemma 3.4 *Suppose G is a graph and k is a positive integer. Let $p = (k-1)/2$ and $\nabla_p(G) \leq m$. Then*

$$\mathrm{col}_k(G) \leq 1 + q_k,$$

where q_k is defined as $q_1 = 2m$ and for $i \geq 1$, $q_{i+1} = q_1 q_i^{2i^2}$.

Proof. The proof of this lemma is similar to the proof of Theorem 4 in [5]. Suppose G is a graph, k is a positive integer, $p = (k-1)/2$ and $\nabla_p(G) \leq m$. We shall construct a linear ordering L on the vertices of G so that for each vertex u, $|R_k(G_L, u)| \leq q_k$, where q_k is defined in the statement of the lemma.

The linear ordering $L = x_1 x_2 \cdots x_n$ is constructed recursively. Suppose that we have constructed the subsequence $x_{i+1} x_{i+2} \cdots x_n$ of L. (If $i = n$, then the sequence is empty.) Let $M = \{x_{i+1}, x_{i+2}, \cdots, x_n\}$ and let $U = V - M$. Let Ω be the probability space, where each point in Ω is a graph $H = (U, F)$. For each pair uv of vertices in

U for which there is a u-v-path P of length at most k with all interior vertices in M, choose one such path P_{uv}. For $z \in M$, let

$$S_z = \{uv : \{u, v\} \subset U, z \in V(P_{uv})\}.$$

Label each $z \in M$ with a random element chosen from S_z. If $S_z = \emptyset$, then z is unlabeled. Let F be the set of pairs uv such that either $uv \in E$ or for all $z \in V(P_{uv})$, z is labeled uv. So if uv is an edge of G, then $uv \in F$ with probability 1. Otherwise,

$$Pr(uv \in F) = \prod_{z \in M \cap V(P_{uv})} \frac{1}{|S_z|}.$$

Let $E[d_H(u)]$ be the expected value of the degree of u in H. Choose $x_i \in U$ so that $E[d_H(x_i)]$ is minimum.

Let H be a random graph defined as above. For each $uv \in F - E$, partition the vertices in P_{uv} into two parts $A(uv, u), A(uv, v)$ so that $z \in A(uv, u)$ if $d_{P_{uv}}(u, z) < d_{P_{uv}}(v, z)$, and $z \in A(uv, v)$ if $d_{P_{uv}}(v, z) < d_{P_{uv}}(u, z)$. If z is in the middle of P_{uv}, then arbitrarily put z in $A(uv, u)$ and $A(uv, v)$. Let $V_u = \cup_{v \in N_H(u)} A(uv, u)$. Then the V_u's are disjoint subsets of $V(G)$, and each induces a connected subgraph of radius at most $k/2$. Let $\mathcal{P} = \{V_u : u \in U\}$. Then $\rho(\mathcal{P}) \leq k/2$ and H is a subgraph of $G/\mathcal{P}$. Thus the minimum degree of H satisfies

$$\delta(H) \leq 2|E'(G/\mathcal{P})|/|\mathcal{P}| \leq 2m.$$

Therefore $E[d_H(x_i)] \leq q_1$.

We now prove by induction on $i \leq k$ that $|R_i(G_L, y)| \leq q_i$ for all vertices y. Fix the time when y was added to the final sequence of L. For $i = 1$, this is true because if $x \in R_1(y)$, then $x \in U$ and $xy \in F$ with probability 1, and $E[d_H(y)] \leq q_1$. Assume this is true for $i \leq t$ and we consider the case that $i = t + 1$. For each $z \in M$, $x \in R_{t+1}(y)$ and $xy \in S_z$, both x, y are in $Q_t(z)$. Thus by Lemma 2.7 and by induction hypothesis,

$$|S_z| \leq |Q_t(z)^2| \leq (q_t)^{2t}.$$

Therefore,

$$\begin{aligned} q_1 &\geq E[d_H(y)] \geq \sum_{x \in R_{t+1}(y)} Pr(xy \in F) \\ &= \sum_{x \in R_{t+1}(y)} \prod_{z \in M \cap V(P_{xy})} \frac{1}{|S_z|} \geq |R_{t+1}(y)|(q_t)^{-2t^2}. \end{aligned}$$

So

$$|R_{t+1}(y)| \leq q_1 (q_t)^{2t^2} = q_{t+1}.$$

□

Corollary 3.5 *There is a function F_k such that*

$$\nabla_{(k-1)/2}(G) + 1 \leq \mathrm{wcol}_k(G) \leq (\mathrm{col}_k(G))^k \leq F_k(\nabla_{(k-1)/2}(G).$$

□

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References

[1] , G. Chen and R. H. Schelp, Graphs with linearly bounded Ramsey numbers, *J. Combin. Theory Ser. B*, 57 (1993), 138-149.

[2] M. DeVos, G. Ding, B. Oporowski, D.P. Sanders, B. Reed, P.D. Seymour and D. Vertigan, Excluding any graph as a minor allows a low tree-width 2-coloring. *J. Combin. Theory Ser. B*, 91 (2004), 25-41.

[3] H.A. Kierstead, A simple competitive graph coloring algorithm, *J. Combin. Theory Ser. B*, 78(2000), 57-68.

[4] H.A.Kierstead and W. T. Trotter, Planar graph coloring with an uncooperative partner, *J. Graph Theory*, 18(1994), 569-584.

[5] H. A. Kierstead and D. Yang, Orderings on graphs and game coloring number, *Order* 20 (2003), 255-264.

[6] H. A. Kierstead and D. Yang, General coloring numbers, grad, and nonrepetitive colorings of graphs, *manuscript* 2007.

[7] A. Kostochka, On the minimum Hadwiger number for graphs with given average degree, *Metody Diskret. Analiz.*,(1982), no. 38, 37-58, in Russian, English translation: AMS Translations (2), 132(1986),15-32.

[8] J. Nešetřil and P. Ossona de Mendez, Tree depth, subgraph coloring and homomorphism bounds, *European Journal of Combinatorics*, 27(2006)(6), 1022-1041.

[9] J. Nešetřil and P. Ossona de Mendez, Grad and classes with bounded expansion I. Decompositions. *KAM-DIMATIA Series*, no. 2005-739.

[10] J. Nešetřil and P. Ossona de Mendez, Grad and Classes with Bounded Expansion II. Algorithmic Aspects, *KAM-DIMATIA Series*, no. 2005-740.

[11] J. Nešetřil and P. Ossona de Mendez, Linear time low tree-width partitions and algorithmic consequences STOC'06. *Proceedings of the 38th Annual ACM Symposium on Theory of Computing*, ACM Press, 2006, 39-400.

[12] A. Thomason, An extremal funcion for contractions of graphs, *Math. Proc. Cambridge Philos. Soc.*, no. 95, (1984), 261-265.