Refined activation strategy for the marking game

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Abstract

This paper introduces a new strategy for playing the marking game on graphs. Using this strategy, we prove that if G is a planar graph, then the game colouring number of G, and hence the game chromatic number of G, is at most 17.

1 Introduction

Suppose $G = (V, E)$ is a graph. The game colouring number of G is defined through a two-person game: the *marking game*. Alice and Bob, with Alice playing first, take turns in playing the game. Each play by either player consists of marking an unmarked vertex of G. The game ends when all vertices are marked. Together the players create a linear order L of the vertices of G in which $x <_L y$ if x is marked before y. The *score* of the game is

$$s = \max_{x \in V(G)} |N^+_{G_L}[x]|,$$

where $N^+_{G_L}[x] = \{y \in N_G[x] : y = x \text{ or } y <_L x\}$. Alice's goal is to minimize the score, while Bob's goal is to maximize it. The *game colouring number* $\mathrm{col_g}(G)$ of G is the least s such that Alice has a strategy that results in a score at most s.

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The game colouring number of a graph was first formally introduced in [19] as a tool in the study of the game chromatic number. The *game chromatic number* $\chi_{\rm g}(G)$ of a graph G is also defined through a two person game. Let G be a finite graph and let X be a set of colours. Alice and Bob, with Alice moving first, take turns in playing the game. Each play by either player consists of colouring an uncoloured vertex of G with a colour from the colour set X so that no two adjacent vertices receive the same colour. Alice wins the game if all the vertices of G are coloured. Otherwise, Bob wins the game. The game chromatic number $\chi_{\rm g}(G)$ of G is the least number of colours in a colour set X for which Alice has a winning strategy.

It is easy to see that for any graph G, $\chi_{\rm g}(G) \leq {\rm col}_{\rm g}(G)$. For many natural classes of graphs, the best known upper bounds for their game chromatic number are obtained by finding upper bounds for their game colouring number. Game colouring number of a graph and its generalization to oriented graphs are also of independent interests, and have been studied extensively in the literature [2, 3, 5-7, 9-11, 13-20].

Suppose $\mathcal{H}$ is a family of graphs. We define the game chromatic number and the game colouring number of $\mathcal{H}$ as

$$\chi_{\rm g}(\mathcal{H}) = \max\{\chi_{\rm g}(G) : G \in \mathcal{H}\},$$

and

$${\rm col}_{\rm g}(\mathcal{H}) = \max\{{\rm col}_{\rm g}(G) : G \in \mathcal{H}\}.$$

We denote by $\mathcal{F}$ the family of forests, by $\mathcal{I}_k$ the family of interval graphs with clique number k, by $\mathcal{P}$ the family of planar graphs, by $\mathcal{Q}$ the family of outer planar graphs, by $\mathcal{PT}_k$ the family of partial k-trees. The exact value of the game colouring numbers of $\mathcal{F}$, $\mathcal{I}_k$, $\mathcal{Q}$ and $\mathcal{PT}_k$ are known. It is proved by Faigle, Kern, Kierstead and Trotter [8] that ${\rm col}_{\rm g}(\mathcal{F}) = 4$, proved by Faigle, Kern, Kierstead and Trotter [8] and Kierstead and Yang [15] that ${\rm col}_{\rm g}(\mathcal{I}_k) = 3k - 2$, proved by Guan and Zhu [9] and Kierstead and Yang [15] that ${\rm col}_{\rm g}(\mathcal{Q}) = 7$, and proved by Zhu [20] and Wu and Zhu [18] that ${\rm col}_{\rm g}(\mathcal{PT}_k) = 3k + 2$ for $k \geq 2$.

Although there are relatively rich results concerning the game chromatic number and game colouring number of graphs, there are very few strategies for either Alice or Bob to play the colouring game or marking game. It is proved in [11] that there is a single strategy, *the activation strategy*, such that if Alice uses this strategy to play the marking game then she achieves the sharp upper bounds on the game colouring numbers of $\mathcal{F}, \mathcal{I}_k, \mathcal{Q}, \mathcal{PK}_k$ as well as the best known upper bounds for many other classes of graphs, including $\mathcal{P}$.

In this paper, we introduce a new strategy, the refined activation strategy, for playing the marking game (it can also be used as a strategy for playing the colouring game). It is quite similar to the activation strategy, however, there are two new ingredients in the recipe. The key idea in the activation strategy is to use a special linear ordering of

$V(G)$ as Alice's preference in activating and marking vertices. In the refined activation strategy, Alice still uses orderings of the vertices as her preference in activating and marking vertices. However, there are two features that are different from the activation strategy. (1): The ordering is a 'dynamic rough ordering'. The vertex set is partitioned into small blocks. Within a block, there maybe noncomparable vertices, the order relation is not transitive and moreover, the order relation between vertices may change from time to time. (2): Each vertex has a preference of its own. If Alice moves from a vertex v to her next target, the preference of v will affect Alice's choice as well.

Kierstead [11] introduced a parameter, the rank $r(G)$, of a graph. By using the activation strategy for playing the marking game, one derives an upper bound for $\mathrm{col}_g(G)$ in terms of $r(G)$, namely $\mathrm{col}_g(G) \le 1 + r(G)$. In Section 2, we introduce a parameter $\gamma(G)$ of a graph, which is a refinement of $r(G)$. Section 3 introduces the refined activation strategy. We prove that if Alice uses this strategy, then the score of the game is at most $1 + \gamma(G)$. Therefore $\mathrm{col}_g(G) \le 1 + \gamma(G)$ for any graph G.

Then we estimate $\gamma(G)$ for planar graphs, which yields a better upper bound for $\mathrm{col}_g(\mathcal{P})$. The game chromatic number and game colouring number of planar graphs is a benchmark problem in the study of the colouring game and marking game. It was conjectured by Bodlaender [1] that $\chi_g(\mathcal{P}) < \infty$. This conjecture is confirmed by Kierstead and Trotter [12], who proved that $\chi_g(\mathcal{P}) \le 33$. This bound is reduced to 30 by Dinski and Zhu [4]. Then by introducing the game colouring number, Zhu [19] proved that $\chi_g(\mathcal{P}) \le \mathrm{col}_g(\mathcal{P}) \le 19$, and this bound is reduced to 18 by Kierstead [11]. Recently, Wu and Zhu [18] proved that $\mathrm{col}_g(\mathcal{P}) \ge 11$. By using the refined activation strategy, this paper proves that $\mathrm{col}_g(\mathcal{P}) \le 17$.

Theorem 1 *If $G = (V, E)$ is a planar graph, then $\chi_g(G) \le \mathrm{col}_g(G) \le 17$.*

2 Refined rank of a graph

Suppose G is a graph, L is a linear ordering of $V(G)$ and u is a vertex of G. An *u-matchable set with respect to* L is a subset $Z \subset N^-_{G_L}[u]$ such that there is a partition $\{X, Y\}$ of Z and there exists a matching M from $X \subset N^-_{G_L}[u]$ to $V^+(u)$ which saturates X and a matching N from $Y \subset N^-_{G_L}(u)$ to $V^+[u]$ which saturates Y. For a vertex u of G, let $m(u, L, G)$ be the size of a largest u-matchable set with respect to L. Let

$$\begin{aligned} r(u, L, G) &= |N^+_{G_L}(u)| + m(u, L, G) \\ r(L, G) &= \max_{u \in V(G)} r(u, L, G). \end{aligned}$$

Then the *rank* of G, introduced by Kierstead in [11], is defined as

$$r(G) = \min\{r(L, G) : L \text{ is a linear ordering of } V(G)\}.$$

It is proved in [11] that for any graph G, $\mathrm{col}_g(G) \leq 1 + r(G)$. In this section, we introduce a parameter, $\gamma(G)$, of a graph G, which is a refinement of $r(G)$. We shall prove that for any graph G, $\mathrm{col}_g(G) \leq 1 + \gamma(G)$.

The rank of a graph is defined through a linear ordering of $V(G)$. The *refined rank* $\gamma(G)$ is defined through a 'dynamic rough ordering' of $V(G)$ and a 'preference function'.

Definition 1 *Suppose $G = (V, E)$ is a graph. A* dynamic rough ordering *of G is a pair $(L_0, \mathcal{P})$ such that L_0 is a digraph on V without opposite arcs, and $\mathcal{P}$ is a partition of V. Each $B \in \mathcal{P}$ is called a* block. *The blocks can be ordered as $B_1, B_2, \cdots, B_m$ such that for any $i < j$, if $x \in B_i$ and $y \in B_j$ then $\overrightarrow{yx} \in L_0$, i.e., $\overrightarrow{yx}$ is an arc of L_0.*

The digraph L_0 is viewed as a rough ordering. In the remaining of this paper, we write $x <_{L_0} y$ if $\overrightarrow{yx}$ is an arc of L_0. We say two vertices x, y are *comparable* in L_0 if and only if either $x <_{L_0} y$ or $y <_{L_0} x$. The digraph L_0 is not really an ordering, because inside a block B_i, there may be noncomparable vertices, the relation $<_{L_0}$ may not transitive, and there may be directed cycles. In particular, if $x <_{L_0} y$ and $y <_{L_0} z$, then it is not necessary that x and z are comparable. However, if we ignore what happens inside the blocks, it becomes a linear ordering.

In the definition, there is nothing which is really dynamic. What we have here are simply a fixed rough ordering L_0 and a partition $\mathcal{P}$. However, we use the adjective 'dynamic' to suggest that the rough ordering used in the strategy will change from time to time, and L_0 is just the initial state of the 'real' dynamic rough ordering.

We write $x \approx y$ if x and y are in the same block of $\mathcal{P}$, and write $x \not\approx y$ otherwise.

Let

$$\begin{array}{rclrcl} V^+_{L_0}(x) &=& \{y : y <_{L_0} x\}, & V^-_{L_0}(x) &=& \{y : x <_{L_0} y\}, \\ V^+_{L_0,\not\approx}(x) &=& \{y \in V^+_{L_0}(x) : x \not\approx y\}, & V^-_{L_0,\not\approx}(x) &=& \{y \in V^-_{L_0}(x) : x \not\approx y\}, \\ V^+_{L_0,\approx}(x) &=& \{y \in V^+_{L_0}(x) : x \approx y\}, & V^-_{L_0,\approx}(x) &=& \{y \in V^-_{L_0}(x) : x \approx y\}, \\ V_{L_0,\approx}(x) &=& V^+_{L_0,\approx}(x) \cup V^-_{L_0,\approx}(x), & V^\times_{L_0}(x) &=& \{y : x \not<_{L_0} y, y \not<_{L_0} x\}. \end{array}$$

Let $V^+_{L_0}[x] = V^+_{L_0}(x) \cup \{x\}$ and $V^-_{L_0}[x] = V^-_{L_0}(x) \cup \{x\}$. If $x \approx y$, then $V^+_{L_0,\not\approx}(x) = V^+_{L_0,\not\approx}(y)$ and $V^-_{L_0,\not\approx}(x) = V^-_{L_0,\not\approx}(y)$. We let $V^+_{L_0,\not\approx}(B_i) = V^+_{L_0,\not\approx}(x)$ and $V^-_{L_0,\not\approx}(B_i) = V^-_{L_0,\not\approx}(x)$ for some (and hence for all) $x \in B_i$.

Given a digraph Q, we denote by $\overline{Q}$ the graph obtained from Q by omitting the orientation of the arcs, i.e., an arc $\overrightarrow{xy}$ of Q becomes an edge xy of $\overline{Q}$.

A *preference function* of $(L_0, \mathcal{P})$ is a mapping ρ which assigns to each vertex $y \in V(G)$ a subset $\rho(y)$ of $N_G(y) \cap V^+_{L_0,\not\approx}(y)$ such that the following holds:

[P1]: For any index i, if $y \in V^-_{L_0,\not\approx}(B_i)$, then $\rho(y) \cap B_i$ contains at most one edge of $\overline{L_0}$.

The set $\rho(y) \cap B_i$ (which could be empty) is called the *y-preferred subset of B_i*. If $\rho(y) \cap B_i$ does contain an edge uv of $\overline{L_0}$, then we call the edge uv a *y-affected edge of B_i*.

For any vertex x, let $\rho^{-1}(x) = \{y : x \in \rho(y)\}$. Note that $\rho^{-1}(x) \subset N_G(x) \cap V^-_{L_0,\not\approx}(x)$. Let $D(x) = (N_G(x) \cap V^-_{L_0,\not\approx}(x)) \setminus \rho^{-1}(x)$.

Suppose $(L_0, \mathcal{P})$ is a dynamic rough ordering of G, ρ is a preference function of $(L_0, \mathcal{P})$, and x is a vertex of G. Assume $x \in B_i$. A subset W of $\rho^{-1}(x) \cup \{x\}$ is called *x-matchable with respect to* $((L_0, \mathcal{P}), \rho)$ if W can be partitioned into three sets $W = X \cup Y \cup Z$ such that the following hold:

1. $X \subset \rho^{-1}(x)$, $Y \subset \rho^{-1}(x) \cup \{x\}$, $Z \subset \rho^{-1}(x)$.

2. There is a matching M of G from X to $V^+_{L_0}[x] \cup V^\times_{L_0}(x)$ which saturates X.

3. There is a matching N of G from Y to $V^+_{L_0,\not\approx}(x) \cup V^\times_{L_0}(x)$ that saturates Y.

4. Z is either empty or $Z = \{z\}$ contains a single element. If $Z = \{z\}$ then $N_G(z) \cap V_{L_0,\approx}(x) \neq \emptyset$.

Let $m(x, (L_0, \mathcal{P}), \rho, G)$ be the size of the largest x-matchable set W with respect to $((L_0, \mathcal{P}), \rho)$. Assume $x \in B_i$. Let

$$
\begin{aligned}
\gamma(x, (L_0, \mathcal{P}), \rho, G) &= |N_G(x) \cap (V^+_{L_0}(x) \cup B_i)| + |D(x)| + m(x, (L_0, \mathcal{P}), \rho, G), \\
\gamma((L_0, \mathcal{P}), \rho, G) &= \max_{x \in V} \gamma(x, (L_0, \mathcal{P}), \rho, G), \\
\gamma(G) &= \min\{\gamma((L_0, \mathcal{P}), \rho, G) : (L_0, \mathcal{P}) \text{ is a dynamic rough ordering of } G \\
&\qquad \text{and } \rho \text{ is a preference function of } (L_0, \mathcal{P})\}.
\end{aligned}
$$

To see that $\gamma(G)$ is a refinement of $r(G)$, it suffices to note that for a linear ordering L_0 of V, let $\mathcal{P}$ be the trivial partition in which each block contains a single vertex. Then $(L_0, \mathcal{P})$ is a dynamic rough ordering. Let $\rho(x) = V^+_{L_0}(x) \cap N_G(x)$ for all x. Then ρ is a preference function of $(L_0, \mathcal{P})$. It follows from the definitions that for any $x \in V$, $r(x, L_0, G) = \gamma(x, (L_0, \mathcal{P}), \rho, G)$. Therefore, for any graph G, we have $\gamma(G) \leq r(G)$.

3 Refined activation strategy

This section gives a strategy for Alice to play the marking game. We need a few notation for the description of the strategy. We need to refer to a digraph L (a rough ordering), which is the 'real' dynamic rough ordering obtained from L_0 by possibly reversing the orientations of some arcs. So L is a living creature, and the letter L always stands for the current digraph L. The strategy will give reversal rules that describe how the arcs of L be reversed. Here we just note the following properties of L, which follow easily from the reversal rules (which will be given later):

(1): Although L and L_0 may have different arcs, we shall always have $\overline{L} = \overline{L_0}$. So two vertices x, y are comparable in L if and only if they are comparable in L_0.

(2): Arcs of L_0 between vertices of different blocks will not be reversed at any time. In other words, the reversal of arcs take place inside the blocks only.

The sets $V_L^+(x), V_L^-(x)$, etc. will be defined similarly as $V_{L_0}^+(x), V_{L_0}^-(x)$, etc., except that in place of L_0 we use the digraph L.

Suppose X is a subset of V. A *minimal element of X with respect to L* is an element $x \in X$ such that for any $y \in X$, $y \not<_L x$. As L may contain directed cycles, for an arbitrary subset X of V, X may not have a minimal element. In case minimal element exists, it may not be unique. The following definition of $\min_v X$ combines the rough ordering and the preference function together in finding a (more or less minimal) element $\min_v X$ of X.

Definition 2 *Suppose x is a vertex of V and X is a nonempty subset of V. Then $\min_v X$ is an element of X defined as follows:*

Let i be the smallest index such that $B_i \cap X \neq \emptyset$. If $X \cap B_i \cap \rho(v) \neq \emptyset$, then $\min_v X$ is an arbitrary (but fixed) minimal element of $X \cap B_i \cap \rho(v)$ with respect to L. Note that by our definition of the preference function, $\rho(v) \cap B_i$ contains at most one arc of L, and hence the minimal element exists. If $X \cap B_i \cap \rho(v) = \emptyset$, then $\min_x X$ is an arbitrary (but fixed) element of $X \cap B_i$.

Note that if each block B_i is a singleton, then L_0 is a linear order and $\min_v X$ is simply the minimum element of X. Indeed, in this case, the refined activation strategy (which we will describe soon) is the same as the activation strategy in [11].

Theorem 2 *For any graph G, $\mathrm{col_g}(G) \leq 1 + \gamma(G)$.*

Proof. Let $(L_0, \mathcal{P})$ be a dynamic rough ordering of G and let ρ be a preference function of $(L_0, \mathcal{P})$ such that $\gamma(G) = \gamma((L_0, \mathcal{P}), \rho, G)$. We shall give a strategy for Alice to play the game that results a score at most $1 + \gamma(G)$.

In the play of the game, Alice will maintain a subset A of *active vertices*. We say a vertex v is *activated* to mean that v is added to A. Once a vertex is activated, it remains active afterwards. Let U be the set of unmarked vertices. To unify the description we consider an equivalent version of the marking game in which Bob plays first by marking a new vertex x_0, which is an isolated vertex, and $x_0 <_{L_0} y$ for all $y \in V$.

Initialization: $A := \emptyset$, $U := V(G)$ and $L := L_0$. If we view the digraph L as a set of arcs, then by *reversing the arc* $\overrightarrow{uv}$ *of* L, we mean let $L := (L \setminus \{\overrightarrow{uv}\}) \cup \{\overrightarrow{vu}\}$.

Suppose Bob has just marked a vertex b and now it is Alice's turn. If all the vertices are marked, then the game is over. Otherwise, let u be an arbitrary unmarked vertex.

- if $N_G(b) \cap V_L^+(b) \cap U \neq \emptyset$ then $x := b$, else $x := u$ end if;

- while $x = b$ or $x \notin A$ do
  $A := A \cup \{x\}$;
  $w := \min_x N_G[x] \cap V_L^+[x] \cap U$;
  if there is an arc $\overrightarrow{uw}$ of L incident to w such that uw is an x-affected edge, then reverse the arc $\overrightarrow{uw}$ end if;
  $x := w$ end do;

- Mark x (i.e., $U := U \setminus \{x\}$) end do;

This strategy is similar to the activation strategy in [11]. Starting from the vertex b which has just been marked by Bob (or starting from any unmarked vertex, if $N_G(b) \cap V_L^+(b) \cap U = \emptyset$), Alice starts to activate vertices. After Alice activated a vertex x, she 'jumps' to the least unmarked 'forward' neighbour w of x, which she will either activate if it is not active yet, or mark if it is already active. The difference between this strategy and the original activation strategy is that the 'least element' refers to a dynamic rough ordering L. Moreover, this dynamic ordering L is 'modified' by the preference of x.

If there is a jump from x to w, we say x *made a contribution* to w, and say w *received a contribution* from x. If X, Y are subsets of V, then we say Y received a contribution from X if a vertex $y \in Y$ received a contribution from a vertex $x \in X$. Observe that only unmarked vertex can receive contributions. If a vertex receives the first contribution, it becomes active. After receiving the second contribution, it becomes marked. So each vertex can receive at most 2 contributions. At the time a vertex x is activated, it will make a contribution to a least unmarked vertex (according the current order with modification through ρ) in $V_L^+(x)$, unless $N_{G_L}^+(x) \cap U$ is empty, in which case x will make a contribution to itself, and be marked.

We shall show that at any time any unmarked vertex is adjacent to at most $\gamma(G)$ active vertices. Since every vertex marked by Bob immediately becomes active and any vertex marked by Alice is already active, this will prove Theorem 2.

Assume x is an unmarked vertex. Let X be the set of active neighbours of x, i.e., $X = A \cap N_G(x)$. We shall determine the maximum possible value of $|X|$.

Assume $x \in B_i$. We partition the set X into three parts.

$$\begin{aligned} X_1 &= X \cap (V^+_{L_0}(x) \cup B_i), \\ X_2 &= X \cap \rho^{-1}(x), \\ X_3 &= X \cap (V^-_{L_0,\not\approx}(x) \setminus \rho^{-1}(x)) = X \cap D(x). \end{aligned}$$

Then $|X_1| \leq |N_G(x) \cap (V^+_{L_0}(x) \cup B_i)|$ and $|X_3| \leq |D(x)|$. It remains to show that $|X_2| \leq m(x, (L_0, \mathcal{P}), \rho, G)$.

Assume $z \in X_2$, and assume that when z is activated, it made a contribution to a vertex $w(z)$. By our strategy $w(z) = min_z N_G[z] \cap V^+_{L'}[z] \cap U'$, where L' is the digraph at the time when z is activated, and U' is the set of unmarked vertices at the time z is activated.

Case 1 No arcs of the form $\overrightarrow{yx}$ have ever been reversed till Bob's last move.

As $\overrightarrow{zx} \in L_0$, we have $\overrightarrow{zx} \in L'$ (as the arc $\overrightarrow{zx}$ is not reversed). Hence $x \in N_G[z] \cap V^+_{L'}[z] \cap U'$. So $w(z)$ is either incomparable with x or $w(z) <_{L'} x$, i.e., $w(z) \in V^+_{L'}[x] \cup V^\times_{L_0}(x) \subset V^+_{L_0}[x] \cup V^\times_{L_0}(x)$ (the last inclusion holds because no arcs of the form $\overrightarrow{yx}$ has been reversed). Let X be the subset of X_2 consisting of those vertices z such that z is the first vertex of X_2 making a contribution to $w(z)$. Let Y be the subset of X_2 consisting of those vertices z such that z is the second vertex of X_2 making a contribution to $w(z)$. As each vertex can receive at most two contributions, $X \cup Y$ is a partition of X_2.

Let $M = \{zw(z) : z \in X\}$, and let $N = \{zw(z) : z \in Y\}$. Then M is a matching from X to $V^+_{L_0}[x] \cup V^\times_{L_0}$, and N is a matching from Y to $V^+_{L_0,\not\approx}[x] \cup V^\times_{L_0}$. Note that the target sets of M and N are different. The reason that for $z \in Y$, we cannot have $w(z) \in V^+_{L_0,\approx}(x)$ is that each vertex $w \in V^+_{L_0,\approx}(x)$ can receive at most one contribution from X_2. Indeed, if $w \in V^+_{L_0,\approx}(x)$, then after w receives the first contribution from X_2, the arc $\overrightarrow{xw}$ will be reversed, and becomes the arc $\overrightarrow{wx}$. The arc $\overrightarrow{wx}$ need to be reversed before the vertex w can receive the second contribution from X_2. But by our assumption, no such arcs have ever been reversed. If x has received at most one contribution, then x is not matched to any vertex of Y, hence N is a matching from Y to $V^+_{L_0,\not\approx}(x) \cup V^\times_{L_0}$. Therefore X_2 is x-matchable with respect to $((L_0, \mathcal{P}), \rho)$, which implies that $|X_2| \leq m(x, (L_0, \mathcal{P}), \rho, G)$. Assume x received two contributions from X_2 (note that if this happens, then this is the moment just before Alice going to mark x), and let $z^* \in Y$ be the vertex from which x receives the second contribution.

When x receives the first contribution, it made a contribution to $w(x) \in V^+_{L_0}(x)$. Let $Y' = (Y \setminus \{z^*\}) \cup \{x\}$, let $N' = (N \setminus \{z^*x\}) \cup \{xw(x)\}$. Then M, N' shows that $X \cup Y'$ is x-matchable with respect to $((L_0, \mathcal{P}), \rho)$. Hence $|X_2| = |X| + |Y| = |X| + |Y'| \leq m(x, (L_0, \mathcal{P}), \rho, G)$.

Case 2 Some arcs of the form $\overrightarrow{yx}$ have been reversed before Bob's last move.

By our strategy, an arc of the form $\overrightarrow{yx}$ is reversed only if x has received a contribution. Moreover, when x receives one contribution, at most one such arc is reversed (this is why we require that the preference function ρ have property [P1]). Since x is unmarked yet, x has received at most one contribution before Bob's last move. So there is a unique arc of the form $\overrightarrow{yx}$, say $\overrightarrow{y^*x}$, has been reversed and becomes $\overrightarrow{xy^*}$. There are two possibilities: (1) $y^* \in V^-_{L_0,\approx}(x)$. In this case y^* may receive one contribution from X_2. Note that after y^* receives one contribution from X_2, the arc $\overrightarrow{xy^*}$ is reversed, and becomes $\overrightarrow{y^*x}$ again. So y^* cannot receive the second contribution from X_2 (as x is unmarked). (2) $y^* \in V^+_{L_0,\approx}(x)$. In this case, the arc between x and y^* in L_0 is $\overrightarrow{xy^*}$. After y^* receives one contribution from X_2, the arc is reversed to $\overrightarrow{y^*x}$. Then after x receives a contribution from X_2, the arc is reversed again and becomes $\overrightarrow{xy^*}$. Now y^* can receive the second contribution. In (1), let z^* be the vertex of X_2 which made a contribution to y^*, if such a vertex exists. In (2), let z^* be the second vertex of X_2 making a contribution to y^*, if such a vertex exists. If the vertex z^* as above exists, then let $Z = \{z^*\}$, otherwise let $Z = \emptyset$. Let X, Y be the partition of $X_2 \setminus \{z^*\}$ as discussed in Case 1. Then the same argument as in Case 1 shows that $|X_2| = |X| + |Y| + |Z| \leq m(x, (L_0, \mathcal{P}), \rho, G)$. ∎

4 Estimation of $\gamma(G)$

To find a good upper bound for $\gamma(G)$ is nontrivial. In this section, we introduce the concept of a bound graph for $(G, L_0, \mathcal{P}, \rho)$, which will be used to derive an upper bound for $\gamma(G)$.

Suppose $(L_0, \mathcal{P})$ is a dynamic rough ordering of G and ρ is a preference function of $(L_0, \mathcal{P})$. Let H be a graph with vertex set V. We say H is a *bound graph* for $(G, L_0, \mathcal{P}, \rho)$ if the following hold:

[B1] G is a subgraph of H.

[B2] If $x \approx y$ and there is a vertex $z \in \rho^{-1}(x) \cap \rho^{-1}(y)$, then $x \sim_H y$.

[B3] If $x \in V^-_{L_0,\not\approx}(y)$ and there is a vertex $z \in \rho^{-1}(x)$ such that $y \sim_G z$, then $x \sim_H y$.

Suppose H is a bound graph for $(G, L_0, \mathcal{P}, \rho)$. Suppose $x \in B_i$. Let

$$\begin{aligned} A(x) &= \{y \in V^+_{L_0,\not\approx}(B_i) \cup V^\times_{L_0}(x) : x \sim_H y\}, \\ B(x) &= \{y \in V^+_{L_0,\approx}(x) : x \sim_H y\}. \\ C(x) &= \{y \in V^-_{L_0,\approx}(x) : x \sim_H y\}. \end{aligned}$$

Let

$$\tau(x) = \begin{cases} 0, & \text{if } B(x) = C(x) = \emptyset \\ 1, & \text{otherwise} \end{cases}.$$

Recall that $D(x) = N_G(x) \cap V^-_{L_0,\not\approx}(x)) \setminus \rho^{-1}(x)$. Let

$$\phi_H(x) = 3|A(x)| + 2|B(x)| + |C(x)| + |D(x)| + \tau(x).$$

Lemma 1 *Suppose $(L_0, \mathcal{P})$ is a dynamic rough ordering of G, ρ is a preference function of $(L_0, \mathcal{P})$, and H is a bound graph for $(G, L_0, \mathcal{P}, \rho)$. Then for any vertex $x \in V$,*

$$\gamma(x, (L_0, \mathcal{P}), \rho, G) \leq \phi_H(x) + 1.$$

Proof. By definition, $|N_G(x) \cap (V^+_{L_0}(x) \cup B_i)| \leq |A(x)| + |B(x)| + |C(x)|$. So it suffices to show that $m(x, (L_0, \mathcal{P}), \rho, G) \leq 2|A(x)| + |B(x)| + \tau(x) + 1$. Let W be an x-matchable subset of $N_G(x) \cap V^-_{L_0,\not\approx}(x)$ with respect to $((L_0, \mathcal{P}), \rho)$ with $|W| = m(x, (L_0, \mathcal{P}), \rho, G)$. Let $W = X \cup Y \cup Z$ be the corresponding partition of W.

By definition, there is a matching M from X to $V^+_{L_0}[x] \cup V^\times_{L_0}(x)$ that saturates X. Assume that M saturates $X' \subset V^+_{L_0}[x] \cup V^\times_{L_0}$. By definition of a bound graph, we know that $X' \setminus \{x\} \subset N_H(x)$. Therefore $X' \setminus \{x\} \subset A(x) \cup B(x)$. Hence $|X| = |X'| \leq |A(x)| + |B(x)| + 1$.

There is a matching N from Y to $V^+_{L_0,\not\approx}(x) \cup V^\times_{L_0}(x)$ that saturates Y. Assume that N saturates $Y' \subset V^+_{L_0,\not\approx}(x) \cup V^\times_{L_0}$. Again, by definition of a bound graph, we know that $Y' \subset N_H(x)$. Therefore $Y' \subset A(x)$. It follows easily from the definition that $|Z| \leq \tau(x)$. Therefore

$$|W| = |X| + |Y| + |Z| \leq 2|A(x)| + |B(x)| + \tau(x) + 1.$$

∎

5 Triangulations of the plane

We shall use Theorem 2 and Lemma 1 to prove Theorem 1. Thus we need to find, for any planar graph G, a dynamic rough ordering $(L_0, \mathcal{P})$, a preference function ρ of

$(L_0, \mathcal{P})$, and a bound H for $(G, L_0, \mathcal{P}, \rho)$, such that for each vertex $x \in V$, $\phi_H(x) \leq 15$. For this purpose, we need a lemma about the structure of plane triangulations.

Suppose R is a plane triangulation and $V(R)$ is partitioned into two sets $C \cup U$, where C (could be an empty) is an independent set of R, and each vertex of C has degree 4 or 5. A *candidate* for (R, C, U) is a triple (B, ρ, Q) such that B is a subset of U, Q is a digraph with vertex set B, and ρ is a mapping which assigns to each vertex $y \in C$ a subset $\rho(y)$ of B. Moreover, the following hold:

[C1] If $\{v_1, v_2, v_3\} \subset B$ is a face of R, then $\{v_1, v_2, v_3\}$ contains at most one arc of Q.

[C2] For any $y \in C$, $|\rho(y) \cap B| \leq 2$.

[C3] If $x, x' \in \rho(y)$ for some $y \in C$, then $x \sim_R x'$.

[C4] If there is a $y \in C$ such that $x \in \rho(y)$ and $x' \in N_R(y) \cap (U \setminus B)$, then $x \sim_R x'$.

Suppose (B, ρ, Q) is a candidate for (R, C, U) and $x \in B$. Let

$$A(x) = (N_R(x) \cap U) \setminus N_Q(x), B(x) = N_Q^+(x), C(x) = N_Q^-(x).$$

Let $D(x) = (N_R(x) \cap C) \setminus \rho^{-1}(x)$.

Let

$$\tau(x) = \begin{cases} 0, & \text{if } C(x) = B(x) = \emptyset \\ 1, & \text{otherwise.} \end{cases}$$

Let

$$\phi(x) = 3|A(x)| + 2|B(x)| + |C(x)| + |D(x)| + \tau(x).$$

We call the candidate (B, ρ, Q) a *valid candidate* if the following holds:

[C5] For all $x \in B$, $\phi(x) \leq 15$.

Theorem 3 *Suppose R is a plane triangulation, $C \cup U$ is a partition of $V(R)$, C is an independent set of R and each vertex of C has degree 4 or 5. If $U \neq \emptyset$, then R has a valid candidate.*

The definition of a valid candidate is a little bit technical. To have a rough idea of this concept before we get to the proof, we consider the special case that $C = \emptyset$. Then we may simply choose a vertex x with $d_R(x) \leq 5$ and let $S = \{x\}$. We do not need to worry about ρ and Q, as $C = \emptyset$ and Q consists of a single vertex. In this case,

$A(x) = N_R(x)$, $B(x) = C(x) = D(x) = \emptyset$ and $\tau(x) = 0$, hence $\phi(x) = 3d_R(x) \leq 15$. So in some sense, Theorem 3 is a generalization of the statement that *each plane triangulation has a vertex of degree at most* 5. For those readers who are familiar with the proof in [11], the argument in [11] actually shows that, if $\phi(x) \leq 15$ is replaced by $\phi(x) \leq 16$, then the corresponding "valid candidate" (B, ρ, Q) exists with B being a single element set and $\rho(y) = \emptyset$ for all $y \in C$ (so one does not need to introduce the digraph Q and the preference function ρ). The main effort of this paper is to reduce $\phi(x) \leq 16$ to $\phi(x) \leq 15$, which then reduces the upper bound for $\mathrm{col_g}(\mathcal{P})$ from 18 to 17.

The remaining of this section is devoted to the proof of Theorem 3. Assume the lemma is not true, and (R, C, U) is a counterexample. First we derive some properties of (R, C, U). Then we shall derive a contradiction by using the discharging method.

For each vertex $x \in V(R)$, let $p(x) = |N_R(x) \cap U|$ and $q(x) = |N_R(v) \cap C|$. Since R is a triangulation, and C is an independent set, it follows that for any $x \in U$, $p(x) \geq q(x)$; for any vertex $x \in C$, $q(x) = 0$. Let Γ be the graph with vertex set C in which $x \sim_\Gamma y$ if $N_R(x) \cap N_R(y)$ contains two adjacent vertices of U. For each vertex $v \in U$, let Γ_v be the subgraph of Γ induced by $N_R(v) \cap C$. The following lemma follows easily from the definition.

Lemma 2 *For each vertex $v \in U$, Γ_v is an induced subgraph of a cycle. If Γ_v is a cycle, then $p(v) = q(v)$. Otherwise $q(v) \leq p(v) - 1$. If Γ_v has a component consisting of at most $p(v) - 2$ vertices, then $q(v) \leq p(v) - 2$.*

Easy calculation shows that if $u \in U$, and $3p(u) + q(u) \leq 15$, then (B, ρ, Q) is a valid candidate for (R, C, U), where $B = \{v\}$, Q is the trivial digraph containing only one vertex, and $\rho(y) = \emptyset$ for all $y \in C$. As (R, C, U) has no valid candidate, we have the following lemma.

Lemma 3 *For every $u \in U'$, $p(u) \geq 4$. Moreover, if $p(u) = 4$, then $q(u) = 4$; if $p(u) = 5$, then $q(u) \geq 1$.*

Definition 3 *Suppose $x \in U$ and $z \in C$ and $x \sim_R z$. We say x and z are* minor neighbours *of each other if $p(x) = 5$ and $q(x) = 1$.*

Definition 4 *Suppose $x \in U$, $z \in C$, $x \sim_R z$ and $p(z) = 4$. Let the other three neighbours of z be u_1, u_2, u_3. We say x and z are* major neighbours *of each other if one of the following holds:*

1. *$p(x) \geq 6$.*

2. *$p(x) = 5$, $3 \leq q(x) \leq 5$ and two of the u_i's are minor neighbours of z.*

3. *$p(x) = 5$, $4 \leq q(x) \leq 5$, one of the u_i's, say u_1, is a minor neighbour of z and moreover, $p(u_2), p(u_3) \leq 5$ and $q(u_2), q(u_3) \leq q(x)$.*

Lemma 4 *Suppose $z \in C$, $p(z) = 4$. If z has two minor neighbours, then z has two major neighbours.*

Proof. Assume y_1, y_2 are two minor neighbours of z. Let u_1, u_2 be the other two neighbours of z. Assume to the contrary that u_1 is not a major neighbour of z. Then $p(u_1) \leq 5$. We shall derive a contradiction by finding a valid candidate. Depending on whether y_1 and y_2 are adjacent or not, we have two cases as depicted in Figure 1. In any case, it is easy to verify that $d_{\Gamma_{u_1}}(z) \leq 1$. By Lemma 2, $q(u_1) \leq p(u_1) - 1$. By Lemma 3 and the definition of major neighbour, we conclude that $p(u_1) = 5$ and $q(u_1) \leq 2$.

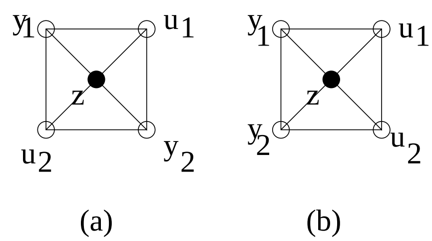

Figure 1: A vertex $z \in C$ with two minor neighbours

(In all the figures of this paper, a filled circle is a vertex of C, and an unfilled circle is a vertex of U.)

First we consider the case that $y_1 \not\sim_R y_2$, as depicted in Figure 1 (a). Let $B = \{u_1, y_1, y_2\}$, let Q be the digraph which consists of arcs $\overrightarrow{y_2u_1}, \overrightarrow{u_1y_1}$, and let $\rho(z) = \{y_2\}$ and $\rho(y) = \emptyset$ for $y \in C \setminus \{z\}$. The digraph Q and the mapping ρ are as depicted in Figure 2 (a). Note that z is not a vertex of Q. We put a dotted line from z to y_2 to indicate that $\rho(z) = \{y_2\}$. We claim that (B, ρ, Q) is a valid candidate.

[C1]: We need to show that no two arcs of Q is contained in a facial triangle of R. Assume $\overrightarrow{y_2u_1}, \overrightarrow{u_1y_1}$ is contained in a facial triangle. Then $N_R(u_1) \cap U = \{y_1, y_2\}$, i.e., $p(u_1) = 2$, in contrary to Lemma 3.

For [C2], [C3], [C4], it suffices to consider $z \in C$ and its neighbours (as $\rho(y) = \emptyset$ for $y \in C \setminus \{z\}$). The verification is straightforward (by referring to Figure 2 (a)) and is left to the readers. The following table verifies [C5] for each vertex v of B.

v	$3\|A(v)\|$	$2\|B(v)\|$	$\|C(v)\|$	$\|D(v)\|$	$\tau(v)$	$\phi(v)$
y_1	12	0	1	1	1	15
y_2	12	2	0	0	1	15
u_1	9	2	1	2	1	15

The numbers in the table are upper bounds for the corresponding parameter. For example, the number 1 at row u_1 and column $|D(v)|$ means that $|D(u_1)| \leq 1$.

We verify this table for u_1 and y_2. We have $|A(u_1)| = p(u_1) - |N_Q(u_1)| = 5 - 2 = 3$, so $3|A(u_1)| = 9$. As $N_Q^+(u_1) = B(u_1) = \{y_1\}$, we have $2|B(u_1)| = 2$. As $N_Q^-(u_1) = C(u_1) = \{y_2\}$, we have $|C(u_1)| = 1$. By definition, as $\rho^{-1}(u_1) = \emptyset$, $|D(u_1)| = q(u_1) \leq 2$. As $C(u_1) \neq \emptyset$ we have $\tau(u_1) = 1$. Therefore $\phi(u_1) \leq 9 + 2 + 1 + 2 + 1 = 15$. Now we consider y_2. Similarly $|A(y_2)| = p(y_2) - |N_Q(y_2)| = 5 - 1 = 4$. From Figure 2 (a), we see that $B(y_2) = N_Q^+(y_2) = \{u_1\}$ and $C(y_2) = N_Q^-(y_2) = \emptyset$. So $2|B(y_2)| = 2$ and $|C(y_2)| = 0$. Since $\rho^{-1}(y_2) = \{z\} = N_R(y_2) \cap C$, we have $D(y_2) = \emptyset$. Therefore $|D(y_2)| = 0$. As $B(y_2) \neq \emptyset$, we have $\tau(y_2) = 1$. Therefore $\phi(y_2) = 12+2+0+0+1 = 15$.

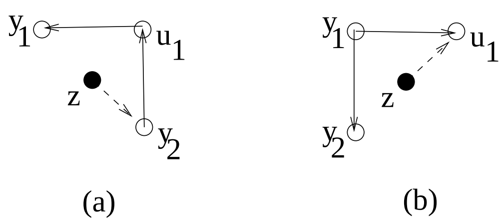

Figure 2: Digraphs in the proof of Lemma 4

Next assume that y_1 and y_2 are not adjacent, as depicted in Figure 1 (b). Let $B = \{u_1, y_1, y_2\}$, let Q be the digraph which consists of arcs $\overrightarrow{y_1u_1}, \overrightarrow{y_1y_2}$, and let $\rho(z) = \{u_1\}$ and $\rho(y) = \emptyset$ for $y \in C \setminus \{z\}$. The digraph Q and the mapping ρ are as depicted in Figure 2 (b). We claim that (B, ρ, Q) is a valid candidate.

[C1] [C2], [C3], [C4] are easily verified as in the previous case. The following table verifies [C5].

v	$3\|A(v)\|$	$2\|B(v)\|$	$\|C(v)\|$	$\|D(v)\|$	$\tau(v)$	$\phi(v)$
y_1	9	4	0	1	1	15
y_2	12	0	1	1	1	15
u_1	12	0	1	1	1	15

∎

Lemma 5 *Suppose $z \in C$, $p(z) = 4$. If z has one minor neighbour, then z has at least one major neighbour.*

Proof. Assume z has one minor neighbour u_1. Let the other neighbour of z be u_2, u_3, u_4 so that (u_1, u_2, u_3, u_4) is a 4-cycle in R. By definition, it suffices to show that one of the neighbours u_i of z has either $p(u_i) \geq 6$ or $p(u_i) = 5$ and $q(u_i) \geq 4$.

Assume to the contrary that for each $i \in \{2,3,4\}$, $p(u_i) \leq 5$, and if $p(u_i) = 5$ then $q(u_i) \leq 3$. Let $B = \{u_1, u_2, u_3, u_4\}$, let Q be the digraph with arcs $\overrightarrow{u_1u_2}, \overrightarrow{u_1u_4}, \overrightarrow{u_2u_3}, \overrightarrow{u_3u_4}$. Let $\rho(z) = \{u_2, u_3\}$ and $\rho(y) = \emptyset$ for $y \in C \setminus \{z\}$. The digraph Q and the mapping ρ are as depicted in Figure 3.

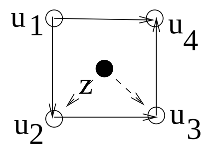

Figure 3: Digraph in the proof of Lemma 5

We claim that (B, ρ, Q) is a valid candidate. Similarly, [C1]-[C4] are easily verified by referring to Figure 3. The following table verifyies [C5].

v	$3\|A(v)\|$	$2\|B(v)\|$	$\|C(v)\|$	$\|D(v)\|$	$\tau(v)$	$\phi(v)$
u_1	9	4	0	1	1	15
u_2	9	2	1	2	1	15
u_3	9	2	1	2	1	15
u_4	9	0	2	3	1	15

∎

Lemma 6 *Suppose $z \in C$ and $p(z) = 5$. Then z has at most 3 minor neighbours.*

Proof. Assume to the contrary that z has four minor neighbours. Then these four minor neighbours form a path, say $P = (u_1, u_2, u_3, u_4)$, of R. Let $B = \{u_1, u_2, u_3\}$, let Q be the digraph with arcs $\overrightarrow{u_2u_1}, \overrightarrow{u_2u_3}$. Let $\rho(y) = \emptyset$ for $y \in C$. We claim that (B, ρ, Q) is a valid candidate. Similarly, we just list a table to verify [C5].

v	$3\|A(v)\|$	$2\|B(v)\|$	$\|C(v)\|$	$\|D(v)\|$	$\tau(v)$	$\phi(v)$
u_1	12	0	1	1	1	15
u_2	9	4	0	1	1	15
u_3	12	0	1	1	1	15

∎

Lemma 7 *If $x \in U$ and $p(x) = q(x) = 5$, then x has at most 3 major neighbours.*

Proof. Assume $x \in U$ and $p(x) = q(x) = 5$. Assume to the contrary that x has 4 major neighbours. Then these four major neighbours form a path of length 3 in Γ_x, say $P = (z_1, z_2, z_3, z_4)$. Since z_i is a major neighbour of x, by definition, $p(z_i) = 4$ and z_i has a minor neighbour. Let the neighbours of x and z_i's be as depicted in Figure 4.

As $q(u_i) \geq 2$, so for $i = 1,2,3,4$, w_i is the minor neighbour of z_i. Since $q(w_i) = 1$ for $i = 1,2,3,4$, for $i = 2,3,4$, z_{i-1} and z_i have degree 1 in Γ_{u_i}. So $\{z_{i-1}, z_i\}$ form a

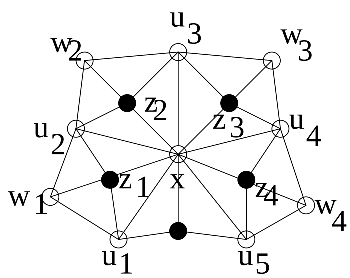

Figure 4: A vertex $x \in U$ with four major neighbours

component of Γ_{u_i}. By Lemma 2, $q(u_i) \leq p(u_i) - 2 = 3$ for $i = 2, 3, 4$. By Definition 4, $p(u_i) \leq 5$ for $i = 1, 2, \cdots, 5$. As $q(u_i) \leq 3$ for $i = 2, 3, 4$, by Lemma 3, $p(u_i) = 5$ for $i = 2, 3, 4$.

Let $B = \{w_1, u_2, w_2, u_3, w_3, x\}$, let Q be the digraph with arcs $\overrightarrow{u_2w_1}, \overrightarrow{w_2u_2}, \overrightarrow{w_2u_3}, \overrightarrow{u_3w_3}, \overrightarrow{u_2x}, \overrightarrow{u_3x}$. Let $\rho(z_1) = \rho(z_2) = \rho(z_3) = \{x\}$ and $\rho(y) = \emptyset$ for $y \in C \setminus \{z_1, z_2, z_3\}$. The digraph Q and the mapping ρ are as depicted in Figure 5. We claim that (B, ρ, Q) is a valid candidate. Similarly, we just list a table to verify

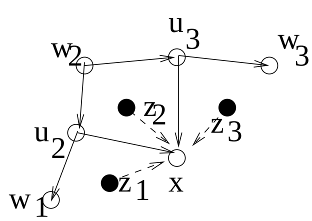

Figure 5: Digraph in the proof of Lemma 7

[C5].

v	$3\|A(v)\|$	$2\|B(v)\|$	$\|C(v)\|$	$\|D(v)\|$	$\tau(v)$	$\phi(v)$
w_1	12	0	1	1	1	15
u_2	6	4	1	3	1	15
w_2	9	4	0	1	1	15
u_3	6	4	1	3	1	15
w_3	12	0	1	1	1	15
x	9	0	2	2	1	14

∎

Lemma 8 *If $x \in U$, $p(x) = 5$ and $q(x) = 4$, then x has at most 2 major neighbours.*

Proof. Assume $x \in U$ and $p(x) = 5$ and $q(x) = 4$. Assume to the contrary that x has 3 major neighbours. Then two of the major neighbours, say z_1, z_2, are adjacent in Γ_x,

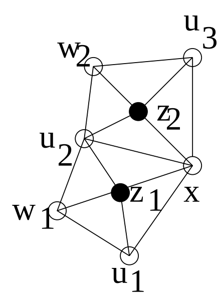

Figure 6: A vertex $x \in U$ with major neighbours z_1, z_2 adjacent in Γ_x

as depicted in Figure 6. By definition, each of z_1, z_2 has at least one minor neighbour. Since $q(u_2) \geq 2$, so u_2 is not a minor neighbour of z_1 or z_2.

There are three cases to be discussed.

Case 1: w_1 is a minor neighbour of z_1 and w_2 is a minor neighbour of z_2. By Definition 4, $p(u_i) \leq 5$ and $q(u_i) \leq q(x) = 4$ for $i = 1, 2, 3$.

Let $B = \{u_1, w_1, u_2, w_2, u_3, x\}$, let Q be the digraphs with arcs $\overrightarrow{w_1u_1}, \overrightarrow{w_1u_2}, \overrightarrow{xu_1}, \overrightarrow{w_2u_2}, \overrightarrow{w_2u_3}, \overrightarrow{xu_3}, \overrightarrow{xu_2}$. Let $\rho(z_1) = \{x, u_1\}$, $\rho(z_2) = \{x, u_3\}$ and $\rho(y) = \emptyset$ for $y \in C \setminus \{z_1, z_2\}$. The digraph Q and the mapping ρ are as depicted in Figure 7 (a). We claim that (B, ρ, Q) is a valid candidate. [C1]-[C4] can be verified

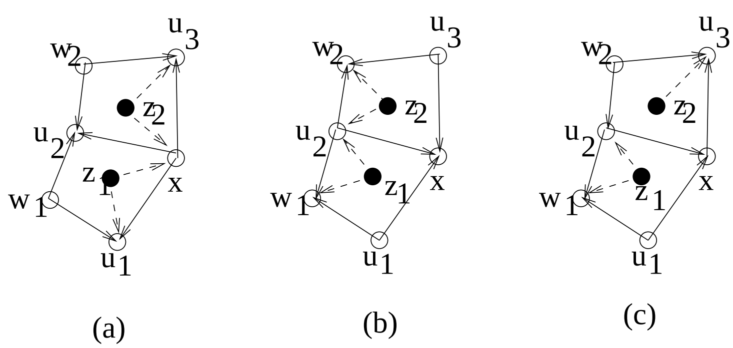

Figure 7: Digraph in the proof of Lemma 8

easily, by referring to Figure 7 (a). The following table verifies [C5].

v	$3\|A(v)\|$	$2\|B(v)\|$	$\|C(v)\|$	$\|D(v)\|$	$\tau(v)$	$\phi(v)$
u_1	9	0	2	3	1	15
w_1	9	4	0	1	1	15
u_2	6	0	3	4	1	14
w_2	9	4	0	1	1	15
u_3	9	0	2	3	1	15
x	6	6	0	2	1	15

Case 2: u_1 is a minor neighbour of z_1 and u_3 is a minor neighbour of z_2. By definition, for $i = 1, 2$, $p(w_i) \leq 5$ and $q(w_i) \leq 4$, and $p(u_2) \leq 5$, $q(u_2) \leq 4$.

Let $B = \{u_1, w_1, u_2, w_2, u_3, x\}$, let Q be the digraphs with arcs $\overrightarrow{u_1w_1}, \overrightarrow{u_2w_1}, \overrightarrow{u_1x}, \overrightarrow{u_2w_2}, \overrightarrow{u_3w_2}, \overrightarrow{u_3x}, \overrightarrow{u_2x}$. Let $\rho(z_1) = \{w_1, u_2\}$, $\rho(z_2) = \{w_2, u_2\}$ and $\rho(y) = \emptyset$ for $y \in C \setminus \{z_1, z_2\}$. The digraph Q and the mapping ρ are as depicted in Figure 7 (b).

We claim that (B, ρ, Q) is a valid candidate. [C1]-[C4] can be verified easily, by referring to Figure 7 (b). The following table verifies [C5].

v	$3\|A(v)\|$	$2\|B(v)\|$	$\|C(v)\|$	$\|D(v)\|$	$\tau(v)$	$\phi(v)$
u_1	9	4	0	1	1	15
w_1	9	0	2	3	1	15
u_2	6	6	0	2	1	15
w_2	9	0	2	3	1	15
u_3	9	4	0	1	1	15
x	6	0	3	4	1	14

Case 3: u_1 is a minor neighbour of z_1 and w_2 is a minor neighbour of z_2. (The case that w_1 is a minor neighbour of z_1 and u_3 is a minor neighbour of z_2 is symmetric.) By Definition 4, $p(u_2), p(u_3), p(w_1) \leq 5$, $q(u_2), q(u_3), q(w_1) \leq 4$.

Let $B = \{u_1, w_1, u_2, w_2, u_3, x\}$, let Q be the digraphs with arcs $\overrightarrow{u_1w_1}, \overrightarrow{u_2w_1}, \overrightarrow{u_1x}, \overrightarrow{w_2u_2}, \overrightarrow{w_2u_3}, \overrightarrow{xu_3}, \overrightarrow{u_2x}$. Let $\rho(z_1) = \{w_1, u_2\}$, $\rho(z_2) = \{u_3\}$ and $\rho(y) = \emptyset$ for $y \in C \setminus \{z_1, z_2\}$. The digraph Q and the mapping ρ are as depicted in Figure 7 (c).

We claim that (B, ρ, Q) is a valid candidate. [C1]-[C4] can be verified easily, by referring to Figure 7 (b). The following table verifies [C5].

v	$3\|A(v)\|$	$2\|B(v)\|$	$\|C(v)\|$	$\|D(v)\|$	$\tau(v)$	$\phi(v)$
u_1	9	4	0	1	1	15
w_1	9	0	2	3	1	15
u_2	6	4	1	3	1	15
w_2	9	4	0	1	1	15
u_3	9	0	2	3	1	15
x	6	2	2	4	1	15

∎

Lemma 9 *If $x \in U$, $p(x) = 5$ and $q(x) = 3$, then x has at most 1 major neighbour.*

Proof. Assume to the contrary that x has two major neighbours, z_1 and z_2. By definition, each z_i has $p(z_i) = 4$ and has two minor neighbours. If z_1 and z_2 are adjacent in Γ_x, then the configuration is as depicted in Figure 6. Then u_1, w_1 are minor neighbours of z_1 and w_2, u_3 are minor neighbours of z_2. Let $B = \{w_1, u_1, x, u_3, w_2\}$, let

Q be the digraphs with arcs $\overrightarrow{u_1w_1}, \overrightarrow{u_1x}, \overrightarrow{u_3w_2}, \overrightarrow{u_3x}$, and let $\rho(y) = \emptyset$ for all $y \in C$. It is easy to verify that (B, ρ, Q) is a valid candidate.

Assume that z_1 and z_2 are not adjacent in Γ_x. Depending on the position of the other neighbour z_3 of x in C, we have two cases, as depicted in Figure 8.

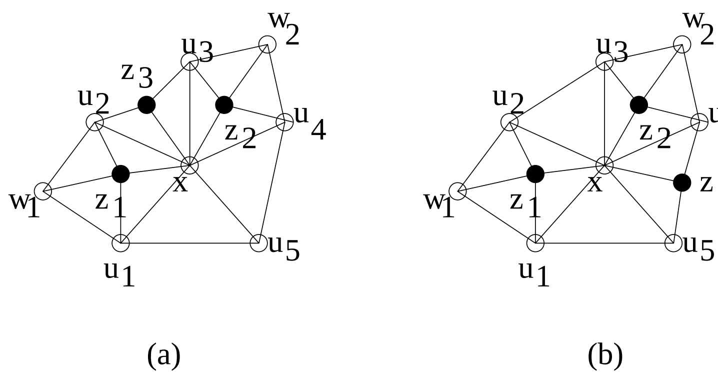

Figure 8: A vertex $x \in U$ with major neighbours z_1, z_2 not adjacent in Γ_x

Case 1 This case is as depicted in Figure 8 (a). As $q(u_2) \geq 2$ and $q(u_3) \geq 2$, we conclude that u_1, w_1 are minor neighbours of z_1 and w_2, u_4 are minor neighbours of z_2. Let $B = \{u_1, w_1, w_2, u_4, x\}$, let Q be the digraph with arcs $\overrightarrow{u_1w_1}, \overrightarrow{u_1x}, \overrightarrow{u_4w_2}, \overrightarrow{u_4x}$. Let $\rho(y) = \emptyset$ for $y \in C$. The digraph Q and the mapping ρ are as depicted in Figure 9 (a). We claim that (B, ρ, Q) is a valid candidate. Below is a table to verify [C5].

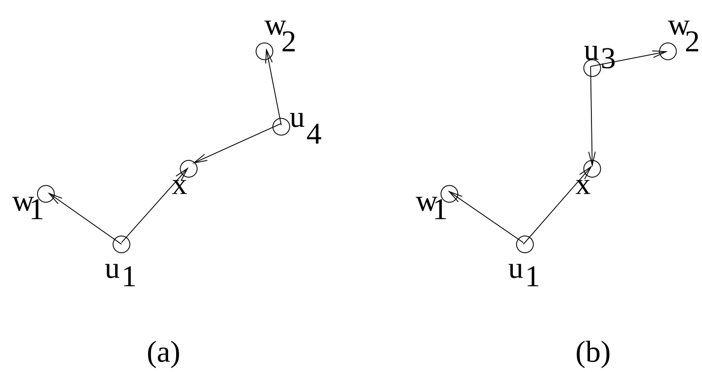

Figure 9: Digraph in the proof of Lemma 9

v	$3\|A(v)\|$	$2\|B(v)\|$	$\|C(v)\|$	$\|D(v)\|$	$\tau(v)$	$\phi(v)$
w_1	12	0	1	1	1	15
u_1	9	4	0	1	1	15
w_2	12	0	1	1	1	15
u_4	9	4	0	1	1	15
x	9	0	2	3	1	15

Case 2 This case is as depicted in Figure 8 (b). As $q(u_4) \geq 2$, we conclude that u_3, w_2 are minor neighbours of z_2. If u_2 is a minor neighbour of z_1, then let $B = \{u_2, u_3, w_2\}$, let Q be the digraph with arcs $\overrightarrow{u_3u_2}, \overrightarrow{u_3w_2}$. Let $\rho(y) = \emptyset$ for $y \in C$. Then (B, ρ, Q) is a valid candidate. Otherwise, w_1, u_1 are minor neighbours of z_1. Let $B = \{u_1, w_1, u_3, w_2, x\}$, let Q be the digraph with arcs $\overrightarrow{u_1w_1}, \overrightarrow{u_1x}, \overrightarrow{u_3w_2}, \overrightarrow{u_3x}$. Let $\rho(y) = \emptyset$ for $y \in C$. The digraph Q and the mapping ρ are as depicted in Figure 9 (b). Then (B, ρ, Q) is a valid candidate. The verifications are similar as above and omitted. ∎

Proof of Theorem 3 Charge each vertex $v \in V(R)$ with a charge $c_0(v) = d_R(v)$. We redistribute the charges according to the following rules:

Suppose $x \in U$ and $z \in C$ and $x \sim_R z$. If x, z are major neighbours of each other, then move a charge of 1 from x to z. If x, z are neither major neighbours nor minor neighbours of each other, then move a charge of $1/2$ from x to z. If x is a minor neighbour of z, then no charge is moved from x to z.

Denote by c^* the new charge assignment. Since $\sum_{x\in V(R)} c^*(x) = \sum_{x \in V(R)} c_0(x) = 6|V(R)| - 12$, there is a vertex x with $c^*(x) < 6$. We shall derive a contradiction by showing that $c^*(x) \geq 6$ for each $x \in V(R)$.

Suppose $x \in U$. If $p(x) \geq 6$, then $c^*(x) \geq p(x)$. Assume $p(x) = 5$. By Lemma 3, $q(x) \geq 1$. If $q(x) = 5$, then $c_0(x) = 10$. By Lemma 7, x has at most three major neighbour, each of which receives a charge of 1 from x, and every other neighbour of x in C receives a charge of $1/2$ from x. So the total charge sent out from x is at most 4, and hence $c^*(x) \geq 6$. If $q(x) = 4$, then $c_0(x) = 9$. By Lemma 8, x has at most two major neighbour, each of which receives a charge of 1 from x, and every other neighbour of x in C receives a charge of $1/2$ from x. So the total charge sent out from x is at most 3, and hence $c^*(x) \geq 6$. If $q(x) = 3$, then $c_0(x) = 8$. By Lemma 9, x has at most 1 major neighbour which receives a charge of 1 from x, and every other neighbour of x in C receives a charge of $1/2$ from x. So the total charge sent out from x is at most 2, and hence $c^*(x) \geq 6$. If $q(x) = 2$, then by definition, x has no major neighbour, so each neighbour of x in C receives a charge of $1/2$ from x. So $c^*(x) = 6$. If $q(x) = 1$, then x has only one minor neighbour in C which receives no charge from x. So $c^*(x) = c_0(x) = 6$.

Assume $p(x) = 4$. Then by Lemma 3, $q(x) = 4$ and hence $c_0(x) = 8$. Each neighbour of x in C receives a charge of $1/2$ from x. So the total charge sent out from x is 2. Hence $c^*(x) = 6$.

Suppose $x \in C$. If $p(x) = 4$, then $c_0(x) = 4$. By Lemmas 4 and 5, either x has no minor neighbours, or has one minor neighbour and at least one major neighbour, or two minor neighbour and two major neighbour. If x has no minor neighbour, then it receives at least $1/2$ from each of its neighbours in U and thus has $c^*(x) = 6$. If x has 1 minor neighbour and at least 1 major neighbour, then it receives a charge of 1 from the major neighbour and receives a charge of at least 1/2 from each of the two other neighbours (which are neither major nor minor neighbours). Hence $c^*(x) \geq 6$. If x has two major neighbours, then it receives a charge of 1 from each of them and hence has $c^*(x) \geq 6$. If $p(x) = 5$, then $c_0(x) = 5$. By Lemma 6, x has at most 3 minor neighbours. So x has at least two neighbours which are not minor, and each of them send 1/2 charge to x. Therefore $c^*(x) \geq 6$.

This completes the proof of Theorem 3. ∎

6 Proof of Theorem 1

This Section proves Theorem 1. It suffices to prove Theorem 1 for plane triangulations.

Suppose G is a plane triangulation. We shall construct a dynamic rough ordering $(L_0, \mathcal{P})$, a preference function ρ of $(L_0, \mathcal{P})$, and a bound graph H for $(G, L_0, \mathcal{P}, \rho)$ as follows.

The blocks of $\mathcal{P}$ are constructed one by one. First we construct B_m, then B_{m-1}, and so on. At the time we construct B_i, we shall construct simultaneously the restriction of the digraph L_0 to B_i, the intersection $\rho(y) \cap B_i$ for each $y \in V^-_{L_0,\not\approx}(B_i)$, and the edges of $H \setminus (B_m \cup B_{m-1} \cup \cdots \cup B_{i+1})$ incident to vertices of B_i. Initially, B_m consists of a single vertex of degree at most 5 in G. The edges of H incident to the vertex of B_m are exactly the edges of G incident to it.

Suppose we have constructed $B_m, B_{m-1}, \cdots, B_{i+1}$. Let $C' = \cup_{j=i+1}^m B_j$, and let $U = V \setminus C'$. Now for each $x \in C'$ and $y \in U$, $y <_{L_0} x$. By our construction of $B_m, B_{m-1}, \cdots, B_{i+1}$, each vertex of C' is adjacent to at most 5 vertices of U. First we delete all edges of G joining vertices of C'. If $z \in C'$ is adjacent to at most three vertices of U, then delete z, and add edges between each pair of non-adjacent neighbours of z in C. Let $C = C' \setminus \{z : |N_G(z) \cap U| \leq 3\}$. If z is adjacent to 4 or 5 vertices of G, then add edges between each pair of non-adjacent 'consecutive' neighbours of z in U. Here consecutive refers to the particular plane embedding of $G \setminus E(C')$. Now the resulting graph is a plane triangulation R. Obviously $C \cup U$ is a partition of $V(R)$, and C is an independent set of R.

By Theorem 3, (R, C, U) has a valid candidate (B, ρ', Q). Let $B_i = B$. Let the restriction of L_0 to B_i be Q. For each vertex $y \in C'$, if $y \in C' \setminus C$, then let $\rho(y) \cap B_i = N_G(y) \cap B_i$; if $y \in C$, then let $\rho(y) \cap B_i = \rho'(y)$. Let the edges of $H \setminus (B_m \cup B_{m-1} \cup \cdots \cup B_{i+1})$ incident to vertices of B_i be exactly the edges of $R \setminus C$ incident to vertices of B_i. For each $x \in B_i$ and $y \in U \setminus B_i$, let $y <_{L_0} x$. Note that by definition of valid candidate, $\phi(x) \leq 15$ for each $x \in B_i$, which implies that x is adjacent to at most 5 vertices of $U \setminus B_i$.

We claim that this process constructs a dynamic rough ordering $(L_0, \mathcal{P})$, a preference function ρ of $(L_0, \mathcal{P})$, and a bound graph H for $(G, L_0, \mathcal{P}, \rho)$, such that for each vertex $x \in V$, $\phi_H(x) \leq 15$. By Lemma 1, this implies that $\gamma(G) \leq 16$, and hence $\mathrm{col_g}(G) \leq 17$.

It follows from the definition that $(L_0, \mathcal{P})$ is a dynamic rough ordering of G. To prove that ρ is a preference function of $(L_0, \mathcal{P})$, we need to show that for any index i, if $y \in V^-_{L_0,\not\approx}(B_i)$, then $B_i \cap \rho(y)$ contains at most one edge of $\overline{L_0}$.

Let C', C, U, B and R be the sets and graph defined as above at the time B_i is constructed. Then $y \in C'$. If $y \in C' \setminus C$ then $|\rho(y)| = |N_G(y) \cap B| \leq |N_R(y) \cap U| \leq 3$. If $|N_R(y) \cap U| \leq 2$, then of course $\rho(y) \cap B$ contains at most one arc of L_0. If $|N_R(y) \cap U| = 3$, then $N_R(y) \cap U$ is a facial triangle of R. By [C1], the facial triangle contains at most one arc of Q. Hence $B_i \cap \rho(y)$ contains at most one arc of L_0.

Assume $y \in C$. Then it follows from [C2] that $|B_i \cap \rho(y)| \leq 2$, and $B_i \cap \rho(y)$ contains at most one arc of L_0. So ρ is a preference function of $(L_0, \mathcal{P})$.

Now we prove that H is a bound graph for $(G, L_0, \mathcal{P}, \rho)$. It is obvious that G is a subgraph of H, i.e., [B1] is satisfied. Assume $x, y \in B_i$ and there is a vertex $z \in \rho^{-1}(x) \cap \rho^{-1}(y)$ such that $x \sim_G z$ and $y \sim_G z$. Let C', C, U, B and R be the sets and graph defined as above at the time B_i is constructed. Then $z \in C'$. If $z \in C' \setminus C$, then by definition of R, we have $x \sim_R y$, hence $x \sim_H y$. Assume $z \in C$. Since $x, y \in \rho(z)$, By [C3], we have $x \sim_R y$ and hence $x \sim_H y$, i.e., [B2] is satisfied. Assume $x \in B_i$ and $y \in V^+_{L_0,\not\approx}(B_i)$ and there is a vertex $z \in \rho^{-1}(x)$ such that $y \sim_G z$. If $z \in C' \setminus C$, then by definition of R, we have $x \sim_R y$ and hence $x \sim_H y$. If $z \in C$, then by [C4], we have $x \sim_R y$ and hence $x \sim_H y$. Thus [B3] is satisfied, and hence H is indeed a bound graph for $(G, L_0, \mathcal{P}, \rho)$.

It remains to show that for each x, we have $\phi_H(x) \leq 15$. This follows from the construction, because if $x \in B_i$ and $B_i = B$, where (B, Q, ρ) is the corresponding valid candidate, then $\phi_H(x) = \phi(x) \leq 15$. ∎

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