The game coloring number of planar graphs

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Abstract

This paper discusses a variation of the game chromatic number of a graph: the game coloring number. This parameter provides an upper bound for the game chromatic number of a graph. We show that the game coloring number of a planar graph is at most 19. This implies that the game chromatic number of a planar graph is at most 19, which improves the previous known upper bound for the game chromatic number of planar graphs.

1 Introduction

This paper is motivated by the investigation of the game chromatic number of graphs. Let G be a finite graph and let X be a set of colors. We consider a coloring problem posed as a two-person game, with one person (Alice) trying to color the graph, and the other (Bob) trying to prevent this from happening. Alice and Bob alternate turns, with Alice having the first move. A move consisting of selecting an uncolored vertex x and assigning it a color from the color set X distinct from the colors assigned previously (by either player) to neighbours of x. If after $n = |V(G)|$ moves, the graph G is colored, Alice is the winner.

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Bob wins if an impass is reached before all vertices in the graph are colored, i.e., there is an uncolored vertex which is adjacent to vertices of all the colors. The *game chromatic number* of a graph $G = (V, E)$, denoted by $\chi_g(G)$, is the least cardinality of a color set X for which Alice has a winning strategy. This parameter is well-defined, since Alice always wins if $|X| = |V|$.

The game chromatic number of a graph was first studied by Bodlaeder [2]. It seems to be very difficult to determine or estimate the game chromatic number of even small graphs. However, for special classes of graphs, non-trivial upper and lower bounds for the game chromatic number are obtained. The easiest case is the class of forests. It was proved by Faigle, Kern, Kierstead and Trotter [9] that the game chromatic number of a forest is at most 4, and that there are forests of game chromatic number 4. The game chromatic number of planar graphs was first studied by Kierstead and Trotter [15]. By using the Four Color Theorem, it was proved in [15] that the game chromatic number of a planar graph is at most 33. It was also claimed in [15] that there are examples of planar graphs with game chromatic number 8. The problem of reducing the gap between the upper and lower bounds for the maximum game chromatic number of planar graphs is posed in [15], (also see [14]). In [7], Dinski and Zhu proved that if a graph has acyclic chromatic number k, then its game chromatic number is at most $k(k+1)$. Since the acyclic chromatic number of a planar graph is at most 5, it follows that the game chromatic number of a planar graph is at most 30.

There is a significant difference between the proof for the upper bound of the game chromatic number of forests and that of planar graphs. For the forests, it was proved [9] that Alice has a strategy to ensure that at any stage of the game, any uncolored vertex has at most three colored neighbours. This of course implies that four colors are enough for Alice to win the game. Indeed, Alice need not choose the color when playing the game. She just needs to select the vertex to be colored. However, in both papers [15] and [7], the winning strategy for Alice needs to carefully choose the color for the vertex to be colored. This leads to the question whether or not Alice has a strategy for playing the coloring game on planar graphs, so that at any stage of the game any uncolored vertex has at most c colored neighbours, for some constant c. This paper discusses this problem, by investigating a variation of the game chromatic number: the game coloring number (defined in Section 2). It is proved that Alice does have such a strategy and that the constant c could be 18. As a consequence, the upper bound for the game chromatic number of planar graphs is reduced to 19.

We remark that the proof of the result in this paper is conceptually simpler than the proofs in [15] and [7]. In [15], the Four Color Theorem is used. In [7], the proof relies on the result that planar graphs have acyclic chromatic number ≤ 5, which is also a difficult result. The proof in this paper only uses a very simple property of planar graphs: the existence of a "light edge" in planar graphs with minimum degree ≥ 3, which follows easily from Euler Formula. Also the strategy for Alice is simpler than those described in [15] and [7]. Given a planar graph, one only need to do some simple preprocessing of the graph, which can be done manually for any fair sized planar graph, and then follow an easy rule in the game.

2 Game coloring number

Let $G = (V, E)$ be a finite graph and let L be a linear order on the vertex set V. For a vertex $x \in V$, the *back degree of x relative to L* is defined as $|\{y \in V : xy \in E \text{ and } \ x > y \text{ in } \ L\}|$. The *back degree of* L is then the maximum back degree of vertices relative to L. The *coloring number*, $\text{col}(G)$, of a graph G is equal to $1+k$, where k is the minimum integer such that there is a linear order L on V which has back degree k. The coloring number was introduced by Erdős and Hajnal [8] and studied in [8, 10, 12, 13, 18] etc. It is obvious (see [14]) that for any graph G, we have $\chi(G) \leq \text{col}(G)$.

The coloring number of a graph $G = (V, E)$ is determined by finding an "optimal" linear order on the vertex set V. Instead of finding an optimal linear order on V, we consider a linear order produced by playing a game. It is a two person game, played by Alice and Bob, who alternate turns and with Alice having the first move. Each move consisting of selecting a vertex (among the remaining vertices) and put it at the end of the linear order formed by the previously selected vertices. Thus Alice select v_1, then Bob select v_2, and Alice select v_3, etc. After all the vertices are selected, we obtain a linear order L on V. Alice's goal is to minimize the back degree of L, while Bob's goal is to maximize the back degree of L. This is a zero-sum two person game, therefore each player has an optimal strategy. We define the *game coloring number* $\text{col}_g(G)$ *of* G to be $1 + k$, where k is the back degree of a linear order L, which is produced by playing the game with both players using their optimal strategies.

Lemma 1 *For any graph G, we have $\text{col}_g(G) \geq \chi_g(G)$.*

Proof. This bound is trivial. Indeed, if there are $\text{col}_g(G)$ colors, Alice just needs to choose the next vertex to color by following her optimal strategy in the ordering game, and then color the chosen vertex by First-Fit. ∎

The next two results are also trivial and was implicit in [11].

Lemma 2 *Suppose $G = (V, E)$ and $E = E_1 \cup E_2$. Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$. Then $\text{col}_g(G) \leq \text{col}_g(G_1) + \Delta(G_2)$, where $\Delta(H)$ denotes the maximum degree of a graph H.*

Proof. Alice may simply use the optimal strategy for G_1. ∎

Lemma 3 *Suppose G is a spanning subgraph of H. Then $\text{col}_g(G) \leq \text{col}_g(H)$.*

3 Decomposition of planar graphs

Planar graphs considered in this paper have no parallel edges and loops. Given a planar graph, we shall construct two directed graphs that cover the edges of G and has some special properties. For this purpose, we need a result, which was proved by Borodin [3]. Let an i-vertex be a vertex of degree i. Let an i,j-edge be an edge joining an i-vertex to a j-vertex. We call an edge e a *light edge* if it is either a $3,j$-edge for some $j \leq 10$, or a $4,j$-edge for some $j \leq 8$ or a $5,j$-edge for some $j \leq 6$.

Lemma 4 *Every planar graph with minimum degree* ≥ 3 *contains a light edge.*

Light edges of planar graphs have been studied in many papers, [5, 6, 17, 20]. It was originally discussed in relation to the Four Color Problem, and later found useful with regards to acyclic colorings [4]. Lemma 4 is sharp in the following sense: There are planar graphs in which each edge is either a $3,10$-edge or a non-light edge; there are planar graphs in which each edge is either a $4,8$-edge or a non-light edge; there are planar graphs in which each edge is either a $5,6$-edge or a non-light edge.

Lemma 5 below is the basis for Alice's strategy, described in the next section. To avoid confusion, we fix a few terms. We shall denote a directed graph by $\vec{G} = (V, \vec{E})$, possibly with indices on the sets. If $e = uv \in \vec{E}$, then we say the edge e is directed *from* u *to* v, u is called an *in-neighbour of* v and v is called an *out-neighbour of* u. The *in-degree* (resp. *out-degree*) of v is the number of in-neighbours (resp. out-neighbours) of v. The *degree* of v is the sum of its in-degree and out-degree. For a directed graph $\vec{G} = (V, \vec{E})$, we denote by $G = (V, E)$ the undirected graph obtained from $\vec{G}$ by omitting the orientations of the edges. In particular, $\vec{E}$ means a set of directed edges, and E means the set of undirected edges obtained from $\vec{E}$ by omitting the orientations.

Lemma 5 *Suppose* $G = (V, E)$ *is a connected planar graph without* $2,2$*-edges or* 1*-vertices. Then there are two directed graphs* $\vec{G}_R = (V, \vec{E}_R)$ *and* $\vec{G}_B = (V, \vec{E}_B)$ *that satisfy the following conditions:*

1. $E \subset E_R \cup E_B$, *and* $E_R \cap E_B = \emptyset$,
2. $\vec{G}_R$ *has maximum degree at most* 8, *and has maximum out-degree at most* 3,
3. $\vec{G}_B$ *is acyclic, and each vertex has out-degree* 2, *except two vertices, say* r, r', *which are joined by a directed edge* $r'r$, *and have out-degrees* 0 *and* 1 *respectively.*

4. Suppose u, v are the two out-neighbours of a vertex x in G_B, then either $uv \in \vec{E}_R \cup \vec{E}_B$, or $vu \in \vec{E}_R \cup \vec{E}_B$.

Proof. The graphs $\vec{G}_R$ and $\vec{G}_B$ are more or less obtained from G by coloring its edges by two colors, "red" and "blue", and assigning an orientation at the same time. Those red edges form the graph $\vec{G}_R$ and those blue edges form the graph $\vec{G}_B$. However, in the process of coloring the edges, we may need to add some edges to G. So $\vec{G}_R$ and $\vec{G}_B$ may contain some edges not belong to G.

In the process of coloring the edges of G, we keep track of a plane graph G_U, which is more or less the subgraph of G induced by the uncolored edges. Again, it may contain edges not belong to G. The following is an algorithm that construct the graphs $\vec{G}_R$ and $\vec{G}_B$ from G:

Initially, let $G_U = G$.

If G_U is isomorphic to K_3, then color all the edges of G_U blue, arbitrarily assign orientations to the edges so that it is acyclic. Otherwise, suppose $|V(G_U)| \geq 4$. If G_U has a vertex, say v, of degree 2, then we do the following:

1. Color the two edges incident to v blue, and oriented these two blue edges from v to the two neighbours.

2. Delete v (together with the two incident edges) from G_U.

3. If w, u are the two neighbours of v and that uw is not an edge of $G_U \cup G_R \cup G_B$, then add the edge uv to G_U.

If G_U contains no vertex of degree 2, then by Lemma 4, G_U has a light edge e. In this case, we color e red, oriented it from an end vertex of degree ≤ 5 to the other end vertex, and delete e from G_U. (Note that each light edge has an end vertex of degree ≤ 5).

Obviously G_U is always a planar graph without 1-vertex, or parallel edges, or loops, or $2,2$-edges, and that the coloring process terminates in $O(|E|)$ steps.

First we show that the red subgraph $\vec{G}_R$ has maximum degree ≤ 8. Indeed, suppose $d_{\vec{G}_R}(v) \geq 1$, let e be the first edge incident to v which is colored red. By the coloring rules, when e is colored red, there are at most 10 uncolored edges incident to v. All the other edges (if any) incident to v have been colored blue. In the process of coloring the remaining edges, we may add edges which are incident to v. However, this happens only if there is a 2-vertex, say u, adjacent to v and that the two edges incident to u are colored blue and deleted from G_U. Therefore we conclude that, after the edge e is colored red, the total number of red edges and uncolored edges incident to v is always at most 10. Since the last two uncolored edges incident to v will be colored blue, we conclude that $d_{\vec{G}_R}(v) \leq 8$.

The same argument shows that $\vec{G}_R$ has maximum out-degree ≤ 3. We omit the details.

The conclusion concerning the blue graph $\vec{G}_B$ follows trivially from the coloring process. Indeed, let $x_1, x_2, \cdots, x_n$ be the order that the vertices being deleted from G_U (recall that each time two edges colored blue, a 2-vertex is deleted from G_U), then for $i \leq n-2$, each x_i has exactly two out-neighbours x_{i_1}, x_{i_2} with $i_1, i_2 > i$ in G_B, and all the other neighbours x_j of x_i are in-neighbours and $j < i$. The last three vertices x_{n-2}, x_{n-1}, x_n form a transitive triangle in G_B, and we let r, r' be the vertices with out-degrees 0 and 1 respectively.

If w, u are the two out-neighbours of v in $\vec{G}_B$, then either $uw \in E$ or uw is added to G_U at certain stage. Therefore, either $uv \in \vec{E}_R \cup \vec{E}_B$, or $vu \in \vec{E}_R \cup \vec{E}_B$. ∎

4 Alice's strategy

In this section, we prove that if $\vec{G}_R$ and $\vec{G}_B$ satisfy the condition of Lemma 5, then the graph $G^* = (V, E_R \cup E_B)$ has game coloring number at most 19. For this purpose, it suffices to give a strategy for Alice, so that no matter how Bob plays the game, the output of the game, which is a linear order on V, has back degree at most 18. Alice will only take the graph $\vec{G}_B$ into consideration when playing the game, although the graph $\vec{G}_R$ is still needed for the proof. In the following we shall concentrate on the blue graph $\vec{G}_B$. We need to define some terms before describe the strategy.

Suppose $x \in V - \{r, r'\}$, and u, v are the two out-neighbours of x in $\vec{G}_B$. By Lemma 5, either $vu \in \vec{E}_R \cup \vec{E}_B$ or $uv \in \vec{E}_R \cup \vec{E}_B$. Assume that $vu \in \vec{E}_R \cup \vec{E}_B$. We call u, v the *parents of* x, call u the *major parent* of x, and v the *minor parent* of x. We call x a *major son* of u, and call it a *minor son* of v. We call the edge xu a *major edge* and call the edge xv a *minor edge*. Two vertices x, y are called *brothers* if x and y have the same parents. Obviously "brotherhood" is an equivalence relation. We call the exceptional edge $r'r$ a major edge, and r' has a single major parent, no minor parent, and r has no parents.

Lemma 6 *For any vertex x, its minor sons partition into k groups of brothers for some integer $k \leq 5$*

Proof. Let $v_1, v_2, \cdots, v_t$ be the minor sons of x, and let $u_1, u_2, \cdots, u_t$ be the major parent of $v_1, v_2, \cdots, v_t$, respectively. By the definition of major parent, we know that $xu_i \in \vec{E}_R \cup \vec{E}_B$. Since $\vec{G} = (V, \vec{E}_R \cup \vec{E}_B)$ has maximum out-degree at most 5 (because $\vec{G}_B$ has maximum out-degree 2 and G_R has maximum out-degree at most 3), we conclude that the multiset $\{u_1, u_2, \cdots, u_t\}$ contains at most 5 distinct elements. If $u_i = u_j$, then v_i and v_j are brothers. Therefore the minor sons of x form k groups of brothers for an integer $k \leq 5$. ∎

Let T be the directed spanning tree of $\vec{G}_B$ induced by the major edges of G_B, In the process of the game, Alice will keep track of a subset of $V(T)$, which is called the *active set*, and denoted by T_a. The set T_a will contain r and always induce a connected subgraph of T, i.e., T_a induces a subtree. The vertices of T_a are called *active vertices*.

Suppose at certain stage, the active set is T_a. We define two operations on directed paths of $\vec{G}_B$, the *extension* and the *switch*, as follows:

Suppose $P = (y_1, y_2, \cdots, y_k)$ is a directed path of $\vec{G}_B$. Let P' be the unique directed path of T connecting y_k to T_a, i.e., the first vertex of P' is y_k and the last vertex of P' is a vertex of T_a, and all the inner vertex (if any) are not in T_a. Recall that T_a induces a subtree of T. Therefore the path P' is indeed unique. The concatenation PP' of P and P' is called the *extension of* P. Since $\vec{G}_B$ is acyclic, PP' is a directed path of $\vec{G}_B$. Note that given a directed path P of $\vec{G}_B$, its extension is unique (as T_a is fixed). Also note that if the last vertex of P is in T_a, then its extension is itself. By the definition, P is allowed to intersect T_a.

Suppose $P = (y_1, y_2, \cdots, y_k)$ is a directed path of $\vec{G}_B$, and suppose that the last edge, $y_{k-1}y_k$, of P is a major edge. Let y' be the minor parent of y_{k-1}. Then the directed path $P' = (y_1, y_2, \cdots, y_{k-1}, y')$ is called the *switch of* P. We say P' is obtained from P by *switching the last edge*. Note that given a directed path P, if the last edge is a major edge and not equal to $r'r$, then its switch is unique. Otherwise its switch is not defined.

At any stage of the game, we say a vertex is a *selected vertex* if it has been selected by either player before that stage. Otherwise, the vertex is *a free vertex* at that stage.

Now we are ready to describe Alice's strategy.

Initially, Alice select r, and set $T_a = \{v_1\}$. Recall that r is the vertex with out-degree 0 in $\vec{G}_B$. Suppose at certain stage of the game, Bob has selected the last vertex x. Then Alice select the next vertex by the following rule:

Let y be the major parent of x, and let $P_1 = (xy)$. Let P_2 be the extension of P_1. Alice will repeat the following procedure until she found the vertex to be selected.

Suppose the presently found directed path is P_{2t} for some $t \geq 1$, and that the last edge of P_{2t} is vu.

If $vu = r'r$, then select any free vertex x such that all its predecessors in G_B have been selected.

If vu is a minor edge, and u is a free vertex, then select u.

If vu is a minor edge, and u is a selected vertex, then select any free vertex such that all its predecessors in G_B have been selected.

If vu is a major edge, and the number of active brothers of v is even and that u is a free vertex, then select u.

If vu is a major edge, and that either v has an odd number of active brothers, *or u is a selected vertex, then let P_{2t+1} be the switch of P_{2t} and let P_{2t+2} be the extension of P_{2t+1}, and go back to repeat the procedure (with P_{2t} replace by P_{2t+2}).*

It is obvious that the procedure will stop in $O(V)$ steps, and hence Alice will eventually select a vertex.

After Alice selected the next vertex, say v, add the vertices of the directed path P_{2t} and the vertex v to T_a, where P_{2t} is the last path found in the procedure above. It is obvious that T_a induces a subtree of T after adding these vertices.

Theorem 1 *If Alice uses the strategy described above, then the back degree of the linear order produced in the game is at most* 18.

We shall prove that at any stage of the game, after Alice finished her move and before Bob takes his next move, any free vertex has at most 17 selected neighbours. First we make a few observations.

Observation 1 *All the selected vertices are active.*

This follows from the strategy.

Observation 2 *If v has an active major son, then v is active.*

This is because the active vertices induces a subtree of T. If v has a major active son, say u, then since v is on the unique path of T connecting u and r (which is an active vertex), it follows that v is active.

Observation 3 *The vertices selected by Alice is either active before she selects it, or all its predecessors have been selected.*

This follows trivially from the strategy.

Observation 4 *If v has two major active sons who are not brothers, then v is a selected vertex.*

Let u_1, u_2 be the first two active sons of v who are not brothers. When u_1 becomes active, v becomes active (cf. Observation 2).

When u_2 becomes active, it means that the path P_{2t} constructed in the procedure above passes through u_2 (at certain moment), and hence it ends at v, because $v \in T_a$ and $u_2 \notin T_a$ at that stage. Now since u_2 has no brothers which are active, by the rule, Alice will select the vertex v, provided that it is not selected before.

Observation 5 *If v has three active major sons, then v is a selected vertex.*

If v has three active major sons and is still not selected, then by Observation 4, these three active major sons of v are brothers. Let u_1, u_2, u_3 be the first three major sons of v who are brothers. When u_1 becomes active, v also becomes active. When u_3 becomes active, it means that the path P_{2t} constructed in the procedure above passes through u_3, and hence it ends at v, because $v \in T_a$ at that stage. Now since u_3 has two active brothers (i.e., an even number of active brothers), by the rule, Alice will select the vertex v, provided that it is not selected before.

Observation 6 *If v has two active minor sons who are brothers, then v is active.*

Let u_1, u_2 be the first two active minor sons of v who are brothers. Let v' be the major parent of u_1 and u_2. When u_1 becomes active, v' also becomes active. When u_2 becomes active, it means that the path P_{2t} constructed in the procedure above passes through u_2, and hence it ends at v', because $v' \in T_a$ at that stage. Now since u_2 has exactly one active brother (i.e., an odd number of active brothers), by the rule, the path P_{2t} is switched, and hence passes through v. Therefore either v is already active at that stage, or becomes active after Alice finishes that move.

Observation 7 *If v has four active minor sons which are brothers, then v is a selected vertex.*

Let u_1, u_2, u_3, u_4 be the first four active minor sons of v who are brothers. Let v' be the major parent of u_1, u_2, u_3 and u_4. When u_1 becomes active, v' also become active. When u_2 becomes active, v becomes active (cf. Observation 7). When u_4 becomes active, it means that the path P_{2t} constructed in the procedure above passes through u_4, and hence it ends at v', because $v' \in T_a$ at that stage. Now since u_4 has three active brothers (i.e., an odd number of active brothers), by the rule, the path P_{2t} is switched and then extended to the path P_{2t+2}. Since $v \in T_a$ at that stage, P_{2t+2} ends at v. By the rule, Alice will select v at that stage, provided that v is not selected before.

Observation 8 *If v has four active minor sons which form two pairs of brothers, then v is a selected vertex.*

The argument for making this observation is similar to that for Observation 7 and we omit the details.

Observation 9 *If v has two active minor sons who are brothers, and one active major son, then v is a selected vertex.*

Suppose u_1 is an active major son of v, and u_2, u_3 are the first two active minor sons of v who are brothers, where u_2 becomes active before u_3 does. Let v' be the major parent of u_2 and u_3.

First we assume that u_1 becomes active before u_3 becomes active. When u_1 becomes active, v becomes active (cf. Observation 2). When u_2 becomes active, v' becomes active. When u_3 becomes active, it means that the path P_{2t} constructed in the procedure above passes through u_3, and hence it ends at v', because $v' \in T_a$ at that stage. Now since u_3 has one active brother (i.e., an odd number of active brothers), by the rule, the path P_{2t} is switched and then extended to the path P_{2t+2}. Since $v \in T_a$ at that stage, P_{2t+2} ends at v. By the rule, Alice will select v at that stage, provided that v is not selected before.

The case u_3 becomes active before u_1 becomes active is similar, and we omit the details.

With these observations, we are ready to count the number of active neighbours of free vertices. Suppose Alice has finished her move, and that x is still a free vertex. We now count the number of active neighbours of x.

Since x has only two parents, x has at most two active parents. Since $\vec{G}_R$ has maximum degree 8, x has at most 8 active neighbours in $\vec{G}_R$.

By Observation 4, x has at most two active major sons.

By Lemma 6, the minor sons of x partition into at most 5 groups of brothers. By Observations 7 and 8, each group has at most one active minor son of x, except one group which may contain (at most) three active minor sons of x. Therefore x has at most 7 active minor sons.

By Observation 9, either x has no active major sons, or x has at most 5 active minor sons (i.e., each group of brothers that are minor sons of x contains at most one active element). Therefore the total number of active sons (major and minor) is at most 7.

Since each neighbour of x is either a parent of x, or a neighbour in $\vec{G}_R$, or a major son, or a minor son, we conclude that x has at most 17 active neighbours. By Observation 1, each selected vertex is an active vertex. Therefore x has at most 17 selected neighbours.

When Bob select a vertex, the selected neighbours of a free vertex increases at most by 1. Thus after Bob's move, each free vertex has at most 18 selected neighbours. This completes the proof of Theorem 1.

Corollary 1 *For any planar graph G, $\mathrm{col}_g(G) \leq 19$.*

Proof. If G is connected, has no 1-vertex, no 2, 2-edges, then by Lemma 5, G has a spanning supergraph G^* which is the union of two graphs $\vec{G}_R$ and $\vec{G}_B$ that satisfy the condition of Theorem 1. By Lemma 3, $\mathrm{col}_g(G) \leq \mathrm{col}_g(G^*) \leq 19$. If G is not connected, or contains 1-vertex, or 2, 2-edges, then it is easy to see that by adding edges to G, we may obtain a connected planar graph G' without 1-vertex and $2,2$-edges. By Lemma 3, we have $\mathrm{col}_g(G) \leq \mathrm{col}_g(G') \leq 19$. ∎

Corollary 2 *For any planar graph G, $\chi_g(G) \leq 19$.*

5 Open questions

This section contains some open questions that relate to the result and concepts in this paper. For the convenience of the readers, we list the definition and give reference to a few terms used in the questions.

arboricity of a graph : the minimum number of spanning forests of a graph G that covers all the edges of G [19];

acyclic chromatic number : the minimum number of colors needed to color the vertex set of a graph so that each color class is an independent set and the union of any two color classes induces a forest [1, 4, 7, 16];

oriented chromatic number : for an oriented graph $\vec{G}$, the oriented chromatic number $\chi_o(\vec{G})$ is the minimum number of colors needed to color the vertex set of $\vec{G}$ so that each color class is an independent set, and the edges between any two color classes are oriented in the same direction. For an unoriented graph G, the oriented chromatic number is the maximum of $\chi_o(\vec{G})$ of all the orientations of G, [16].

By Lemma 1 we know $\chi_g(G) \leq \mathrm{col}_g(G)$. Indeed, for many classes of graphs, the best upper bound for the game chromatic number are obtained by investigating the game coloring number, although the term "game coloring number" was not used before. For example, it is proved in [11] that outerplanar graphs have game coloring number at most 7. This is also the best known upper bound for the game chromatic number of outerplanar graphs. Also for interval graphs, the upper bound of the game chromatic number given in [9] is also derived from an upper bound of the game coloring number of such graphs. In some sense, the parameter $\mathrm{col}_g(G)$ is easier to handle. The parameter $\chi_g(G)$ exhibits some strange properties. For example, it is not hereditary, i.e., a subgraph may have a larger game chromatic number. Also the following "naive" question has puzzled this author for quite some time:

Question 1 *Suppose $\chi_g(G) = k$. It is true that for any $k' > k$, if the color set X has cardinality k', then Alice has a winning strategy for the coloring game played on G?*

One would expect the answer to be "yes", because having more colors seems to be to the advantage for Alice. However, to my knowledge, this is an open problem. The following “easier” (probably not easier at all) question is also open:

Suppose $\chi_g(G) = k$. Is there any particular number, say $f(k)$, which depends on k only and is not equal to k, such that Alice has a winning strategy for the coloring game played on G with $f(k)$ colors ?

Compared to such strangeness of the game chromatic number, the game coloring number behaves very naturally. This probably contributes to the fact that the upper bounds for the game chromatic number of many classes of graphs are obtained by investigating the game coloring number. This naturally leads to the question that whether or not the game coloring number of a graph is bounded above by a function of the game chromatic number. The answer is “no”, as the complete bipartite graph $K_{n,n}$ has game chromatic number 3, and game coloring number $n+1$. However, this phenomenon may be due to the fact that the game chromatic number is not hereditary. Therefore, the “correct” question should restrict to hereditary classes of graphs.

Question 2 *Suppose a hereditary class of graphs (i.e., closed under taking (not necessarily induced) subgraphs) has bounded game chromatic number. Is it true that this class of graph also has bounded game coloring number ?*

The result in this paper seems to provide (very weak) evidence that supports a ”yes” answer.

Question 3 *Suppose a hereditary class of graphs has bounded game chromatic number. Is it true that the arboricity of this class of graphs is also bounded?*

Since a graph of bounded game coloring number certainly has a bounded arboricity, a ”yes” answer to Question 2 implies a ”yes” answer to Question 3.

Note that a class of graphs of bounded arboricity may have unbounded game chromatic number [9]. However, it is possible that the arboricity (of a hereditary class of graphs) is bounded by a function of the game chromatic number. One may compare this to the relation among the arboricity, the acyclic chromatic number and the oriented chromatic number. It is well-known that graphs of bounded arboricity may have unbounded acyclic chromatic number and unbounded oriented chromatic number, however, it was proved in [16] a graph of bounded acyclic chromatic number or bounded oriented chromatic number does have bounded arboricity.

For the relation between the acyclic chromatic number and the game chromatic number, it was proved in [7] that a class of graphs of bounded acyclic chromatic number have bounded game chromatic number. We repeat a question asked in [7]:

Question 4 *Is it true that a hereditary class of graphs of bounded game chromatic number also has bounded acyclic chromatic number?*

The following question is related to Question 2:

Question 5 *Is it true that a class of graphs of bounded acyclic chromatic number also has bounded game coloring number?*

Since we know that a hereditary class of graphs of bounded acyclic chromatic number has bounded game chromatic number, a "yes" answer to Question 2 implies a "yes" answer to Question 5. The result in this paper exhibit an example of a hereditary class of graphs which are first shown to have bounded game chromatic number, and then shown to have bounded coloring number. The class of partial k-trees and the class of graphs embeddable on a surface of genus $g \geq$ are also first shown to have bounded game chromatic number [7], and in a paper in preparation [21], these classes of graphs are shown to have bounded game coloring number.

Question 6 *Is it true that a class of graphs of bounded game coloring number also has bounded acyclic chromatic number?*

This question is related to Question 4. Because graphs of bounded game coloring number has bounded game chromatic number, a "yes" answer to Question 4 implies a "yes" answer to Question 6.

The maximum game chromatic number as well as the maximum game coloring number of a planar graph is now known to lie in the range from 8 to 19. It would be nice if the gap could be further reduced. Without any solid evidence to support any conjecture on the exact value of the maximum game chromatic number and the maximum game coloring number of a planar graph, it is the author's feeling that the exact value might be closer to the upper bound. However, new ideas and techniques might be needed to improve the lower bound.

Remark The result in this paper has been generalized to graphs embeddable on higher surfaces. It is proved in [21] that graphs embeddable on an orientable surface of genus $g \geq 1$ have game coloring number at most $\lfloor \frac{1}{2}(3\sqrt{1+48g}+23) \rfloor$.

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