

Relaxed game chromatic number of graphs

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March 2000.[†]

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Abstract

This paper discusses a coloring game on graphs. Let k, d be non-negative integers and C a set of k colors. Two persons, Alice and Bob, alternately color the vertices of G with colors from C , with Alice having the first move. A color i is legal for an uncolored vertex x if by coloring x with color i , the subgraph

*This research was partially supported by the National Science Council under grant NSC88-2115-M-110-001.

[†]1991 *Mathematics Subject Classification*. 05C20, 05C35, 05C15

Key words and phrases. Relaxed game chromatic number, game chromatic number, game coloring number, outerplanar graph.

of G induced by those vertices of color i has maximum degree at most d . Each move of Alice or Bob colors an uncolored vertex with a legal color. The game is over if either all vertices are colored, or no more vertices can be colored with a legal color. Alice's goal is to produce a legal coloring which colors all the vertices of G , and Bob's goal is to prevent this from happening. We shall prove that if G is a forest, then for $k = 3, d \geq 1$, Alice has a winning strategy. If G is an outerplanar graph, then for $k = 6$ and $d \geq 1$, Alice has a winning strategy.

1 Introduction

This paper investigates a variation of the game chromatic number of a graph. The game chromatic number is defined through a two person game. Let $G = (V, E)$ be a graph, and let C be a set of k colors. Two persons, say Alice and Bob, alternately color the vertices of G with colors from the color set C , with Alice having the first move. A color i is legal for an uncolored vertex x if x has no neighbour colored with color i . In each move, Alice and Bob must color an uncolored vertex with a legal color. The game is over if either all the vertices are colored or no legal color is available for the uncolored vertices. Alice's goal is to color all the vertices of G , and Bob's goal is to prevent this from happening. The game chromatic number $\chi_g(G)$ of G is the least number of colors for which Alice has a winning strategy in coloring G . The concept of the game chromatic number of a graph was introduced in [2], and has attracted some recent attention [2, 3, 7, 8, 10, 12, 13, 14, 17, 18].

This paper discusses the relaxed game chromatic number. The parameter is also defined through a two person game. The rules are almost the same as above. The only difference is in the definition of a legal color for an uncolored vertex.

Suppose $d \geq 0$ is an integer. In a d -relaxed coloring game (played on a graph G with color set C), a color $i \in C$ is legal for an uncolored vertex $x \in V(G)$ if by coloring x with color i , each vertex of color i is adjacent to at most d vertices of color i . The d -relaxed game chromatic number $\chi_g^{(d)}(G)$ of G is the least cardinality of a color set C for which Alice has a winning strategy for the d -relaxed coloring game played on G with color set C .

The concept of the relaxed coloring game is a mixture of the concept of the coloring game and the concept of relaxed coloring (also called improper coloring or defective coloring) of graphs. Given two integers $k \geq 1$ and $d \geq 0$, a graph G is called (k, d) -colorable if its vertices can be colored with k colors in such a way that each vertex is adjacent to at most d vertices with the same color as itself. Relaxed coloring of graphs has been studied extensively in the literature [1, 4, 5, 6, 9, 15, 16]. It was shown in [4] (without using the Four Color Theorem) that every planar graph is $(3, 2)$ -colorable

and $(4, 1)$ -colorable, and every outerplanar graph is $(2, 2)$ -colorable and there exists an outerplanar graph that is not $(2, 1)$ -colorable.

A $(k, 0)$ -coloring of a graph G is the same as a k -coloring of G . Similarly, the 0-relaxed coloring game is the same as the coloring game, and the 0-relaxed game chromatic number of a graph G is the same as its game chromatic number. For convenience, we call a d -relaxed coloring game with k colors a (k, d) -coloring game.

It is natural that the study of the d -relaxed game chromatic number has some similarities with the study of the game chromatic number. However, there are also some substantial differences. A first glance at the definition seems to suggest that if $d' \geq d$, then $\chi_g^{(d')}(G) \leq \chi_g^{(d)}(G)$. However, this is not always true (although it is true in most known cases). For example, $\chi_g^{(0)}(K_{n,n}) = \chi_g(K_{n,n}) = 3$ (if $n \geq 2$), but $\chi_g^{(1)}(K_{n,n}) = n$.

In the study of game chromatic numbers of graphs, a very useful technique is to study the game coloring number, which gives an upper bound for the game chromatic number. It is defined through a two person game as follows: two persons, say Alice and Bob, alternately select vertices of G , to form a linear ordering L of the vertices of G , so that $x \preceq y$ in L if x is selected before y . The *back degree* of a vertex x with respect to L is the number of neighbours of x which precedes x in L , and the *back degree of L* is the maximum of the back degrees of the vertices of G with respect to L . Alice's goal is to minimize the back degree of L , and Bob's goal is to maximize it. The game coloring number $\text{col}_g(G)$ of G is equal to $1 + k$, where k is the back degree of L when both Alice and Bob use their optimal strategies in playing the game.

It is easy to see [17] that $\chi_g(G) \leq \text{col}_g(G)$ for any graph G . And the argument that shows that $\chi_g(G) \leq \text{col}_g(G)$ also shows that $\chi_g^{(d)}(G) \leq \text{col}_g(G)$. For many classes of graphs, the best upper bounds for their game chromatic numbers are derived by studying their game coloring number. For example, for any forest T , $\chi_g(T) \leq \text{col}_g(T) \leq 4$ [8]; for every outerplanar graph G , $\chi_g(G) \leq \text{col}_g(G) \leq 7$ [10]; for every planar graph G , $\chi_g(G) \leq \text{col}_g(G) \leq 18$ [12, 17]; for every partial k -tree G , $\chi_g(G) \leq \text{col}_g(G) \leq 3k + 2$ [18]; etc.

In the study of the relaxed game chromatic number, besides the fact that $\chi_g^{(d)}(G) \leq \text{col}_g(G)$, it is not clear whether the game coloring number can be used to derive better upper bounds for $\chi_g^{(d)}(G)$ when $d \geq 1$. For example, given an integer $k \geq 1$, it is unknown if there exists an integer d such that for any graph G with $\text{col}_g(G) \leq k$, $\chi_g^{(d)}(G) \leq k - 1$.

In this paper, we prove that if G is a forest and $d \geq 1$ then $\chi_g^{(d)}(G) \leq 3$; if G is an outerplanar graph and $d \geq 1$ then $\chi_g^{(d)}(G) \leq 6$.

2 Coloring trees

Bodlaender [2] proved that $\chi_g(G) \leq 5$ if G is a tree and showed that there is a tree G with $\chi_g(G) = 4$. Faigle, Kern, Kierstead and Trotter [8] improved the upper bound to 4 for trees, which is therefore sharp, and remarked that this result extends easily to forests. In this section we prove that $\chi_g^{(d)} \leq 3$ if G is a forest and $d \geq 1$, and we prove that this is sharp when $d = 1$; that is, there is a forest G such that $\chi_g^{(1)}(G) = 3$.

Theorem 1 *If T is a forest and $d \geq 1$, then $\chi_g^{(d)}(T) \leq 3$, i.e., for the $(3, d)$ -coloring game on T , Alice has a winning strategy.*

Proof. Alice's strategy for this game is similar to the strategy for the ordinary coloring game as in [8].

Suppose in the process of the game, the forest T is partially colored. We obtain a collection of subtrees of T as follows: for each colored vertex x with degree k , we split x into k colored vertices (with the same color as x), say x_1, x_2, \dots, x_k , each of which is incident with exactly one edge that was originally incident with x . After splitting each colored vertex of T , we obtain a collection of smaller partially colored trees, say T_1, T_2, \dots, T_m , the union of whose edge sets is the edge set of T . These subtrees T_i are the *trunks* of T . Note that an uncolored leaf of a trunk necessarily corresponds to a leaf of T .

Alice's goal in picking the next vertex to color is simply to ensure that after she colored the picked vertex, each of the trunks of the partially colored T has at most two colored leaves. Suppose Alice can achieve this goal. Then after Bob colors a vertex, then each trunk T_i of the partially colored T has at most two colored leaves, except for one trunk which may have three colored leaves. Moreover, if the trunk T_i has three colored leaves, then one of the three colored leaves has just been colored by Bob, which we call the *new colored leaf* of T_i .

It is easy to see that Alice can achieve the above goal: if after Bob's move, there is a trunk T_i containing three colored leaves, then Alice picks the vertex which lies in the intersection of the paths joining these three colored leaves; otherwise, if there is a trunk T_i containing two colored leaves and T_i has more than one edge, then Alice picks any vertex which lies in the path joining these two colored leaves; otherwise, Alice picks any vertex. This strategy is the same as the strategy for playing the ordinary coloring game.

Suppose Alice has chosen vertex $u \in T_i$ by the strategy above. Alice chooses the color for u as follows: if the colored leaves of T_i used at most two colors, then Alice colors u with a color which has not been used by any colored leaves of T_i . Otherwise,

T_i has three colored leaves which use three distinct colors. Then Alice colors u with the color of the new colored leaf.

To prove that this strategy works, it suffices to show that in case T_i has three colored leaves which use three distinct colors, then the color of the new colored leaf is a legal color for u . This is obvious, because the new colored leaf has no other colored neighbours. ■

If $d = 1$, then the upper bound in Theorem 1 on $\chi_g^{(1)}(G)$ for forests is sharp. Namely, we have the following result:

Theorem 2 *There exists a tree T for which $\chi_g^{(1)}(T) = 3$, i.e., Bob has a winning strategy for the $(2, 1)$ -coloring game on T .*

Proof. Suppose Alice and Bob are playing the $(2, 1)$ -coloring game on a tree T . First we observe that if after Alice's move, there is a partially colored tree as depicted in Fig. 1 below, with $n \geq 5$, then Bob wins the game. (In Fig. 1, the numbers and u_i, v_i are the names of the vertices. The two colors are a and b . A vertex labeled a indicates that vertex is colored with color a , a vertex labeled x means that vertex is colored with either a or b . Other vertices are uncolored.)

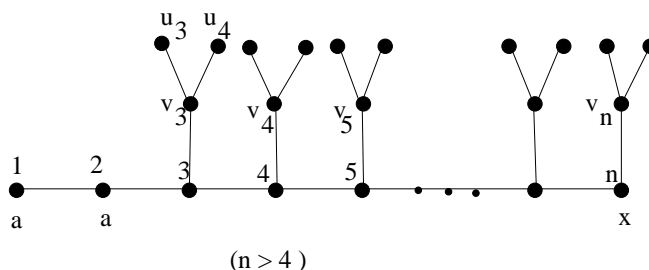


Figure 1: A partially colored tree for which Bob has a winning strategy

First we consider the case $n = 5$. If $x = b$, then Bob colors vertex 4 with color b , and he wins the game because vertex 3 cannot be colored with any color. If $x = a$, then Bob colors vertex v_3 with color b ; this forces Alice to color vertex 3 with color b (for otherwise Bob would color u_3 or u_4 with color b so that no color is legal for vertex 3). Then Bob colors vertex v_5 with color a , and he wins the game because vertex 4 cannot be colored. Suppose $n \geq 6$. Then Bob colors vertex v_3 with color b , which forces Alice to color vertex 3 with color b , and Bob wins the game by induction.

It remains to show that Bob can force a partially colored tree as depicted in Fig. 1 after Alice's move. This can be achieved as follows: Assume the tree T is sufficiently large (for example, a quaternary tree, that is, a tree in which every vertex has degree

1 or 4, of depth 100). After Alice colors the first vertex v_1 , Bob colors a vertex v_2 which is far away from v_1 . No matter which vertex v_3 is colored by Alice in the third move, two of the three vertices v_1, v_2, v_3 have a large distance, and no vertex on the path connecting these two vertices is colored. Without loss of generality, assume these two vertices are v_1 and v_2 . Now Bob colors a vertex u_1 so that the u_1v_1 -path, u_1v_2 -path and the v_1v_2 -path intersect at a single vertex w , and the distance from w to each of u_1, v_1, v_2 is large. Now it is Alice's turn to color. If Alice colors a vertex w' on the union of the above mentioned three paths which is close to w , then Bob colors a neighbour of w' the same color as w' . Then no matter which vertex Alice colors next, there is a partially colored tree as depicted in Fig. 1. If the vertex colored by Alice is not on the union of the above three paths, or the vertex is far from w , then Bob colors w , and in the next move, Bob can force a partially colored tree as depicted in Fig. 1. We leave the details for the readers to check. ■

It is unknown if there exists an integer $d \geq 2$ such that the d -relaxed game chromatic numbers of trees are at most 2.

It was shown in [17] that if the edges of a graph G can be decomposed as $E(G) = E(G_1) \cup E(G_2)$, where G_1 has game coloring number k and G_2 has maximum degree Δ , then $\text{col}_g(G) \leq k + \Delta$. For many classes of graphs an upper bound for their game coloring number is derived using this fact. This makes one wonder if there are some such relations for the relaxed game chromatic number. In general, it is unlikely that such a relation exists. However, the following question might have a positive answer.

Question 1 *Suppose $E(G) = E(G_1) \cup E(G_2)$, G_1 is a tree and G_2 has maximum degree Δ . Is it true that $\chi_g^{(d)}(G) \leq 3 + \Delta$ for $d \geq 1$?*

In the next section, we discuss the d -relaxed game chromatic numbers of outerplanar graphs. The result seems to support a positive answer to the question above. However, the proof heavily relies on the structure of outerplanar graphs, instead of relying on decomposition of the graphs.

3 Coloring outerplanar graphs

The game chromatic number and game coloring number of outerplanar graphs were studied in [10] and [13]. It was proved in [10] that if G is an outerplanar graph then $\chi_g(G) \leq \text{col}_g(G) \leq 7$. On the other hand, it was mentioned in [13] that there exists an outerplanar graph G such that $\chi_g(G) = 6$. It remains an open question whether or not $\chi_g(G) \leq 6$ for all outerplanar graphs. In our language, it remains open whether or not Alice has a winning strategy for the $(6, 0)$ -coloring game on outerplanar graphs.

In this section, we consider the $(6, d)$ -coloring game (for $d \geq 1$) on outerplanar graphs and prove that for such games, Alice does have a winning strategy.

Let G be a 2-connected triangulated outerplanar graph, which is an outerplanar graph each of whose inner faces is a triangle. We produce an ordering of the vertices of G as follows: start from an edge incident to the infinite face, label the two vertices v_1, v_2 . Let v_3 be the other vertex of the triangle (inner face) incident to the edge v_1v_2 . Suppose we have labeled vertices v_1, v_2, \dots, v_i , and there are unlabeled vertices. Then choose a triangle which contains only one unlabeled vertex and label it v_{i+1} . This method produces a labeling v_1, v_2, \dots, v_n of $V(G)$ such that for each j ($3 \leq j \leq n$), v_j is adjacent to two labeled vertices v_{j_1}, v_{j_2} with $j_1 < j_2 < j$. We call v_j a *parent* of v_i if $v_i \sim v_j$ and $j < i$. The ordering constructed above has the following properties:

1. for $i \geq 3$, the vertex v_i has exactly two parents and these two parents are adjacent;
2. if $i \neq j$, then v_i and v_j cannot have the same two parents.

For each $i \geq 3$, suppose v_{i_1}, v_{i_2} are the two parents of v_i . If $i_1 < i_2$, we call v_{i_1} the *major parent* of v_i , and call v_{i_2} the *minor parent* of v_i . The vertex v_i is called a *major child* of v_{i_1} and a *minor child* of v_{i_2} . For each vertex x of G , we shall denote by $f(x)$ the major parent of x , and denote by $m(x)$ the minor parent of x . For convenience, we let $f(v_1) = f(v_2) = v_1 = m(v_1) = m(v_2)$.

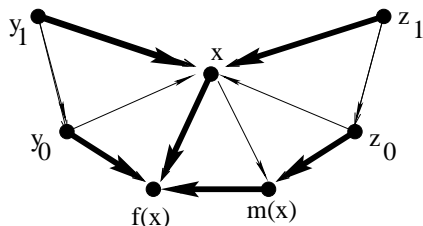


Figure 2: The parents and children of x

Note that if two vertices of G are joined by an edge, then one is a parent of the other. If w is a minor child of x , then $f(w)$ is a parent of x . Since any vertex can have at most two parents, a vertex x can have at most two minor children, one (say y_0) with major parent $f(x)$ and the other (say z_0) with major parent $m(x)$. The children of x belong to at most two paths $\dots y_1 y_0 x$ and $\dots z_1 z_0 x$ in which each vertex is a minor child of the next – see Fig. 2, in which an arrow from a to b denotes that a is a child of b , and thick lines denote major children. (The line from $m(x)$ to $f(x)$ could be either thick or thin.)

Theorem 3 *Suppose $G = (V, E)$ is a 2-connected triangulated outerplanar graph and $d \geq 1$. For the $(6, d)$ -coloring game on G , Alice has a winning strategy.*

Proof. Let G be a 2-connected triangulated outerplanar graph and let v_1, v_2, \dots, v_n be an ordering of the vertices of G which has the properties listed above. We say v_i is less than v_j if $i < j$.

We shall first describe the strategy for Alice to pick the vertex to be colored. Let U denote the set of uncolored vertices. Alice maintains a subset $A \subseteq V$ of *active* vertices. Initially $A = \emptyset$. When a new vertex is put into A , we say x is activated. Once a vertex is activated, it remains active forever. Initially, Alice colors v_1 and activates v_1 . Now suppose that Bob has colored the vertex b . Alice updates A and chooses the next vertex x to be colored by using the following strategy:

Alice will jump from vertex to vertex until she finds the vertex she wants to color. The so called “jumps” are done by applying the following rules successively:

First Alice jumps to b . Assume Alice has jumped to a vertex x .

Rule 1. If x is active and uncolored, then she colors x ;

Rule 2. If x is inactive, uncolored, and both $f(x)$ and $m(x)$ are colored, then she activates x and colors x ;

Rule 3. If $x = v_1$, then Alice colors the least uncolored vertex (and activates it if it is not active yet);

Rule 4. If none of the above is true, then Alice activates x (if x is inactive), and jumps to either $f(x)$ or $m(x)$ (by following the Jumping Rule below) and returns to Rule 1.

Jumping Rule: *If $f(x)$ is uncolored, or $f(x)$ and x are colored the same color, then jump to $f(x)$; otherwise, jump to $m(x)$.*

After choosing the vertex x to be colored, Alice finds a legal color for x as follows: if the colored neighbours of x use at most 5 colors, then she colors x with any color not used by its colored neighbours; if the colored neighbours of x use 6 colors, then we shall prove that one of the colors of a child of x is legal for x . Alice will color x with such a legal color.

To prove the existence of such a legal color for x , we construct a directed graph D with vertex set $V(D) = V(G)$ as follows: Consider all Alice’s moves before the current move (i.e., before the move in which she chooses x). Put a directed edge from

v to v' if in a certain step, Alice jumped from v to v' and v' is not colored before this jump. Parallel directed edges are allowed, i.e., if Alice jumped twice from v to v' before v' being colored, then there are two directed edges from v to v' .

By Alice's strategy, if Alice jumps once to a vertex v' then v' is activated, and if she jumps twice to v' then v' is colored. So in the directed graph D defined above, each vertex has in-degree at most 2, and each uncolored vertex has in-degree at most 1. Since x is uncolored, x has in-degree at most 1 in D .

By our strategy, each time a major child w of x is activated, Alice will jump from w to x , which will result in a directed edge from w to x . Since x has in-degree at most 1, so x has at most 1 active major child. Hence x has at most 5 active neighbours: two parents $f(x)$ and $m(x)$, two minor children, say u_1, u_2 , and one major child, say u_3 . Each colored vertex is either active, or has just been colored by Bob. If the colored neighbours of x use at most 5 colors then Alice will color x with a color not used by its colored neighbours. Assume x has 6 colored neighbours and they use 6 distinct colors. Then these colored neighbours are $f(x), m(x), u_1, u_2, u_3$ as described above, plus another major child u_4 which has just been colored by Bob. We shall prove that one of the colors of u_1 and u_2 is a legal color for x .

Note that by the argument above, Alice would not color a vertex the same color as a child if it has an uncolored parent.

Assume to the contrary that neither of the colors of u_1 and u_2 is a legal color for x . Then for $i = 1, 2$, u_i has a colored neighbour w_i which has the same color as u_i . Note that the parents of u_i are contained in $\{f(x), m(x), x\}$. So w_i is a child of u_i .

Claim 1: For $i \in \{1, 2\}$, among the two vertices w_i, u_i , the last colored vertex is colored by Bob.

Proof: If w_i is colored after u_i , then w_i is not colored by Alice, because Alice would never color a vertex the same color as a parent. If u_i is colored after w_i , then u_i is not colored by Alice, because u_i has an uncolored parent x so Alice would not color u_i the same color as a child. This completes the proof of Claim 1.

Let $A = \{f(x), m(x), x\}$ and $B = \{u_i, w_i : i = 1, 2\} \cup \{u_3\}$. We consider the subgraph of D induced by $A \cup B$. Since x is uncolored, the sum of the in-degrees of $f(x), m(x)$ and x is at most 5.

Claim 2: For $i \in \{1, 2\}$, in the move of Alice in which she activates the first vertex of $\{u_i, w_i\}$, she jumps once from B to A , and that jump adds a directed edge from B to A . After Bob colors the last vertex of $\{u_i, w_i\}$, in Alice's next move, she will also jump once from B to A , which adds a directed edge from B to A . Moreover, in the move of Alice in which she activates u_3 , Alice will jump from u_3 to x , which adds a directed edge from u_3 to x .

Proof: If the first activated vertex of $\{u_i, w_i\}$ is u_i , then after activating u_i , Alice jumps from u_i to $f(u_i) \in A$ if $f(u_i)$ is uncolored, or jumps to $m(u_i) = x \in A$ if $f(u_i)$ is colored. So this jump adds a directed edge from B to A . Assume the first activated vertex of $\{u_i, w_i\}$ is w_i . If w_i is a major child of u_i , then after activating w_i , Alice will jump from w_i to u_i , activate u_i , and then jump from u_i to $f(u_i) \in A$ or to $m(u_i) = x \in A$ (as discussed above). If w_i is a minor child of u_i , then after activating w_i , Alice will jump to $f(w_i) \in A$ if $f(w_i)$ is not colored, or jump to $m(w_i) = u_i$ and then jump to $f(u_i) \in A$ or jump to $x \in A$ (as discussed above). In any case, a directed edge from B to A is added in this move.

Assume the last colored vertex of $\{u_i, w_i\}$ is u_i (by Claim 1, it is colored by Bob). After Bob's colors u_i , Alice will jump from u_i to $f(u_i) \in A$ if $f(u_i)$ is uncolored, or to $m(u_i) = x \in A$ otherwise. Assume the last colored vertex of $\{u_i, w_i\}$ is w_i . If $f(w_i)$ is uncolored, then Alice will jump from w_i to $f(w_i) \in A$ (note that in this case $f(w_i) \neq u_i$ because u_i is already colored). If $f(w_i)$ is colored and $f(w_i) = u_i$, then w_i and $f(w_i)$ are colored the same color. So by the Jumping Rule, Alice will jump from w_i to u_i and then jump from u_i to $f(u_i) \in A$ if $f(u_i)$ is uncolored, or to $m(u_i) = x \in A$ if $f(u_i)$ is colored. If $f(w_i) \neq u_i$, then Alice will jump from w_i to $m(w_i) = u_i$ and then jump from u_i to $f(u_i) \in A$ if $f(u_i)$ is uncolored, or to $m(u_i) = x \in A$ if $f(u_i)$ is colored. In any case this jump adds a directed edge from B to A .

It is obvious that after Alice activates u_3 , she immediately jumps to x (as x is uncolored). This completes the proof of Claim 2.

Note that, by Claim 1, among the two vertices w_i, u_i , the last colored vertex is colored by Bob. The move in which Alice activates the first vertex of $\{u_i, w_i\}$ precedes the move of Bob in which he colors the last vertex of $\{u_i, w_i\}$. So for each $i \in \{1, 2\}$, there are two directed edges from $\{u_i, w_i\}$ to A , by Claim 2. Also the move in which Alice activates u_3 is different from the other moves discussed above. So there are 5 directed edges from B to A . Moreover, after Alice activates x , she will jump from x to $f(x)$ or $m(x)$ which results in another directed edge (for otherwise Alice will color x). But this is contrary to the fact that the in-degree sum of $f(x), m(x)$ and x is at most 5. ■

Note that if H is a subgraph of G , then it is not necessarily true that $\chi_g^{(d)}(H) \leq \chi_g^{(d)}(G)$. However, by a careful analysis of the proof of Theorem 3, it is not difficult to see that the result remains true for subgraphs of 2-connected triangulated outerplanar graphs. We choose to prove it for 2-connected triangulated outerplanar graphs because it makes the description easier. So the proof actually establishes the following result:

Theorem 4 *If G is an outerplanar graph and $d \geq 1$, then $\chi_g^{(d)}(G) \leq 6$, i.e., for the $(6, d)$ -coloring game, Alice has a winning strategy.*

It is unknown whether or not the upper bound in Theorem 4 is sharp for any $d \geq 1$.

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