The game coloring number of pseudo partial k-trees

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July, 1998.†

Running Head: Game coloring pseudo k-tree.

Abstract

This paper introduces a new class of graphs: (a,b)-pseudo partial k-trees. In some sense, the parameters (a,b) measure a graph's deviation from being a partial k-tree. In particular, a $(0,0)$-pseudo partial k-tree is just a partial k-tree. We discuss the game coloring number (as well as the game chromatic number) of (a,b)-pseudo partial k-trees, and prove that the game coloring number of an (a,b)-pseudo partial k-tree is at most $3k+2a+b+2$. In particular, the game coloring number of a partial k-tree is at most $3k+2$. This reduces considerably the previous known upper bound for the game chromatic number of a partial k-tree. Then we describe another strategy for Alice for playing the game, which gives a better upper bound for the game coloring number of (a,b)-pseudo partial 2-trees. Namely we prove that the game coloring number of an (a,b)-pseudo partial 2-tree is at most $a+b+8$. By using this result, we prove that the game coloring number of a graph embeddable on an orientable surface of genus $g \geq 1$ is at most $\lfloor \frac{1}{2}(3\sqrt{1+48g}+23) \rfloor$. This is the first upper bound for the game coloring number of such graphs, and it also improves considerably the previous known upper bound for the game chromatic number of such graphs. Moreover, the strategy for Alice introduced in this paper

*This research was partially supported by the National Science under grant NSC87-2115-M-110-004.

†*1991 Mathematics Subject Classification.* 05C20, 05C35, 05C15

Key words and phrases. Game chromatic number, game coloring number, chordal graph, (a,b)-pseudo chordal graphs, (a,b)-pseudo partial k-trees.

This copy was printed on June 4, 1999

for playing the game can be considered as a common generalization of the previous known strategies for playing the game on forests, interval graphs, outerplanar graphs and planar graphs.

1 Introduction

Let G be a finite graph and let X be a set of colors. We consider the following *coloring game* played by two players Alice and Bob with Alice playing first. The players take turns coloring the vertices of G with colors from X in such a way that no adjacent vertices have the same color. If after $n = |V(G)|$ moves, all the vertices of G is colored, Alice is the winner. Bob wins if at some time one of the players has no legal move, and yet there are uncolored vertices. This means that each of the uncolored vertices is adjacent to vertices of all the colors. So Alice's goal is to produce a proper coloring of all the vertices of G, and Bob tries to prevent this from happening. The *game chromatic number* $\chi_g(G)$ of a graph $G = (V, E)$ is the least cardinality of a color set X for which Alice has a winning strategy. This parameter is well-defined, since Alice always wins if $|X| = |V|$.

The game coloring number of a graph is a variation of the game chromatic number of a graph. Instead of playing a coloring game on G with a color set X, consider the following *ordering game*, also played by two players Alice and Bob with Alice playing first. The players take turns choosing vertices from the set of unchosen vertices. This create a linear order L of the vertices of G with $x < y$ if and only if x is chosen before y. Given a linear order L on V, the *back degree* of a vertex x *relative to* L is equal to the number of neighbours of x which proceeds x in L. The *back degree of* L is the maximum back degree of a vertex relative to L. Alice's goal is to minimize the back degree of L, while Bob's goal is to maximize the back degree of L. This is a zero-sum two person game. Therefore each player has an optimal strategy. The *game coloring number* $\mathrm{col}_g(G)$ of G is defined to be $1 + k$, where k is the back degree of a linear order L, which is produced by playing the game with both players using their optimal strategies. Equivalently, $\mathrm{col}_g(G)$ is the smallest (largest) integer t for which Alice (Bob) has a strategy to ensure that the linear order produced by playing the game has backdegree at most (at least) $t - 1$.

It is easy to see that $\chi_g(G) \leq \mathrm{col}_g(G)$ for any graph G. Indeed, suppose $\mathrm{col}_g(G) = 1 + k$, then Alice has a strategy playing the ordering game, so that no matter how Bob plays the game, the resulting linear order has back degree at most k. Now Alice can play the coloring game on G by using the same strategy to choose the vertex to be colored and by using the First-Fit to choose the color for that vertex. By using such a strategy, whenever a vertex is chosen to be colored (by either player), it has at most k colored neighbours. Therefore $1 + k$ colors suffices to color all the vertices.

There are graphs G such that $\chi_g(G)$ are much smaller than $\mathrm{col}_g(G)$. The complete bipartite graph $K_{n,n}$ has game chromatic number 3, and game coloring number $1+n$. However, for many "natural" classes of graphs, the best known upper bounds for their game coloring numbers are also the best known upper bounds for their game chromatic numbers. It was proved by Faigle, Kern, Kierstead and Trotter [4] that the game chromatic number, as well as the game coloring number, of a forest is at most 4, and that the game chromatic number, as well as the game coloring number, of an interval graph G is at most $3\omega(G) - 2$. The game chromatic number of planar graphs was first studied by Kierstead and Trotter [8]. It was proved in [8] that the game chromatic number of a planar graph is at most 33, and this upper bound was reduced to 30 by Dinski and Zhu [3]. Both arguments in [8] and [3] provide no upper bound for the game coloring number of planar graphs. However, recently Zhu [12] proved that the game coloring number of a planar graphs is at most 19, and very recently this upper bound is further reduced by Kierstead [7] to 18. It was also proved in [8] that the game chromatic number of an outerplanar graph is at most 8. The upper bound is reduced by Guan and Zhu [5] to 7, again by showing that outerplanar graphs have game coloring number at most 7.

In proving the upper bounds for the game coloring numbers of the above mentioned classes of graphs, one needs to design a strategy for Alice, so that no matter how Bob plays the game, the resulting linear order has back degree bounded by a given number. It turns out all the presently known strategies for Alice for the various classes of graphs have many common features: all the strategies use some "tree structure" of the graphs, keep track of a set of active vertices, and selecting the next vertex in a similar manner. By exploring these commom features, we shall present in this paper a unified strategy for Alice, for playing the game on all the above mentioned classes of graphs. This is achieved by introducing a new class of graphs, the class of (a, b)-pseudo partial k-trees, which contains all the above mentioned classes of graphs as subclasses. The parameters (a, b), in some sense, measure a graph's deviation from being a partial k-tree. In particular, a $(0, 0)$-pseudo partial k-tree is just a partial k-tree. On the other hand, for $0 < a \leq b$, an (a, b)-pseudo partial k-tree may have arbitrary large treewidth. We shall present a strategy for Alice for playing the ordering game on (a, b)-pseudo partial k-trees, and prove that by using this strategy, Alice can make sure that the resulting linear order has back degree at most $3k+2a+b+1$, hence any (a, b)-pseudo partial k-tree has game coloring number at most $3k+2a+b+2$. In particular, the game coloring number (and hence the game chromatic number) of a partial k-tree is at most $3k + 2$. This reduces considerably the previous known upper bound for the game chromatic number of partial k-trees, which was $(k + 1)(k + 2)$, and was obtained in [3][1]. Thus, the present best known upper

[1]the author recently learned that Kierstead and Tuza have proved before that the game coloring number (although the name "game coloring number" was not used) of a chordal graph is at most $6k - 8$, where k is the clique number.

bound for the game coloring number of partial k-trees is again the best known upper bound for the game chromatic number of partial k-trees.

For (a, b)-pseudo partial 2-trees, we present another strategy for Alice which gives a better upper bound for the game coloring number. Namely, we shall prove that the game coloring number of an (a, b)-pseudo partial 2-tree is at most $a+b+8$. We then apply this result to deduce an upper bound for the game coloring number of graphs embeddable on orientable surfaces. Let S_g be an orientable surface of genus $g \geq 1$. It was first proved in [8] that the game chromatic number of graphs embeddable on S_g is bounded by a constant. However, no explicit upper bound was given in [8] (as Ramsey theorem was used in the proof, the upper bound is probably very large). In [3], it was proved that the game chromatic number of any graph embeddable on S_g is at most $(4g + 4)(4g + 5)$. We remark that no upper bound was known for the game coloring number of graphs embeddable on S_g. In this paper, we give an upper bound for the game coloring number of such graphs, which also reduce the quadratic form upper bound for the game chromatic number to a bound of square root order. Namely we shall prove that the game coloring number (and hence the game chromatic number) of any graph embeddable on S_g is at most $\lfloor \frac{1}{2}(3\sqrt{1 + 48g} + 23) \rfloor$. Since the chromatic number of a graph embeddable on S_g could be $\lfloor \frac{1}{2}(\sqrt{1 + 48g} + 7) \rfloor$, which is an obvious lower bound for the maximum game chromatic number of graphs embeddable on S_g, the upper bound we obtained here is of the correct order, although the coefficient is probably not sharp.

It can also be derived from the result on (a, b)-pseudo partial 2-trees that planar graphs have game coloring number at most 19, which was first proved in [12].

Finally we remark that the strategy for Alice described in this paper can be regarded as a common generalization of all the previous known strategies for Alice for the various classes of graphs. If the strategy of this paper is applied to each of the classes of graphs for which an upper bound for the game chromatic number is known (i.e., forests, outerplanar graphs, interval graphs, planar graphs, partial k-trees, chordal graphs, graphs embeddable on a fixed surface), it achieves the best known upper bound for the game chromatic number and the game coloring number.

2 Pseudo chordal graphs and pseudo partial k-trees

A graph $G = (V, E)$ is a *chordal graph* if every cycle of G of length ≥ 4 has a chord. An equivalent definition of a chordal graph $G = (V, E)$ is that there is a linear order, say $v_1, v_2, \cdots, v_n$, on the vertex set V, such that for each i, the set $\{v_j : j < i, v_j v_i \in E\}$ induces a complete subgraph of G. By orienting the edges of G in such a way that an edge $v_i v_j$ is directed from v_i to v_j if and only if $i > j$, we obtain an oriented graph $\vec{G} = (V, \vec{E})$ which is acyclic and for each vertex v_i, its outneighbours induce a transitive tournament. The converse is also true, i.e., a graph $G = (V, E)$ is a chordal graph if and only if G has an orientation $\vec{G} = (V, \vec{E})$ which is acyclic and the outneighbours of each vertex induce a transitive tournament.

For an oriented graph $\vec{G}$ and a vertex u of $\vec{G}$, we denote the neighbours of u by $N_{\vec{G}}(u)$, the outneighbours of u by $N^+_{\vec{G}}(u)$, and the inneighbours of u by $N^-_{\vec{G}}(u)$. We denote the degree, outdegree and indegree of u by $d_{\vec{G}}(u)$, $d^+_{\vec{G}}(u)$ and $d^-_{\vec{G}}(u)$, respectively. When the oriented graph $\vec{G}$ is clear from the context we will drop the subscript.

We shall now define two classes of graphs: (a, b)-pseudo chordal graphs and (a, b)-pseudo partial k-trees.

Definition 1 *Suppose a, b are integers such that $0 \leq a \leq b$. A connected graph $G = (V, E)$ is called an* (a, b)-pseudo chordal graph *if there are two oriented graphs $\vec{G}_1 = (V, \vec{E}_1)$ and $\vec{G}_2 = (V, \vec{E}_2)$ on the same vertex set V such that the following is true:*

- *$E_1 \cap E_2 = \emptyset$ and $E = E_1 \cup E_2$. Where E_i is the set of edges obtained from $\vec{E}_i$ by omitting the orientations.*

- *$\vec{G}_1$ is acyclic and has a single sink r.*

- *$\vec{G}_2$ has maximum outdegree $\leq a$, and has maximum degree $\leq b$.*

- *Let $N^+(x) = N^+_{\vec{G}_1}(x)$ be the set of outneighbours of x in $\vec{G}_1$. Let $\vec{G}^* = (V, \vec{E}_1 \cup \vec{E}_2)$. Then $N^+(x)$ induces a transitive tournament in $\vec{G}^*$.*

Definition 2 *A graph G is called an* (a, b)-pseudo partial k-tree *if it is a subgraph of an (a, b)-pseudo chordal graph in which the directed graph $\vec{G}_1$ in the definition has maximum outdegree $\leq k$.*

Note that any induced subgraph of an (a, b)-pseudo chordal graph is still an (a, b)-pseudo chordal graph. Therefore, an (a, b)-pseudo partial k-tree can be equivalently defined as a spanning subgraph of an (a, b)-pseudo chordal graph in which the directed graph $\vec{G}_1$ in the definition has maximum outdegree $\leq k$.

It follows from the definition that if $a = 0$ (hence $b = 0$), then a $(0, 0)$-pseudo chordal graph is simply a chordal graph, and a $(0, 0)$-pseudo partial k-tree is simply a partial k-tree. However, for some $0 < a \leq b$, there are (a, b)-pseudo k-trees which have arbitrarily large treewidth. For example, a result proved in [12] is equivalent to the statement that every planar graph is a $(3, 8)$-pseudo partial 2-tree. Nevertheless, for fixed a, b, the class of (a, b)-pseudo chordal graph does have some similarities with the class of chordal graphs, and the class of (a, b)-pseudo partial k-trees does have similarities with the class of partial k-trees. We shall explore such similarities, and use them to derive upper bounds for the game coloring number of the class of pseudo chordal graphs, as well as the class of pseudo partial k-trees.

Theorem 1 *Suppose G is an (a, b)-pseudo chordal graph. Then* $\mathrm{col}_g(G) \leq 3k + 2a + b + 2$. *Here k is the maximum outdegree of a vertex of $\vec{G}_1$ in the definition above.*

To prove Theorem 1, it suffices to give a strategy for Alice so that no matter how Bob plays the game, the resulting linear order has back degree at most $3k + 2a + b + 1$.

Suppose $G = (V, E)$ is an (a, b)-pseudo chordal graph, and that $\vec{G}_1 = (V, \vec{E}_1)$, $\vec{G}_2 = (V, \vec{E}_2)$ and $\vec{G}^*$ are oriented graphs as in Definition 1. When playing the game, Alice will only take the oriented graph $\vec{G}_1$ into consideration. For each vertex x, the outneighbours of x in $\vec{G}_1$ induce a transitive tournament in $\vec{G}^*$. We call an outneighbour y of x the jth outneighbour of x if y has outdegree $j - 1$ in this transitive tournament. (which is a subdigraph of $\vec{G}^* = (V, \vec{E}_1 \cup \vec{E}_2)$).

For $x \neq r$, let $f(x)$ be the first outneighbour of x and let $l(x)$ be the last outneighbour of x. Let T be the spanning directed tree of $\vec{G}_1$ induced by the edges $xf(x)$. The sink r of $\vec{G}_1$ is also the sink of T, and r is also called the *root of* T.

In the process of the game, Alice will keep track of a subset T_a of V, which is called the active set. The set T_a contains the root r, and will always induce a subtree of T. Suppose at a certain stage, the active set is T_a. We define two operations, the *extension* and the *switch*, on the set of directed paths of $\vec{G}_1$ (with respect to T_a):

Given a directed path $P = (v_1, v_2, \cdots, v_h)$ of $\vec{G}_1$, let P' be the unique directed path of T connecting v_h to T_a, i.e., the first vertex of P' is v_h and the last vertex of P' is a vertex of T_a, and all the inner vertices (if any) are not in T_a. Recall that T_a induces a subtree of T and contains the root r. Therefore the path P' exists and is unique. The concatenation PP' of P and P' is called the *extension of* P. Since $\vec{G}_1$ is acyclic, we know that PP' is indeed a directed path of $\vec{G}_1$. Note that given a directed path P of $\vec{G}_1$, its extension is unique (as T_a is fixed).

Also note that if the last vertex of P is in T_a, then its extension is itself. By the definition, P is allowed to intersect T_a.

Suppose $P = (v_1, v_2, \cdots, v_h)$ is a directed path of $\vec{G}_1$. If v_h is the jth outneighbour of v_{h-1} and $v_h \neq l(v_{h-1})$, i.e., v_h is not the last outneighbour of v_{h-1}, then the switch of P is the directed path P' obtained from P by replacing the last edge by $v_{h-1}u$, i.e., $P' = (v_1, v_2, \cdots, v_{h-1}, u)$, where u is the $(j+1)$th outneighbour of v_{h-1}.

At any stage of the game, we say a vertex is a *selected vertex* if it has been selected by either player before that stage. Otherwise, the vertex is *a free vertex* at that stage.

Now we are ready to describe Alice's strategy. In the process of the game, we shall construct an auxiliary subgraph $\vec{H}$ of $\vec{G}_1$. The graph $\vec{H}$ is not needed in the description of the strategy, but only needed in the proof below.

Initially, Alice select r, and set $T_a = \{r\}$. Let $\vec{H}$ be the single vertex graph induced by $\{r\}$. Suppose at certain stage of the game, the active set is T_a, and Bob has selected a vertex x. Then Alice select the next vertex by the following rule:

Let $P_1 = (xf(x))$. Let P_2 be the extension of P_1. Alice will repeat the following procedure until she finds the vertex to be selected.

Suppose the presently found directed path is P_{2t} for some $t \geq 1$, and that the last edge of P_{2t} is vu.

(1): If u is a free vertex, then select u.

(2): If u is a selected vertex, $u = l(v)$ and v is a free vertex, then select v.

(3): If u is a selected vertex, $u = l(v)$, and v is also a selected vertex, then select any free vertex all of its outneighbours are selected.

(4): If u is a selected vertex, and $u \neq l(v)$, then let P_{2t+1} be the switch of P_{2t} and let P_{2t+2} be the extension of P_{2t+1}, and go back to repeat the procedure (with P_{2t} replaced by P_{2t+2}).

It is obvious that the procedure will stop in $O(|V|)$ steps, and hence Alice will eventually select a vertex.

Let P_{2j} be the last directed path found in the above procedure. After Alice selected the next vertex, say v, add the vertices of the directed path P_{2j} and the vertex v to T_a. It is obvious that T_a induces a subtree of T after adding these vertices. Add all the vertices and directed edges of the path P_{2j} to $\vec{H}$, except in case the vertex v is selected by Rule (2) or (3), then the last edge of P_{2j} is not added to $\vec{H}$ (while the other edges and vertices are still added to $\vec{H}$).

Theorem 2 *Let k be the maximum outdegree of a vertex of $\vec{G}_1$. If Alice uses the strategy as described above, then the back degree of the linear order produced in the game is at most $3k + 2a + b + 1$.*

Proof. First we show that at any moment, after Alice finished her move and before Bob takes his next move, any free vertex has at most $3k + 2a + b$ selected neighbours.

First we make a few observations.

Observation 1 *All the selected vertices are active.*

This follows from the strategy.

Observation 2 *Each vertex of $\vec{H}$ has indegree at most 2. Moreover, if x has indegree 2 in $\vec{H}$, then x is a selected vertex.*

Given a vertex x of $V(\vec{H})$. Let e_1 and e_2 be the first two in-edges of x in $\vec{H}$. Note that the graph $\vec{H}$ is obtained by successively adding vertices and edges. Here we assume that e_1 is the first in-edge of x that is added to $\vec{H}$, and e_2 is the second in-edge of x added to $\vec{H}$.

When e_1 is added to $\vec{H}$, the vertex x becomes active (or is active already at that stage). When e_2 is added to $\vec{H}$, the path P_{2t} will terminate at x when it reaches x, because $x \in T_a$ at that stage. Therefore by the rule x will be selected by Alice. It also follows from the strategy and the construction of $\vec{H}$ that after a vertex x is selected, then no more in-edges of x will be added to $\vec{H}$, because whenever the path P_{2t} reaches x, it is either switched off, or the next vertex is selected by Rules 2 or 3.

Observation 3 *The vertices selected by Alice is either active before she selects it, or all its parents have been selected.*

This follows trivially from the strategy.

Observation 4 *If x is an inneighbour of y in $\vec{H}$ and that y is the jth outneighbour of x, then for every $j' < j$, the j'th outneighbour of x is a selected vertex.*

If x is an inneighbour of y in $\vec{H}$, then there is a path P_{2t} constructed in the procedure that passes through the edge xy. However, all the edges of the paths P_{2t} are edges of the form $xf(x)$, except in the case that a switch operation is applied. But we apply a switch operation only if the end vertex of that path is a selected vertex. Therefore if y is the jth outneighbour of x and $j \geq 2$, and

y' is the $(j-1)$th outneighbour of x, then the path P_{2t} is obtained from a path that ends with the edge xy' by an operation of switch (and extension afterwards). Hence y' is a selected vertex. Similarly for all $j' < j$, the j'th outneighbour of x is a selected vertex.

Observation 5 *If x is an active vertex, then either x and all its outneighbours are selected, or it has outdegree at least 1 in $\vec{H}$.*

Suppose x becomes active at a certain stage. Then either it is selected by Alice using Rule 2 or 3, or a path P_{2t} passes through it. In the former case, x and all its parents are selected, and in the latter case, it has outdegree at least 1 in $\vec{H}$.

With these observations, we are ready to count the number of active neighbours of free vertices. Suppose Alice has finished her move, and that x is still a free vertex. We now count the number of active neighbours of x in $\vec{G}^*$.

Let R be the set of outneighbours of x in $\vec{G}_1$ and let R' be the set of outneighbours of x in $\vec{G}_2$. Then $|R| \leq k$ and $|R'| \leq a$.

Let S be the set of active inneighbours of x in $\vec{G}_1$. Then by Observation 5, each vertex of S has outdegree at least 1 in $\vec{H}$ (because x is not selected yet). By Observation 4, if a vertex $y \in S$ is not an inneighbour of x, then y is an inneighbour of one of the outneighbours of x in $\vec{G}^*$, which means that y is an inneighbour of a vertex in $R \cup R'$. Thus each vertex of S is either an inneighbour of x or is an inneighbour of a vertex of $R \cup R'$. By Observation 3, x has at most one inneighbour in $\vec{H}$, and moreover, if x has an inneighbour in $\vec{H}$, then x is active, which, by Observation 5, implies that x itself is an inneighbour of a vertex in $R \cup R'$.

Combining these arguments, we conclude that each vertex of $R \cup R'$ has at most two inneighbours in S, and each vertex of S is the inneighbour of at least one vertex in $R \cup R'$, with possibly one exception: one vertex of S is an inneighbour of x (and hence is not an inneighbour of any vertex of $R \cup R'$). But in this case, x is an inneighbour of a vertex of $R \cup R'$, and hence that vertex of $R \cup R'$ has at most one inneighbour in S. Therefore $|S| \leq 2|R \cup R'| \leq 2(k+a)$.

The active neighbours of x are either a vertex in R or a neighbour in $\vec{G}_2$ or a vertex in S. Therefore x has at most $3k + 2a + b$ active neighbours in G.

By Observation 1, each selected vertex is an active vertex. Therefore, any free vertex x has at most $3k + 2a + b$ selected neighbours.

After Bob takes his move, i.e., after he selects a vertex, any free vertex has at most $3k + 2a + b + 1$ selected neighbours. Therefore when a vertex is selected by either player, it has at most $3k + 2a + b + 1$ neighbours selected before, which means that the resulting linear order has back degree at most $3k + 2a + b + 1$. This completes the proof of Theorem 2 as well as the proof of Theorem 1. ∎

We note that if G is not connected, but each connected component of G is an (a, b)-pseudo chordal graph with the maximum outdegree of $\vec{G}_1 \leq k$, then it is still true that $\mathrm{col}_g(G) \leq 3k + 2a + b + 2$. Indeed, in case G is not connected, Alice may play the game on each connected component separately, using the same strategy. The only difference is that in some connected components, Bob may have the first move. It is not difficult to see that even if Bob moves first, Alice can still ensure that by following the same rules, the resulting linear order has back degree at most $3k + 2a + b + 1$.

It is easy to see that if H is a spanning subgraph of G then $\mathrm{col}_g(H) \leq \mathrm{col}_g(G)$ (cf. [12]). Therefore we have the following corollary:

Corollary 1 *For any (a, b)-pseudo partial k-tree G, $\chi_g(G) \leq \mathrm{col}_g(G) \leq 3k + 2a + b + 2$.*

Corollary 2 *For any partial k-tree G, $\chi_g(G) \leq \mathrm{col}_g(G) \leq 3k + 2$.*

This upper bound improves considerably the previous known upper bound, which was of the quadratic form $(k + 1)(k + 2)$, [3].

For $k = 1$, i.e., for forest, this upper bound is not sharp. However, it can be proved that if Alice uses the strategy described here, the outcome of the game has back degree at most 3, which gives the sharp upper bound for the game coloring number as well as the best game chromatic number of forests. For $k = 2$, we do not know if the upper bound is sharp. Outer-planar graphs are special partial 2-trees. It follows from Corollary 2 that the game coloring number of an outer-planar graph is at most 8. It was proved in [5] that the game coloring number of an outer-planar graph is at most 7. Again we remark that if Alice uses the strategy in this paper for outer-planar graphs, it can be shown that the linear order resulted from the game has back degree at most 6. Therefore the result for outer-planar graphs is just as good as that in [5]. For interval graphs, it follows from Corollary 2 that an interval graph of clique size ω has game coloring number at most $3\omega - 1$. This is not as good as the upper bound for the game coloring for interval graphs obtained in [4], which was $3\omega - 2$. However, it can be proved that if Alice uses the strategy described here, she can achieve the upper bound $3\omega - 2$. From this point of view, Alice's strategy described above can be regarded as a common generalization for the strategies for forests and interval graphs presented in [4] and for outer-planar graphs presented in [5]. For general partial 2-trees, if Alice uses the strategy of this paper, the linear order resulted from the game could have back degree 7.

3 Pseudo 2-trees

In this section, we give another strategy for Alice to play the game on (a,b)-pseudo partial 2-trees. This strategy gives a better upper bound on the game coloring number of (a,b)-pseudo partial 2-trees. Namely we shall prove the following result:

Theorem 3 *If G is an (a,b)-pseudo partial 2-tree, then* $\mathrm{col}_g(G) \leq a+b+8$.

The rest of this section is devoted to the proof of Theorem 3. The strategy for Alice and the proof are closely parallel to that given in [12], although only planar graphs were treated in [12].

Let $G=(V,E)$ be a fixed (a,b)-pseudo partial 2-tree. Let $\vec{G}_1=(V,\vec{E}_1)$ and $\vec{G}_2=(V,\vec{E}_2)$ be the oriented graphs as in Definition 1, where each vertex of $\vec{G}_1$ has outdegree at most 2.

Given a vertex $x \neq r$, We call the outneighbours of x in $\vec{G}_1$ the *parents of* x. Let $f(x)$ be the first outneighbour of x in $\vec{G}_1$ and let $s(x)$ be the second outneighbour of x in $\vec{G}_1$, if x does have one. Then we call $f(x)$ be *major parent* of x, and call $s(x)$ the *minor parent* of x. The vertex x is called a *major son* of $f(x)$ and a *minor son* of $s(x)$. The edge $xf(x)$ is called a *major edge* and the edge $xs(x)$ is called a *minor edge.*

Note that any vertex has at most one major parent and one minor parent, but a vertex may have many major sons and many minor sons. Two vertices with the same parents (i.e., the same major parent as well as the same minor parent) are called *brothers.*

Lemma 1 *For any vertex x, its minor sons partition into k groups of brothers for some integer $k \leq a+2$*

Proof. Let $v_1, v_2, \cdots, v_t$ be the minor sons of x, and let $u_1, u_2, \cdots, u_t$ be the major parent of $v_1, v_2, \cdots, v_t$, respectively. By the definition of major parent, we know that $xu_i \in \vec{E}_1 \cup \vec{E}_2$. Since $\vec{G}^* = (V, \vec{E}_1 \cup \vec{E}_2)$ has maximum outdegree at most $a+2$ (because $\vec{G}_1$ has maximum outdegree 2 and $\vec{G}_2$ has maximum outdegree at most a), we conclude that the multiset $\{u_1, u_2, \cdots, u_t\}$ contains at most $a+2$ distinct elements. If $u_i = u_j$, then v_i and v_j are brothers. Therefore the minor sons of x form k groups of brothers for an integer $k \leq a+2$. ∎

Let T be the directed spanning tree of $\vec{G}_1$ induced by the major edges of $\vec{G}_1$. Similar to the strategy in Section 2, in the process of the game, Alice will keep track of a subset T_a of $V(T)$ of active vertices, which contains the root r and will always induces a subtree of T. We shall also use the two operations, extension and switch, defined on directed paths of $\vec{G}_1$.

Initially, Alice selects the root r of T, and set $T_a = \{r\}$. Suppose at certain stage of the game, Bob has selected the last vertex x. Then Alice selects the next vertex by the following rule:

Let $P_1 = (xf(x))$. Let P_2 be the extension of P_1. Alice will repeat the following procedure until she finds the vertex to be selected.

Suppose the presently found directed path is P_{2t} for some $t \geq 1$, and that the last edge of P_{2t} is vu.

If u is the last outneighbour of v, and u is a free vertex, then select u.

If u is the last outneighbour of v, and u is a selected vertex, then select any free vertex such that all its parents in $\vec{G}_1$ have been selected.

If u is not the last outneighbour of v, but v has an even number of active brothers and u is a free vertex, then select u.

If u is not the last outneighbour of v, and that either v has an odd number of active brothers, or u is a selected vertex, then let P_{2t+1} be the switch of P_{2t} and let P_{2t+2} be the extension of P_{2t+1}, and go back to repeat the procedure (with P_{2t} replace by P_{2t+2}).

It is obvious that the procedure will stop in $O(|V|)$ steps, and hence Alice will eventually select a vertex.

Let P_{2j} be the last directed path found in the above procedure. After Alice selected the next vertex, say v, add the vertices of the directed path P_{2j} and the vertex v to T_a. It is obvious that T_a induces a subtree of T after adding these vertices.

Theorem 4 *If Alice uses the strategy described above, then the back degree of the linear order produced in the game is at most $a + b + 7$.*

We shall prove that at any stage of the game, after Alice finished her move and before Bob takes his next move, any free vertex has at most $a+b+6$ selected neighbours. First we make a few observations.

Observation 6 *All the selected vertices are active.*

This follows from the strategy.

Observation 7 *If v has an active major son, then v is active.*

This is because the active vertices induces a subtree of T. If v has a major active son, say u, then since v is on the unique path of T connecting u and r (which is an active vertex), it follows that v is active.

Observation 8 *The vertices selected by Alice are either active before she selects them, or all their parents have been selected.*

This follows trivially from the strategy.

Observation 9 *If v has two major active sons who are not brothers, then v is a selected vertex.*

Let u_1, u_2 be the first two active sons of v who are not brothers. When u_1 becomes active, v becomes active (cf. Observation 7). When u_2 becomes active, it means that the path P_{2t} constructed in the procedure above pass through u_2, and hence it ends at v, because $v \in T_a$ and $u_2 \notin T_a$ at that stage. Now since u_2 has no brother which is active, by the rule, Alice will select the vertex v, provided that it is not selected before.

Observation 10 *If v has three active major sons, then v is a selected vertex.*

If v has three active major sons and is still not selected, then by Observation 4, these three active major sons of v are brothers. Let u_1, u_2, u_3 be the first three major sons of v who are brothers. When u_1 becomes active, v also becomes active. When u_3 becomes active, it means that the path P_{2t} constructed in the procedure above pass through u_3, and hence it ends at v, because $v \in T_a$ at that stage. Now since u_3 has two active brothers (i.e., an even number of active brothers), by the rule, Alice will select the vertex v, provided that it is not selected before.

Observation 11 *If v has two active minor sons who are brothers, then v is active.*

Let u_1, u_2 be the first two active minor sons of v who are brothers. Let v' be the major parent of u_1 and u_2. When u_1 becomes active, v' also becomes active. When u_2 becomes active, it means that the path P_{2t} constructed in the procedure above pass through u_2, and hence it ends at v', because $v' \in T_a$ at that stage. Now since u_2 has exactly one active brother (i.e., an odd number of active brothers), by the rule, the path P_{2t} is switched, and hence pass through v. Therefore either v is already active at that stage, or becomes active after Alice finishes that move.

Observation 12 *If v has four active minor sons which are brothers, then v is a selected vertex.*

Let u_1, u_2, u_3, u_4 be the first four active minor sons of v who are brothers. Let v' be the major parent of u_1, u_2, u_3 and u_4. When u_1 becomes active, v' also become active. When u_2 becomes active, v becomes active (cf. Observation 7). When u_4 becomes active, it means that the path P_{2t} constructed in the procedure above pass through u_4, and hence it ends at v', because $v' \in T_a$ at that stage. Now since u_4 has three active brothers (i.e., an odd number of active brothers), by the rule, the path P_{2t} is switched and then extended to the path P_{2t+2}. Since $v \in T_a$ at that stage, P_{2t+2} ends at v. By the rule, Alice will select v at that stage, provided that v is not selected before.

Observation 13 *If v has four active minor sons which form two pairs of brothers, then v is a selected vertex.*

The argument for making this observation is similar to that for Observation 7 and we omit the details.

Observation 14 *If v has two active minor sons who are brothers, and one active major son, then v is a selected vertex.*

Suppose u_1 is an active major son of v, and u_2, u_3 are the first two active minor sons of v who are brothers, where u_2 becomes active before u_3 does. Let v' be the major parent of u_2 and u_3.

First we assume that u_1 becomes active before u_3 becomes active. When u_1 becomes active, v becomes active (cf. Observation 7). When u_2 becomes active, v' becomes active. When u_3 becomes active, it means that the path P_{2t} constructed in the procedure above passes through u_3, and hence it ends at v', because $v' \in T_a$ at that stage. Now since u_3 has one active brother (i.e., an odd number of active brothers), by the rule, the path P_{2t} is switched and then extended to the path P_{2t+2}. Since $v \in T_a$ at that stage, P_{2t+2} ends at v. By the rule, Alice will select v at that stage, provided that v is not selected before.

The case u_3 becomes active before u_1 becomes active is similar, and we omit the details.

With these observations, we are ready to count the number of active neighbours of free vertices. Suppose Alice has finished her move, and that x is still a free vertex. We now count the number of active neighbours of x.

Since x has only two parents (in $\vec{G}_1$), x has at most two active parents. Since $\vec{G}_2$ has maximum degree b, x has at most b active neighbours in $\vec{G}_2$.

By Observation 4, x has at most two active major sons.

By Lemma 1, the minor sons of x partition into at most $a + 2$ groups of brothers. By Observations 7 and 8, each group has at most one active minor son of x, except one group which may contain (at most) three active minor sons of x. Therefore x has at most $a+4$ active minor sons.

By Observation 14, either x has no active major sons, or x has at most $a+2$ active minor sons (i.e., each group of brothers that are minor sons of x contains at most one active element). Therefore the total number of active sons (major and minor) is at most $a+4$.

Since each neighbour of x is either a parent of x, or a neighbour in $\vec{G}_2$, or a major son, or a minor son, we conclude that x has at most $a+b+6$ active neighbours. By Observation 6, each selected vertex is an active vertex. Therefore x has at most $a+b+6$ selected neighbours.

When Bob selects a vertex, the selected neighbours of a free vertex increases at most by 1. Thus after Bob's move, each free vertex has at most $a+b+7$ selected neighbours. This completes the proof of Theorem 4.

4 Graphs embeddable on surfaces

In this section, we use the result in Section 3 to derive upper bounds for the game coloring numbers of graphs embeddable on surfaces of certain genus. The argument is parallel to the argument in [12]. Indeed, a result in [12] is equivalent to the statement that every planar graph is a $(3,8)$-pseudo partial 2-tree. Then it follows from Theorem 3 that the game coloring number of a planar graph is at most 19. In this section, we consider graphs embeddable on higher surfaces.

Definition 3 *We call a class $\mathcal{G}$ of graphs an (a,b)-class if it has the following properties:*

1. *$K_3 \in \mathcal{G}$;*

2. *Any graph $G \in \mathcal{G}$ has minimum degree ≥ 2;*

3. *If $G \in \mathcal{G}$ and has minimum degree ≥ 3, then G has an edge $e = uv$ such that $d(u) \leq a+2$ and $d(v) \leq b+2$;*

4. *Suppose $G \in \mathcal{G}$ and $v \in V(G)$ has degree 2 and the two neighbours of v are not adjacent. Let G' be the graph obtained from G by deleting v and the two edges incident to v, and add an edge connecting the two neighbours of v. Then $G' \in \mathcal{G}$;*

5. *If $G \in \mathcal{G}$ then any subgraph of G of minimum degree ≥ 2 is also a member of $\mathcal{G}$.*

Lemma 2 *If $\mathcal{G}$ is an (a, b)-class, then every graph $G \in \mathcal{G}$ is an (a, b)-pseudo partial 2-tree.*

Proof. Let $G = (V, E)$ be a member of $\mathcal{G}$. We shall construct two oriented graphs $\vec{G}_R = (V, \vec{E}_R)$ and $\vec{G}_B = (V, \vec{E}_B)$ such that the following hold:

1. $E \subset E_R \cup E_B$, and $E_R \cap E_B = \emptyset$,

2. $\vec{G}_R$ has maximum degree at most b, and has maximum outdegree at most a,

3. $\vec{G}_B$ is acyclic, and each vertex has outdegree ≤ 2.

4. Suppose u, v are the two outneighbours of a vertex x in G_B, then either $uv \in \vec{E}_R \cup \vec{E}_B$, or $vu \in \vec{E}_R \cup \vec{E}_B$.

The graphs $\vec{G}_R$ and $\vec{G}_B$ are more or less obtained from G by coloring its edges by two colors, "red" and "blue", and assigning an orientation to the edge at the same time. Those red edges form the graph $\vec{G}_R$ and those blue edges form the graph $\vec{G}_B$. However, in the process of coloring the edges, we may need to add some edges to G. So $\vec{G}_R$ and $\vec{G}_B$ may contain some edges not belong to G.

In the process of coloring the edges of G, we keep track of a graph $G_U \in \mathcal{G}$, which is more or less the subgraph of G induced by the uncolored edges. Again, it may contain edges not belong to G. The following is an algorithm that construct the graphs $\vec{G}_R$ and $\vec{G}_B$ from G:

Initially, let $G_U = G$, and let G_R and G_B be empty graphs.

If G_U is isomorphic to K_3, then color all the edges of G_U blue, arbitrarily assign orientations to the edges so that it is acyclic. Otherwise, suppose $|V(G_U)| \geq 4$. If G_U has a vertex, say v, of degree 2, then we do the following:

1. Color the two edges incident to v blue, and oriented these two blue edges from v to the two neighbours.

2. Delete v (together with the two incident edges) from G_U.

3. If w, u are the two neighbours of v and that uw is not an edge of $G_U \cup G_R \cup G_B$, then add the edge uv to G_U.

If G_U contains no vertex of degree 2, then by definition of $\mathcal{G}$, G_U has an edge $e = uv$ such that $d(u) \leq a + 2$ and $d(v) \leq b + 2$. In this case, we color e red, oriented it from u to v, and delete e from G_U.

It follows from the definition that after one step of such operation, the resulting graph G_U is still a member of $\mathcal{G}$.

First we show that the red subgraph $\vec{G}_R$ has maximum degree $\leq b$. Suppose $d_{\vec{G}_R}(v) \geq 1$. Let e be the first edge incident to v which is colored red. By the coloring rules, when e is colored red, there are at most $b + 2$ uncolored edges incident to v. All the other edges (if any) incident to v have been colored blue. In the process of coloring the remaining edges, we may add edges which are incident to v. However, this happens only if there is a 2-vertex, say u, adjacent to v and that the two edges incident to u are colored blue and deleted from G_U. Therefore we conclude that, after the edge e is colored red, the total number of red edges and uncolored edges incident to v is always at most $b + 2$. Since the last two uncolored edges incident to v will be colored blue, we conclude that $d_{\vec{G}_R}(v) \leq b$.

The same argument shows that $\vec{G}_R$ has maximum outdegree $\leq a$. Suppose $d^+_{\vec{G}_R}(v) \geq 1$. Let e be the first out-edge of v which is colored red. By the coloring and orienting rules, when e is colored red and oriented from v to the other end, there are at most $a + 2$ uncolored edges incident to v. All the other edges (if any) incident to v have been colored blue or colored red and oriented towards v. In the process of coloring the remaining edges, the total number of red edges oriented from v to its other end together with the uncolored edges incident to v is always at most $a + 2$. Since the last two uncolored edges incident to v will be colored blue, it follows that $d^+_{\vec{G}_R}(v) \leq a$.

The conclusion concerning the blue graph $\vec{G}_B$ follows trivially from the coloring process. Indeed, let $x_1, x_2, \cdots, x_n$ be the order that the vertices being deleted from G_U (recall that each time two edges colored blue, a 2-vertex is deleted from G_U), then for $i \leq n - 2$, each x_i has exactly two outneighbours x_{i_1}, x_{i_2} with $i_1, i_2 > i$ in G_B, and all the other neighbours x_j of x_i are inneighbours and $j < i$. The last three vertices x_{n-2}, x_{n-1}, x_n form a transitive triangle in G_B.

If w, u are the two outneighbours of v in $\vec{G}_B$, then either $uw \in E$ or uw is added to G_U at certain stage. Therefore, either $uv \in \vec{E}_R \cup \vec{E}_B$, or $vu \in \vec{E}_R \cup \vec{E}_B$. ∎

Let S be any surface. Let $\mathcal{G}(S)$ be the set of graphs of minimum degree ≥ 2 and embeddable on S. Then it is easy to see that $\mathcal{G}(S)$ satisfies (1), (2), (4), (5) of Definition 3. Thus if for some integers a, b, we can prove that every graph $G \in \mathcal{G}(S)$ of minimum degree ≥ 3 contains an edge $e = uv$ with $d(u) \leq a+2$ and $d(v) \leq b + 2$, then we could conclude that $\mathcal{G}(S)$ is an (a, b)-class.

Corollary 3 *If S is the sphere or the projective plane, then $\mathcal{G}(S)$ is a $(3, 8)$-class. Therefore each graph embeddable on the sphere or on the projective plane* *has game coloring number at most* 19.

Proof. It was proved by Kotzig [10] that every planar graph of minimum degree 3 has an edge $e = uv$ with $d(u) \leq 5$ and $d(v) \leq 10$. Therefore any planar graph of minimum degree ≥ 2 is a $(3,8)$-pseudo partial 2-tree, and hence has game coloring number at most 19. Moreover, any planar graph is a spanning subgraph of a planar graph with minimum degree ≥ 2. Therefore any planar graph has game coloring number at most 19.

It was proved by Sanders [11] that every graph embeddable on the projective plane of minimum degree 3 has an edge $e = uv$ with $d(u) \leq 5$ and $d(v) \leq 10$. Thus for the same reason as above, every graph embeddable on the projective plane has game coloring number at most 19. ∎

The result that planar graphs have game coloring number at most 19 was first proved in [12].

Let S_g be an orientable surface of genus $g \geq 1$, i.e., the sphere with g handles. If $G = (V, E)$ is a graph embedded on S_g, then by Euler formula, we have $|V| + |F| - |E| = 2 - 2g$, where F is the set of faces of G. When each face is incident to at least three edges, we have $6|V| - 2|E| \geq 12 - 12g$, which implies that

$$\Sigma_{v \in V}(d(v) - 6) \leq 12(g - 1).$$

Let $H(g) = \lfloor \frac{7+\sqrt{1+48g}}{2} \rfloor$ be the Heawood number of the orientable surface of genus g. It follows from Euler formula that every graph embeddable on an orientable surface of genus g has a vertex of degree at most $H(g) - 1$, provided that $g \geq 1$.

In the remaining part, let $g \geq 1$ be a fixed integer, and we consider graphs embeddable on S_g. We call an edge $e = uv$ of G a *light edge* if $d(u) \leq 5 + \sqrt{1 + 48g}$ and $d(v) \leq H(g) - 1$.

Lemma 3 *Suppose G is a graph without loop, but may have multiple edges. If G can be embedded on S_g so that each face is bounded by at least three edges, then G has a light edge.*

Proof. Assume to the contrary that there is a graph G embedded on S_g which has no light edges. We may assume that G is maximal in the sense that adding any edge would either create a light edge, or not embeddable on S in the manner as required. Then we conclude that each face of G must be a triangle. Indeed, if $v_1, v_2, v_3, \cdots, v_k$ is the vertices of a face with $k \geq 4$. Then $G + v_1v_3$ is still embeddable on S_g with each face bounded by at least three edges. By the maximality of G, we conclude that $G + v_1v_3$ has a light edge, hence v_1v_3 is a light edge. Thus at least one of the vertices v_1, v_3 has degree at most $H(g) - 1$. Without loss of generality, we may assume that $d(v_1) \leq H(g)-1$. For the same reason, $G+v_2v_k$ has a light edge which implies that one of v_2, v_k has degree at most $H(g)-1$. But then either v_1v_2 or v_1v_k is a light edge of G, contrary to our assumption. Therefore each face of G is a triangle.

For a vertex v of G, define the charge $c(v)$ to be the degree of v, i.e., $c(v) = d_G(v)$. We now *redistribute* the charges of the vertices as follows: Each vertex of degree ≥ 7 send a charge of 1 to each of its neighbours whose degree is ≤ 5. Let the new charge of vertex v be $c'(v)$. Let $f(v) = c'(v) - 6$.

If $d(v) \leq 5$, then each of its neighbours have degree $\geq 6+\sqrt{1+48g} > 7$ (as G has no light edges), and hence $c'(v) \geq 6$ and $f(v) \geq 0$. If $6 \leq d(v) \leq 5+\sqrt{1+48g}$, then none of its neighbours has degree ≤ 5, hence $f(v) = d(v) - 6 \geq 0$. Assume now that $d(v) = k \geq 6+\sqrt{1+48g}$. Let $v_1, v_2, \cdots, v_k$ be the k neighbours of v, that are ordered in such a way that v_i and v_{i+1} are incident to a same face (the summation of the indices is modulo k). Since each face of G is a triangle, we have v_i is adjacent to v_{i+1}. Because G has no light edges, we conclude that for each i, at least one of the vertices v_i, v_{i+1} has degree $\geq H(g) > 5$. Therefore $c'(v) \geq \lceil d(v)/2 \rceil$. Hence $f(v) \geq d(v)/2 - 6 \geq (6+\sqrt{1+48g})/2 - 6$.

Let u be a vertex of G of maximum degree. Then $d(u) \geq 6+\sqrt{1+48g}$. At least $\lceil d(u)/2 \rceil$ of the neighbours of u have degree $\geq H(g)$. For each such neighbour v of u, we have

$$f(v) \geq \min\{H(g)-6, \frac{6+\sqrt{1+48g}}{2} - 6\} = \frac{\sqrt{1+48g}}{2} - 3.$$

As each other vertex v has $f(v) \geq 0$, we conclude that

$$\Sigma_{v \in V} f(v) \geq (\frac{\sqrt{1+48g}}{2} + 4)(\frac{\sqrt{1+48g}}{2} - 3).$$

On the other hand

$$\Sigma_{v \in V} f(v) = \Sigma_{v \in V}(c'(v) - 6) = \Sigma_{v \in V}(d(v) - 6) \leq 12(g-1).$$

Therefore

$$12(g-1) \geq (\frac{\sqrt{1+48g}}{2} + 4)(\frac{\sqrt{1+48g}}{2} - 3),$$

which is a contradiction. Therefore G does have a light edge. ∎

Corollary 4 *If G is a simple graph (i.e., without loops or parallel edges) embeddable on S_g and G has minimum degree ≥ 3, then G has a light edge.*

Corollary 5 *Given an integer $g \geq 1$, let S_g be the orientable surface of genus g. Then $\mathcal{G}(S_g)$ is a $(\lfloor \frac{1+\sqrt{1+48g}}{2} \rfloor, \lfloor 3+\sqrt{1+48g} \rfloor)$-class. Therefore each graph G embeddable on S_g has*

$$\chi_g(G) \leq \mathrm{col}_g(G) \leq \lfloor \frac{1}{2}(3\sqrt{1+48g}+23) \rfloor.$$

The proof of Corollary 5 is similar to the proof of Corollary 3, and we omit the details.

We note that no upper bounds for the game coloring number for graphs embeddable on given surfaces were known before, although an upper bound for the game chromatic number of such graphs were given in [3]. It was proved in [3] that graphs embeddable on S_g have game chromatic number at most $(4g+4)(4g+5)$. This upper bound is quadratic in terms of g, where the upper bound of Corollary 5 is of the order of the square root of g. Even for small g, the bound in Corollary 5 is much better. For example, the upper bound for the game chromatic number of toroidal graphs given in [3] is 72, where the upper bound deduced from Corollary 5 is 22. Moreover, since the maximum chromatic number of a graph embeddable on S_g is equal to $H(g) = \lfloor \frac{7+\sqrt{1+48g}}{2} \rfloor$, which is a trivial lower bound for the maximum game chromatic number of such graphs, we conclude that the maximum game chromatic number of a graph embeddable on S_g has order $\sqrt{g}$. However, the correct coefficient remains unknown.

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