Game Chromatic Number of Outerplanar Graphs*

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Appeared JGT 30(1999),67-70.

Abstract

This note proves that the game chromatic number of an outerplanar graph is at most 7. This improves the previous known upper bound of the game chromatic number of outerplanar graphs.

1991 Mathematics Subject Classification. 05C20, 05C35, 05C15
Keywords. Game chromatic number, outerplanar graph, 2-tree.

Let G be a finite graph and let X be a set of colors. We consider a modified graph coloring problem posed as a two-person game, with one person (Alice) trying to color the graph, and the other (Bob) trying to prevent this from happening. Alice and Bob alternate turns, with Alice having the first move. A move consisting of selecting a previously uncolored vertex x and assigning it a color from the color set X distinct from the colors assigned previously (by either player) to neighbours of x. If after $n = |V(G)|$ moves, the graph

*This research was supported in part by the National Science Council of the Republic of China under grant NSC87-2115-M-110-004/006.

G is colored, Alice is the winner. Bob wins if an impasse is reached before all vertices in the graph are colored, i.e., for every uncolored vertex x and every color α from X, x is adjacent to a vertex having color α. The *game chromatic number* of a graph $G = (V, E)$, denoted by $\chi_g(G)$, is the least cardinality of a color set X for which Alice has a winning strategy. This parameter is well-defined, since Alice always wins if $|X| = |V|$.

The game chromatic number of a graph is introduced by Bodlaender in [1], where the game chromatic number of a tree is discussed, and it is conjectured that the game chromatic number of a planar graph is bounded. It is proved by Kierstead and Trotter [4] that the maximum of the game chromatic number of a forest is 4; the maximum game chromatic number of a planar graph is between 8 and 33, and that the maximum game chromatic number an outerplanar graph is between 6 and 8. Recently Dinski and Zhu [2] proved that if a graph G has acyclic chromatic number n, then its game chromatic number is at most $(n+1)n$. As planar graphs have acyclic chromatic number at most 5, this reduces the upper bound of the maximum game chromatic number of a planar graph from 33 to 30. In this short note, we reduce the upper bound of the maximum game chromatic number of outerplanar graphs from 8 to 7. A graph G is an *outerplanar graph* if G can be embedded on the plane so that every vertex is incident to the infinite face. A *triangulated outerplanar graph* is an outerplanar graph in which every face is a triangle, except possibly the infinite face. A *triangulation* of an outerplanar graph G is any triangulated outerplanar graph G' obtained from G by adding edges. A graph G is a 2-*tree* if there is an ordering $v_1, v_2, \cdots, v_n$ such that v_1 is adjacent to v_2, and for every $i \geq 3$ the vertex v_i has exactly two neighbours v_j with $j < i$ and these two neighbours are adjacent. It is well-kown and also easy to see that a triangulated outerplanar graph is a 2-tree. However, a 2-tree may not be an outerplanar graph.

Theorem 1 *If G is an outerplanar graph then $\chi_g(G) \leq 7$.*

Let G be an outerplanar graph. Let X be a set of 7 colors. Before describing a winning strategy for Alice, we discuss the structure of G. Let G' be the triangulation of G. Let $v_1, v_2, \cdots, v_n$ be an ordering of the vertices G' such that v_1 is adjacent to v_2, and for every $i \geq 3$ the vertex v_i has exactly two neighbours v_j with $j < i$. Moreover, we require that v_1v_2 is an edge incident to the infinite face of G'. For each $i \geq 3$, let $i_1 < i_2 < i$ be the indices of these two neighbours of v_i. We may view $i \to i_1$ and $i \to i_2$ as mappings from $V(G) - \{v_1, v_2\}$ to $V(G)$, which are well-defined. The following properties of these mappings are straightforward:

1. v_{i_1} is adjacent to v_{i_2} in G'; and

2. if $i \neq j$, then $\{i_1, i_2\} \neq \{j_1, j_2\}$.

Property (1) is common to all 2-trees. Property (2) is special to triangulated outerplanar graphs. Indeed, to ensure that the ordering has property (2), the first two vertices v_1 and v_2 should form an edge incident to the infinite face of G'. The following lemma follows from Property (2):

Lemma 1 *For any vertex v_k, there are at most two vertices, say v_i and v_j such that $i_2 = j_2 = k$.*

Proof. Assume to the contrary that there are three vertices v_i, v_j, v_h such that $i_2 = j_2 = h_2 = k$. Then by Property (2), the three indices i_1, j_1, h_1 are pairwise distinct. By Property (1), v_k is adjacent to each of $v_{i_1}, v_{j_1}, v_{h_1}$. However, it follows from the definition that $i_1, j_1, h_1 < k$, contrary to the fact

v_k has at most two neighbours v_s with $s < k$. This completes the proof of Lemma 1. ∎

Now we delete all the edges of the form $v_i v_{i_2}$ from the graph G', and denote the resulting graph by T. Since each vertex v_i ($i \geq 2$) has exactly one neighbour v_j in T with $j < i$, the graph T is indeed a tree. By Lemma 1, each vertex v_i of G' is incident to at most three deleted edges, one of the form $v_i v_{i_2}$ and two of the form $v_j v_{j_2}$ with $j_2 = i$.

We shall now describe the strategy of Alice. The graphs G' and T are auxiliary graphs used by Alice only for the purpose of determining which vertex to color. The graph G is the one that will be colored.

In the game, Alice will pick the vertex to be colored by considering the structure of the tree T. In other words, when Alice determines the next vertex to be colored, the edges of $G - T$ is invisible to her. Only when she picks the color for that vertex, she will consider those edges.

Suppose in the process of the game, the tree T is partially colored. We define a *trunk* of T to be a maximal subtree T' of T such that every colored vertex of T' is a leaf of T'. Note that the collection of trunks of T is uniquely determined by the partial coloring of T. Indeed, the collection of trunks of T can be obtained as follows: For each colored vertex x with degree d, we may split x into d colored vertices (with the same color as x), say $x_1, x_2, \cdots, x_d$, so that each of the x_i is incident with exactly one of the d edges that was originally incident with x. After splitting each of the colored vertices of T, we obtain a collection of smaller partially colored trees, say $T_1, T_2, \cdots, T_m$. The union of the edges of T_i's is equal to the edge set of T. In each of the T_i's, only some of the leaves may be colored. These subtrees T_i are the trunks of the partially colored T.

Alice's goal in picking the next vertex to color is simply to ensure that after she colored the picked vertex, each of the trunks of the partially colored T has at most two colored leaves. Suppose Alice can achieve this goal. Then after Bob's color a vertex, then each trunk T_i of the partially colored T has at most three colored leaves. Therefore, at any moment of the game, each uncolored vertex has at most three colored neighbours in T. By Lemma 1, any vertex of G has at most three neighbours in $G - T$. Therefore altogether each uncolored vertex has at most six colored neighbours in G, and hence can be colored by a color not used by any of its colored neighbours. Hence Alice will win the game.

It remains to show that Alice can pick the next vertex to color in such a way that after coloring that vertex, each of the trunks T_i of the partially colored T has at most two colored leaves. We shall prove by induction on the number of steps that Alice can achieve this goal.

At the beginning of the game this is certainly true. Suppose this is true at step k. After Alice's move, each of the trunk T_i's has at most two colored leaves. Now suppose Bob color a vertex x of trunk T_j. In case x is not a leaf of T_j, then after x being colored, T_j is partitioned further into smaller trunks. No matter how the trunk T_j is partitioned, among these smaller trunks, at most one of them has three colored leaves. Assume there is a trunk, say T_{ji} which contains three colored leaves, say x, y, z. (The case there is no such trunk is trivial). Let P_{xy}, P_{yz} and P_{xz} be the x-y-path, y-z-path and x-z-path of T_{ji}, respectively. Then the intersection of P_{xy}, P_{yz} and P_{xz} consists of exactly one vertex, say u. Alice simply colors vertex u. This move will partition T_{ji} into smaller trunks, each having at most two colored leaves. Therefore Alice indeed can achieve the goal that after her move, each of the trunks of the partially colored T has at most two colored leaves. Hence Alice has a winning strategy. This completes the proof of Theorem 1.

The proof of Theorem 1 can be easily adopted to prove the following more general result:

Theorem 2 *Suppose $G = (V, E)$ is a graph and $E = E_1 \cup E_2$. Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$. If*

1. *$\Delta(G_1) = d$, and*
2. *for the coloring game played on G_2, Alice has a strategy such that in the process of the game any uncolored vertex has at most k colored neighbours,*

then $\chi_g(G) \leq k + d + 1$.

Note that Condition (2) above implies that the game chromatic number of G_2 is at most $k+1$. However, if Condition (2) is replaced by the condition that "$\chi_g(G_2) \leq k + 1$", the statement above would be false.

Kierstead and Trotter [4] have proved that there are outerplanar graphs with game chromatic number at least 6. It remains an open question whether or not there are outerplanar graphs with game chromatic number 7.

Remark Recently, X. Zhu has proved that the game chromatic number of a planar graph is at most 19 [5], the game chromatic number of a partial k-tree is at most $3k + 2$ [6] and that for $g \geq 1$, any graph embeddable on the orientable surface of genus g has game chromatic number at most $\lfloor \frac{3}{2}\sqrt{1 + 48g} + 11 + \frac{1}{2} \rfloor$ [7].

References

[1] H. L. Bodlaender. On the complexity of some coloring games. *International Journal on Foundations of Computer Science*, 2: 133–147, 1991.

[2] T. Dinski and Xuding Zhu. Game chromatic number of graphs. *Discrete Mathematics*, to appear.

[3] U. Faigle, U. Kern, H. Kierstead, and W. T. Trotter. On the game chromatic number of some classes of graphs. *Ars Combinatoria*, 35:143–150, 1993.

[4] H. A. Kierstead and W. T. Trotter. Planar graph coloring with an uncooperative partner. *Journal of Graph Theory*, 18(6):569–584, 1994.

[5] X. Zhu. Game coloring number of planar graphs. Preprint, 1998.

[6] X. Zhu. Game coloring number of chordal graphs. Preprint, 1998.

[7] X. Zhu. Game coloring number of pseudo partial k-trees. Preprint, 1998.