Edge-partitions of Planar Graphs and Their Game Coloring Numbers

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May 6, 2000

Abstract

Let G be a planar graph and let $g(G)$ and $\Delta(G)$ be its girth and maximum degree, respectively. We show that G has an edge-partition into a forest and a subgraph H so that (i) $\Delta(H) \leq 4$ if $g(G) \geq 5$; (ii) $\Delta(H) \leq 2$ if $g(G) \geq 7$; (iii) $\Delta(H) \leq 1$ if $g(G) \geq 11$; (iv) $\Delta(H) \leq 7$ if G does not contain 4-cycles (though it may contain 3-cycles). These results are applied to find the following upper bounds for the game coloring number $\mathrm{col}_g(G)$ of a planar graph G: (i) $\mathrm{col}_g(G) \leq 8$ if $g(G) \geq 5$; (ii) $\mathrm{col}_g(G) \leq 6$ if $g(G) \geq 7$; (iii) $\mathrm{col}_g(G) \leq 5$ if $g(G) \geq 11$; (iv) $\mathrm{col}_g(G) \leq 11$ if G does not contain 4-cycles (though it may contain 3-cycles).

Keywords: Planar graph, girth, light edge, game chromatic number, game coloring number, decomposition.

*Partially supported by the National Science Council of R.O.C. under grant NSC88-2115-M-110-001.

1 Introduction

All graphs considered in this paper are finite, loopless, and without multiple edges unless otherwise stated. For a plane graph G, let $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$, and $\delta(G)$ denote the vertex set, the edge set, the face set, the maximum degree, and the minimum degree of G, respectively. For $x \in V(G) \cup F(G)$, let $d_G(x)$ (or $d(x)$) denote the *degree* of x in G. The degree of a face f is the number of edge-steps in the boundary walk of f. A *k-vertex* (or *k-face*) of G is a vertex (or face) of degree k. For $f \in F(G)$, we use $V(f)$ to denote the set of vertices on the boundary of f. The *girth* $g(G)$ of a graph G is the length of a shortest cycle of G.

The game chromatic number of a graph is defined through a two-person graph coloring game. Suppose that G is a graph and C is a set of colors. Alice and Bob take turns coloring the vertices of G with colors from C so that no two adjacent vertices receive the same color. Alice has the first move and she wins the game if all the vertices of G are colored. Bob wins the game if at a certain time there is an uncolored vertex which cannot be colored properly, i.e., the uncolored vertex is adjacent to vertices of all colors. The *game chromatic number* $\chi_g(G)$ of G is the least cardinality of a set C of colors for which Alice has a winning strategy.

The game coloring number of a graph G is defined through a two-person graph ordering game. Alice and Bob take turns choosing vertices from the set of unchosen vertices of G. This defines a linear order L of the vertices of G with $x < y$ if and only if x is chosen before y. The *back degree* of a vertex x with respect to L is the number of its neighbors y in G such that $y < x$. The *back degree* of L is the maximum back degree of a vertex of G with respect to L. Alice's goal is to minimize the back degree of L and Bob's goal is to maximize it. The *game coloring number* $\mathrm{col}_g(G)$ of G is defined to be $k+1$, where k is the minimum integer such that Alice has a strategy for the graph ordering game to ensure that the back degree of L is at most k. Equivalently, k is the maximum integer such that Bob has a strategy for the graph ordering game to ensure that the back degree of L is at least k.

The concept of the game chromatic number of a graph was introduced by Bodlaender [1]. The concept of the game coloring number of a graph was first formally defined and investigated by Zhu [17]. It is easy to see that $\chi_g(G) \le \mathrm{col}_g(G)$ for any graph G. There exist graphs G such that $\chi_g(G)$ is much smaller than $\text{col}_g(G)$. However, for many classes of graphs, the best known upper bounds for their game chromatic numbers are also the best known upper bounds for their game coloring numbers. Faigle et al. [7] proved that $\chi_g(T) \leq \text{col}_g(T) \leq 4$ for every forest T and $\chi_g(G) \leq \text{col}_g(G) \leq 3\omega(G) - 2$ for every interval graph G, where $\omega(G)$ is the clique number of G. Kierstead and Trotter [11] studied the game chromatic number of planar graphs and proved that 33 is an upper bound. Dinski and Zhu [6] reduced their upper bound to 30. Both [6] and [11] did not provide any upper bound for the game coloring number of planar graphs. Recently, Zhu [17] proved that the game coloring number of a planar graph is at most 19. This upper bound was further reduced by Kierstead [12] to 18. Kierstead and Trotter [11] also proved that the game chromatic number of an outerplanar graph is at most 8. This upper bound has been reduced by Guan and Zhu [8]. They actually showed that the game coloring number of an outerplanar graph is at most 7.

In this paper, we study the upper bounds for game coloring numbers of planar graphs whose girths have prescribed lower bounds. More precisely, we will establish the following.

Main Theorem *Let G be a planar graph. Then the following statements hold.*

(i) $\chi_g(G) \leq \text{col}_g(G) \leq 8$ *if* $g(G) \geq 5$.

(ii) $\chi_g(G) \leq \text{col}_g(G) \leq 6$ *if* $g(G) \geq 7$.

(iii) $\chi_g(G) \leq \text{col}_g(G) \leq 5$ *if* $g(G) \geq 11$.

(iv) $\chi_g(G) \leq \text{col}_g(G) \leq 11$ *if G does not contain 4-cycles.*

To prove these results, we rely on appropriate edge-partitions of planar graphs. An *edge-partition* of a graph G is a decomposition of G into subgraphs $G_1, G_2, \ldots, G_m$ such that $E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_m)$ and $E(G_i) \cap E(G_j) = \emptyset$ for $i \neq j$. Edge-partitions of planar graphs have been studied extensively. Nash-Williams [14] gave a necessary and sufficient condition for a graph to have an edge-partition into a fixed number of forests. His results imply that every planar graph has an edge-partition into three forests and every planar graph without 3-cycles has an edge-partition into two forests. Kampen [9] proved that every maximal planar graph G has an edge-partition into three trees of order $|G| - 1$. For a maximal bipartite planar graphs G, Ringel [15] further showed that G has an edge-partition into two trees of order $|G| - 1$. Wu [16] proved that every planar graph G with $\Delta(G) \neq 7$ has an edge-partition into $\lceil (\Delta(G) + 1)/2 \rceil$ linear forests. Kedlaya [10] proved that every planar graph has an edge-partition into two graphs, each of which has no subgraph homeomorphic to the complete graph K_4.

In this paper, we consider the edge-partitions of planar graphs of suitably large girth into a forest and a graph of bounded maximum degree. The existence of such edge-partitions will be proved by using a vertex-inserting technique. Afterwards, the following easy lemma, which was first explicitly stated in Zhu [17], can be applied to planar graphs to obtain our Main Theorem.

Lemma 1 *Suppose that the graph G has an edge-partition into two subgraphs G_1 and G_2. Then* $\mathrm{col}_g(G) \leq \mathrm{col}_g(G_1) + \Delta(G_2)$.

2 Light Edges

Suppose G is a planar graph. We will denote by n, m, q the number of vertices, edges and faces of G, respectively. Let $E_i = \{xy \in E(G) \mid d(x) \geq i \text{ and } d(y) \geq i\}$ and $V_i = \{x \mid d(x) \geq i\}$. Let n_i and f_i denote the number of i-vertices and the number of i-faces, respectively. Let $n_i(v)$ denote the number of i-vertices that are adjacent to v, and $f_i(v)$ the number of i-faces that are incident to v. If the graph under consideration is G', then we use $n', m', q', n'_i, f'_i, n'_i(v), f'_i(v)$ denote the corresponding parameters.

An edge $e = xy$ is called an (a, b)-edge if $d(x) = a$ and $d(y) = b$. For an edge $e = xy$, Let $M(e) = \max\{d(x), d(y)\}$. Let $M^*(G) = \min\{M(e) \mid e \in E(G)\}$. It was proved by Borodin [3] that a planar graph G with $\delta(G) \geq 3$ contains either a $(5, j)$-edge for some $j \leq 6$, or a $(4, j)$-edge for some $j \leq 8$ or a $(3, j)$-edge for some $j \leq 10$. It follows that if $\delta(G) \geq 3$, then $M^*(G) \leq 10$; if $\delta(G) \geq 4$, then $M^*(G) \leq 8$; if $\delta(G) = 5$, then $M^*(G) \leq 6$. In [4], Borodin improved his own result to show that every planar graph G with $\delta(G) \geq 4$ contains either a $(4, j)$-edge for some $j \leq 7$ or a $(5, j)$-edge for some $j \leq 6$, thus $M^*(G) \leq 7$. Kotzig [13] proved that, if G is a planar graph without 3-cycles and $\delta(G) = 3$, then there is an edge xy such that $d(x) + d(y) \leq 8$, which implies that $M^*(G) \leq 5$.

The condition that $\delta(G) \geq 3$ is essential in these results. Without this condition, the parameter $M^*(G)$ is not bounded by a constant. In fact, $M(e) = n$ for any edge e of the planar bipartite graph $K_{2,n}$. However, we will show in this paper that, if G does not contain 4-cycles, then $M^*(G)$ is bounded by a constant even when $\delta(G) = 2$.

Theorem 2 *Let G be a connected planar graph with $\delta(G) \geq 2$. If $g(G) \geq 2k+1$ for some integer $k \geq 2$. Then $M^*(G) \leq \lceil \frac{k+3}{k-1} \rceil$.*

Proof. Suppose the theorem is false. Let G be a counterexample, i.e., G is a connected plane graph with $\delta(G) \geq 2$, $g(G) \geq 2k+1$, $M(e) \geq \lceil \frac{k+3}{k-1} \rceil + 1$ for every edge $e \in E(G)$. Let $s = \lceil \frac{k+3}{k-1} \rceil$. For each edge $e \in E_{s+1}$, insert a new vertex into the edge e, i.e., subdivide the edge e. Let G' be the resulting graph. Then G' is a bipartite plane graph, with $A = \{x : d(x) \leq s\}$ and $B = \{x : d(x) \geq s+1\}$ as the two parts. Therefore $\sum_{i=2}^{s} in'_i = m' = \sum_{i \geq s+1} in'_i$. Hence $\sum_{i=2}^{s} n'_i \leq m'/2$ and $\sum_{i \geq s+1} n'_i \leq m'/(s+1)$. This implies that

$$n' \leq \frac{m'}{2} + \frac{m'}{s+1} \leq \frac{m'}{2} + \frac{(k-1)m'}{2k+2} = \frac{km'}{k+1}.$$

Since G has girth at least $2k+1$ and G' is bipartite, it follows that G' has girth at least $2k+2$ and hence $2m' \geq (2k+2)q'$. So $q' \leq m'/(k+1)$. Therefore

$$n' + q' \leq \frac{km'}{k+1} + \frac{m'}{k+1} = m',$$

contradicting Euler's formula. □

Corollary 3 *Suppose G is a connected planar graph with $\delta(G) \geq 2$. Then $M^*(G) \leq 5$ if $g(G) \geq 5$, $M^*(G) \leq 3$ if $g(G) \geq 7$ and $M^*(G) \leq 2$ if $g(G) \geq 11$.*

Corollary 3 is sharp. Let G be obtained from the icosahedron by subdividing each edge once. Then $g(G) = 6$, $\delta(G) = 2$ and $M^*(G) = 5$. Let H be obtained from the dodecahedron by subdividing each edge once. Then $g(H) = 10$, $\delta(H) = 2$ and $M^*(G) = 3$.

Theorem 4 *If G is a connected planar graph with $\delta(G) \geq 2$ and does not contain 4-cycles, then $M^*(G) \leq 8$.*

Proof. Suppose that G is a counterexample, i.e., G is a connected plane graph without 4-cycles, with $\delta(G) \geq 2$ and $M(e) \geq 9$ for every edge e. For $v \in V_9$, let $f_3^*(v) = |\{f \in F(G) \mid d(f) = 3,\ v \in V(f)$ and $V(f)$ contains a 2-vertex$\}|$. As G does not contain two adjacent 3-faces (for otherwise G contains a C_4), it follows that $f_3^*(v) \geq n_2(v) - (d(v) - 2f_3(v))$. Also G has no 2-vertex lying on a common boundary of a 3-face and a 5-face (for otherwise G contains a C_4). Therefore,

$$\begin{aligned} f_5(v) &\leq d(v) - f_3(v) - \lceil f_3^*(v)/2 \rceil \\ &\leq d(v) - f_3(v) - (n_2(v) - d(v) + 2f_3(v))/2. \end{aligned}$$

Hence $4f_3(v)+2f_5(v)+n_2(v) \leq 3d(v)$ and $\sum_{v\in V_9}(4f_3(v)+2f_5(v)+n_2(v)) \leq \sum_{v\in V_9} 3d(v)$. Note that each 3-face is incident to at least 2 vertices of V_9, each 5-face is incident to at least 3 vertices of V_9 and each 2-vertex is adjacent to exactly 2 vertices of V_9. Let $t = |E_9|$. Then we have $4 \times 2f_3 + 2 \times 3f_5 + 2n_2 \leq 3(m + t)$, hence

$$\frac{4}{3}f_3 + f_5 \leq \frac{m}{2} + \frac{t}{2} - \frac{n_2}{3}. \tag{1}$$

Since

$$\sum_{i\geq 3} if_i = 2m \tag{2}$$

and $f_4 = 0$, we add (1) and (2) together to get $6(f_3 + f_5 + f_6) + \sum_{i\geq 7} if_i \leq (5m/2) + (t/2) - (n_2/3) + (5f_3/3)$. This implies that

$$q = \sum_{i\geq 3} f_i \leq \frac{5}{12}m + \frac{t}{12} - \frac{n_2}{18} + \frac{5}{18}f_3. \tag{3}$$

If we insert a new vertex to every edge $e \in E_9$, then the resulting graph G' is a bipartite plane graph, with $A = \{x : d(x) \leq 8\}$ and $B = \{x : d(x) \geq 9\}$ as the two parts. So $\sum_{i=2}^{8} in'_i = m' = \sum_{i\geq 9} in'_i = m'$. Also $n'_2 = n_2 + t$, $m' = m + t$ and $n'_i = n_i$ for $i \geq 3$. Therefore,

$$\sum_{i=2}^{8} in_i = m' - 2t = m - t, \tag{4}$$

and

$$\sum_{i\geq 9} in_i = m' = m + t. \tag{5}$$

By (4), we have $(2 + \frac{2}{17})n_2 + 3n_3 + \cdots + 8n_8 = m - t + \frac{2}{17}n_2$. Hence

$$n_2 + n_3 + \cdots + n_8 \leq \frac{17}{36}m - \frac{17}{36}t + \frac{n_2}{18}. \tag{6}$$

By (5), we have

$$\sum_{i\geq 9} n_i \leq \frac{m}{9} + \frac{t}{9}. \tag{7}$$

Adding (3), (6) and (7) together, we conclude that

$$\begin{aligned} n+q &\leq \frac{5}{12}m + \frac{17}{36}m + \frac{1}{9}m + \frac{1}{12}t - \frac{17}{36}t + \frac{1}{9}t + \frac{5}{18}f_3 \\ &= m - \frac{5}{18}t + \frac{5}{18}f_3. \end{aligned}$$

Since every 3-face contains an edge $e \in E_9$, we know $f_3 \leq t$. Therefore $n + q \leq m$, contradicting Euler's formula. □

We do not know if the upper bound for $M^*(G)$ in Theorem 4 is sharp. However, it cannot be smaller than 3 since an icosidodecahedron is a planar graph G with $M^*(G) = 3$ and containing no 4-cycles.

3 Edge-partitions

Let G be a graph and k a positive integer. We say that G possesses *property* P_k if one of the following conditions holds.

(1) $\delta(G) \leq 1$.

(2) $M^*(G) \leq k$.

A graph G is said to be P_k*-hereditary* if each subgraph H of G has property P_k.

Theorem 5 *For $k \geq 1$, let G be a P_k-hereditary graph. Then G has an edge-partition into two graphs T and H such that T is a forest and H is a graph with $\Delta(H) \leq k-1$.*

Proof. We proceed by induction on the number $|E|$ of edges of G. When $|E| \leq 1$, the result is trivial. Let G be a P_k-hereditary graph with $|E| \geq 2$. Without loss of generality, we may assume that G is connected.

If $\delta(G) = 1$, let v be an 1-vertex of G and $vu \in E(G)$. By the induction hypothesis, $G - vu$ has a required edge-partition $T' \cup H'$. We obtain the required edge-partition of G by setting $T = T' + vu$ and $H = H'$.

Now we assume that $\delta(G) \geq 2$. If $M^*(G) \leq k$, choose $xy \in E(G)$ such that $d(x), d(y) \leq k$. Let $G' = G - xy$. By the induction hypothesis, we may construct a required edge-partition $T' \cup H'$ of G so that x and y are incident to at least one edge of T'. This implies that $d_{H'}(x) \leq d_G(x) - 2 \leq k - 2$ and $d_{H'}(y) \leq k - 2$. Let $T = T'$ and $H = H' + xy$. Then $G = T \cup H$ is an edge-partition satisfying the requirements of the theorem. □

Corollary 6 *Suppose that G is a planar graph. Then G has an edge-partition into a forest T and a subgraph H so that*

(i) $\Delta(H) \leq 4$ *if* $g(G) \geq 5$;

(ii) $\Delta(H) \leq 2$ *if* $g(G) \geq 7$;

(iii) $\Delta(H) \leq 1$ *if* $g(G) \geq 11$;

(iv) $\Delta(H) \leq 7$ *if G does not contain 4-cycles.*

Proof. It follows from Corollary 3 and Theorem 4 that

1. G is P_5-hereditary if $g(G) \geq 5$;
2. G is P_3-hereditary if $g(G) \geq 7$;
3. G is P_2-hereditary if $g(G) \geq 11$.
4. G is P_8-hereditary if G does not contain 4-cycles.

Then the corollary follows from Theorem 5. □

Since the game coloring number of a forest is at most 4, our Main Theorem follows from Corollary 6 and Lemma 1.

4 Open Problems

Our results show that if G is a planar graph of girth at least 5, then G has an edge-partition into a forest and a graph of bounded maximum degree. This conclusion may fail for planar graphs with girth ≤ 4. For an integer $g \geq 5$, let $\sigma(g)$ denote the least integer k such that every planar graph G with girth g has an edge-partition into a forest T and a graph H with $\Delta(H) \leq k$.

Problem 1 *Determine the exact value $\sigma(g)$ for $g \geq 5$.*

It follows from our results that $\sigma(g) \leq 4$ for $g = 5$ and 6, $\sigma(g) \leq 2$ for $7 \leq g \leq 10$, and $\sigma(g) \leq 1$ for $g \geq 11$. It is obvious that $\sigma(g) > 0$ for any g. So $\sigma(g) = 1$ for

$g \geq 11$. Note that every nonempty forest T contains at least two 1-vertices. Therefore $\Delta(H) \geq \delta(G) - 1$. Since there are planar graphs G with girth 5 and $\delta(G) = 3$, e.g., a dodecahedron, we have $\sigma(5) \geq 2$.

For an integer $k \geq 1$, let $\lambda(k)$ denote the least integer n such that every planar graph G with girth $g(G) \geq n$ has an edge-partition into a forest T and a graph H with $\Delta(H) = k$.

Problem 2 *Determine the exact value $\lambda(k)$ for $k \geq 1$.*

It follows from our results that $6 \leq \lambda(1) \leq 11$, $5 \leq \lambda(k) \leq 7$ for $2 \leq k \leq 3$, $5 \leq \lambda(4) \leq 6$, and $\lambda(k) = 5$ for all $k \geq 5$.

Conjecture 3 *Every planar graph G has an edge-partition into a forest T and a graph H with $\Delta(H) \leq \lceil \Delta(G)/2 \rceil + 1$.*

The upper bound $\lceil \Delta(G)/2 \rceil + 1$ cannot be replaced by a fixed integer and is sharp if the conjecture is true. For example, every 4-regular or 5-regular planar graph may attain this bound. The following theorem proves a statement weaker than Conjecture 3.

Theorem 7 *Every planar graph G has an edge-partition into a forest T and a graph H with $\Delta(H) \leq \lceil \Delta(G)/2 \rceil + 5$.*

It is easy to see that Theorem 7 holds trivially when $\Delta(G) \leq 13$. Instead of proving Theorem 7, we establish the stronger Theorem 9. We need the following lemma of Borodin [2] to get the result.

Lemma 8 *Let G be a planar graph with $\delta(G) \geq 2$. Then either there is an edge $xy \in E(G)$ with $d(x) + d(y) \leq 15$, or there is a 2-alternating cycle, i.e., an even cycle $C = v_1 v_2 v_3 \cdots v_{2n} v_1$ such that $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 2$.*

Theorem 9 *Every planar graph G has an edge-partition into a forest T and a graph H such that $d_H(v) \leq \max\{12, \lceil d_G(v)/2 \rceil + 5\}$ for every $v \in V(H)$.*

Proof. We prove this theorem by induction on the number of edges. If $|E(G)| \leq 12$, the result holds obviously. Let G a connected planar graph with $|E(G)| \geq 13$. If G

contains an 1-vertex v and $vu \in E(G)$, then $G - vu$ has an edge-partition $T' \cup H'$ satisfying the theorem by the induction hypothesis. We can extend an edge-partition of $G - vu$ to a required edge-partition of G by adding the edge vu to H'.

Assume $\delta(G) \geq 2$. By Lemma 8, either G has an edge $e = xy$ such that $d(x) + d(y) \leq 15$, or G has a 2-alternating cycle. If G has an edge xy such that $d(x) + d(y) \leq 15$, then $d(x) \leq 13$ and $d(y) \leq 13$. Let $T' \cup H'$ be an edge-partition of $G' = G - xy$, where T' is a forest and $d_{H'}(v) \leq \max\{12, \lceil d_{G'}(v)/2 \rceil + 5\}$ for every $v \in V(H')$. We may assume that both x and y are incident to at least one edge of T'. Let $H = H' + xy$ and $T = T'$. Then $d_H(x) \leq d_{H'}(x) + 1 \leq 12 - 1 + 1 = 12$ and similarly $d_H(y) \leq 12$. Since $d_H(t) = d_{H'}(t)$ for all $t \in V(H) - \{x, y\}$, $H \cup T$ is the required edge-partition of G.

If G has a 2-alternating cycle $C = v_1 v_2 v_3 \cdots v_{2s} v_1$ with $d_G(v_i) = 2$ for $i = 1, 3, \ldots, 2s-1$, where $s \geq 2$, then let $G' = G - E(C)$. By the induction hypothesis, G' has an edge-partition $T' \cup H'$ with properties stated in the theorem. We may assume that every v_i, $i = 2, 4, \ldots, 2s$, is incident to at least one edge of T'. We set $T = T' \cup \{v_1v_2, v_3v_4, \ldots, v_{2s-1}v_{2s}\}$ and $H = H' \cup \{v_2v_3, v_4v_5, \ldots, v_{2s}v_1\}$. Thus T and H constitute an edge-partition of G. It is obvious that T is a forest. Let v be an arbitrary vertex of H. If $v \notin V(C)$, then $d_H(v) = d_{H'}(v) \leq \max\{12, \lceil d_{G'}(v)/2 \rceil + 5\} = \max\{12, \lceil d_G(v)/2 \rceil + 5\}$. If $v = v_i$ and i is odd, then $d_H(v_i) = 1$. Suppose now that $v = v_i$ and i is even. It is easy to see that $d_H(v) = d_{H'}(v) + 1$ and $d_G(v) = d_{G'}(v) + 2$. For convenience, we write $t(v) = \max\{12, \lceil d_G(v)/2 \rceil + 5\}$ and $t'(v) = \max\{12, \lceil d_{G'}(v)/2 \rceil + 5\}$. If $d_{H'}(v) \leq 11$, then $d_H(v) \leq 12 \leq t(v)$. If $d_{H'}(v) \geq 13$, then $t'(v) = \lceil d_{G'}(v)/2 \rceil + 5$, hence $d_{H'}(v) \leq t'(v) = \lceil d_{G'}(v)/2 \rceil + 5$. Therefore $d_H(v) = d_{H'}(v) + 1 \leq \lceil d_{G'}(v)/2 \rceil + 5 + 1 = \lceil d_G(v)/2 \rceil + 5 \leq t(v)$. Assume $d_{H'}(v) = 12$. Since v is incident to at least one edge of T', $d_{G'}(v) = d_{H'}(v) + d_{T'}(v) \geq 12 + 1 = 13$ and thus $d_G(v) \geq 15$. Consequently, $d_H(v) = 13 = 8 + 5 \leq \lceil d_G(v)/2 \rceil + 5 = t(v)$. □

Using the following Lemma by Wu [16], a proof similar to Theorem 9 can be worked out to establish Theorem 11. We also can show that Conjecture 3 holds for all planar graphs of girth at least 5.

Lemma 10 *Let G be a planar graph with $\delta(G) \geq 2$. If $4 \leq g(G) \leq 6$, then either there is an edge $xy \in E(G)$ such that $d(x) + d(y) \leq 17 - 2g(G)$, or there is a 2-alternating cycle.*

Theorem 11 *Every planar graph G containing no 3-cycles has an edge-partition into a forest T and a graph H with $\Delta(H) \leq \lceil \Delta(G)/2 \rceil + 2$.*

Corollary 12 *Let G be a planar graph. Then $\chi_g(G) \leq \mathrm{col}_g(G) \leq \lceil \Delta(G)/2 \rceil + 9$. Moreover, $\chi_g(G) \leq \mathrm{col}_g(G) \leq \lceil \Delta(G)/2 \rceil + 6$ if G contains no 3-cycles.*

Let γ denote the least integer k such that every planar graph G has an edge-partition into two forests T_1, T_2 and a graph H with $\Delta(H) \leq k$.

Problem 4 *Determine the exact value of γ.*

The following theorem shows that $\gamma \leq 8$.

Theorem 13 *Every planar graph G has an edge-partition into two forests T_1, T_2 and a graph H with $\Delta(H) \leq 8$.*

Proof. We prove this theorem by induction on $|G| + |E|$. Suppose that G is a planar graph. If $\delta(G) \leq 2$, let v be a $\delta(G)$-vertex. Any required edge-partition of $G - v$ can be easily extended to a required edge-partition of G. If $\delta(G) \geq 3$, then there is an edge $xy \in E(G)$ such that $d(x) + d(y) \leq 13$ by a result of Borodin [2]. This implies that $d(x), d(y) \leq 10$. By the induction hypothesis, $G - xy$ has an edge-partition $T_1' \cup T_2' \cup H'$ such that T_1' and T_2' are forests and H' satisfies $\Delta(H') \leq 8$. We may assume that both x and y are incident to at least one edge of each of T_1' and T_2'. Let $T_1 = T_1'$, $T_2 = T_2'$ and $H = H' + xy$. Then $T_1 \cup T_2 \cup H$ is a required edge-partition of G. □

There exist planar graphs G such that $\Delta(H) \geq 2$ whenever G is edge-partitioned into T_1, T_2 and H. This can be seen by counting the number of edges of G, T_1, T_2 and H. Consequently, $\gamma \geq 2$. Guan and Zhu [8] proved that every outerplanar graph has an edge-partition into a forest T and a graph H with $\Delta(H) \leq 3$. Chartrand, Geller and Hedetniemi [5] conjectured that every planar graph has an edge-partition into two outerplanar graphs. If their conjecture holds, then we have $\gamma \leq 6$.

Acknowledgment. This work was done while the fifth author was visiting the Institute of Mathematics, Academia Sinica, Taipei. The financial support provided by the Institute is greatly appreciated.

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