# Circular game chromatic number of graphs

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**Abstract**

In a circular $r$-colouring game on $G$, Alice and Bob take turns colour the vertices of $G$ with colours from the circle $S(r)$ of perimeter $r$. Colours assigned to adjacent vertices need to have distance at least 1 in $S(r)$. Alice wins the game if all vertices are coloured, and Bob wins the game if some uncoloured vertices have no legal colour. The circular game chromatic number $\chi_{cg}(G)$ of $G$ is the infimum of those real numbers $r$ for which Alice has a winning strategy in the circular $r$-colouring game on $G$. This paper proves that for any graph $G$, $\chi_{cg}(G) \leq 2\mathrm{col}_g(G) - 2$, where $\mathrm{col}_g(G)$ is the game colouring number of $G$. This upper bound is shown to be sharp for forests. It is also shown that for any graph $G$, $\chi_{cg}(G) \leq 2\chi_a(G)(\chi_a(G) + 1)$, where $\chi_a(G)$ is the acyclic chromatic number of $G$. This paper also determines the exact value of the circular game chromatic number of some special graphs, including complete graphs, paths and cycles.

**Keywords:** circular game chromatic number, game colouring number, acyclic chromatic number, graphs, forests, planar graphs.

## 1 Introduction

This paper studies the circular game chromatic number of graphs. For a positive real number $r$, let $S(r)$ denote the circle obtained from the interval $[0, r]$ by identifying

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0 and $r$ into a single point. For any $x \in \mathbb{R}$, $[x]_r \in [0, r)$ denotes the remainder of $x$ upon division of $r$. For $a, b \in S(r)$, the interval $[a, b]_r$ is defined as $[a, b]_r = \{x \in S(r) : 0 \leq [x-a]_r \leq [b-a]_r\}$. The length of the interval $[a, b]_r$ is equal to $[b-a]_r$. Two points $a, b \in S(r)$ partition $S(r)$ into two arcs: $[a, b]_r$ and $[b, a]_r$. The *distance* between $a$ and $b$, denoted by $|a-b|_r$, is the length of the shorter arc. In other words, $|a-b|_r = \min\{[a-b]_r, [b-a]_r\} = \min\{|a-b|, r-|a-b|\}$. For a graph $G$, a circular $r$-colouring of $G$ is a mapping $f : V(G) \to S(r)$ such that for any edge $xy$ of $G$, $|f(x) - f(y)|_r \geq 1$. The *circular chromatic number* $\chi_c(G)$ of $G$ is the least $r$ for which $G$ has a circular $r$-colouring.

The circular chromatic number of graphs was first introduced by Vince [7] as the "star chromatic number". The concept has attracted considerable attention [11, 12] in the past decade. It is known that for any graph $G$, $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$ and hence $\chi(G) = \lceil \chi_c(G) \rceil$, and for any rational number $r \geq 2$, there is a graph $G$ with $\chi_c(G) = r$. So $\chi_c(G)$ determines $\chi(G)$, however, graphs with the same chromatic number can have distinct circular chromatic number. In this sense, $\chi_c(G)$ is viewed as a refinement of $\chi(G)$, and $\chi(G)$ is viewed as an approximation of $\chi_c(G)$.

The circular game chromatic number of $G$ is defined through a game. Given a graph $G$ and a real number $r$, the *circular $r$-colouring game on $G$* is a two-person game played by Alice and Bob. The two players alternate their turns. At each turn, a player picks an uncoloured vertex $x$ and assign a legal colour $f(x) \in S(r)$ to $x$, where a colour $a \in S(r)$ is legal for an uncoloured vertex $x$ if no colour from the interval $(a-1, a+1)_r$ is assigned to any neighbour of $x$ in previous moves (by either player). The game ends if either all vertices of $G$ are coloured or there is an uncoloured vertex $x$ which has no legal colour. In the former case, Alice wins the game. In the latter case, Bob wins the game. We also need to specify who has the first move. This could be crucial to the outcome of the game. However, most of the results proved in this paper remain true whether Alice or Bob has the first move. For convenience, we shall assume that Alice has the first move, except that in the last section, we also consider the case that Bob has the first move.

The *circular game chromatic number* $\chi_{cg}(G)$ of $G$ is the infimum of those $r$ for which Alice has a winning strategy for the circular $r$-colouring game on $G$. Observe that $\chi_{cg}(G)$ is well-defined, since if $r \geq 2\Delta(G)$, it is obvious that Alice has a winning strategy for the circular $r$-colouring game on $G$.

Note that if Alice wins then the players produce an $r$-circular colouring of $G$. It follows that $\chi_{cg}(G) \geq \chi_c(G)$. And the equality may holds for some graphs. For example, it is easy to check that $\chi_{cg}(K_{1,m}) = \chi_c(K_{1,m}) = 2$.

The circular colouring game chromatic number is a variation of the game chromatic number. Suppose $G$ is a graph and $X$ is a set of colours. Alice and Bob take turns to colour the vertices of $G$ with colours from $X$, with Alice having the first move. At each turn, a player colours an uncoloured vertex $x$ with a legal colour from $X$, where a colour $a \in X$ is legal for $x$ if no neighbour of $x$ is coloured with colour $a$. The game ends if either all the vertices of $G$ are coloured or there is an uncoloured vertex $x$ which has no legal colour. In the former case, Alice wins the game, and in the latter case Bob wins the game. The *game chromatic number* $\chi_g(G)$ of $G$ is the least number of colours contained in $X$ so that Alice has a winning strategy for the colouring game on $G$ with colour set $X$.

The colouring game on planar graphs was first introduced by Steven Brams in [4], and later the game was re-invented by Bodlaneder in [2], where the game chromatic number of an arbitrary graph is defined (see [1] for a survey). The game chromatic number of various classes of graphs have been studied extensively in the literature.

For a class $\mathcal{K}$ of graphs, let $\chi_g(\mathcal{K}) = \max\{\chi_g(G) : G \in \mathcal{K}\}$. A benchmark problem concerning the game chromatic number of graphs is to determine $\chi_g(\mathcal{P})$ where $\mathcal{P}$ is the class of planar graphs. It is now known that $8 \leq \chi_g(\mathcal{P}) \leq 17$ [6, 9]. The other classes of graphs for which the game chromatic number have been studied include outerplanar graphs [5], forests, partial $k$-trees, interval graphs, $(a, b)$-pseudo partial $k$-trees [10, 8], Cartesian product of graphs [13], etc.

In the study of the game chromatic number of graphs, the marking game on graphs was introduced in [9]. Two players, Alice and Bob, alternately mark unmarked vertices of $G$. The game ends when all vertices are marked. For a vertex $x$ of $G$, let $b(x)$ be the number of neighbours of $x$ that are marked before $x$ is marked. The score of the game is $1 + \max\{b(x) : x \in V(G)\}$. Alice's goal is to minimize the score of the game, and Bob's goal is to maximize the score. The *game colouring number* $\text{col}_g(G)$ of $G$ is the minimum $s$ such that Alice has a strategy that ensures that the resulting score is at most $s$. The game colouring number was introduced as a tool in the study of the game chromatic number (as it is easy to see that for any graph $G$, $\chi_g(G) \leq \text{col}_g(G)$). However, the game colouring number itself is also of independent interest and has been studied extensively in the literature.

In this paper, we first study some basic properties of the circular game chromatic number of graphs. We explore the relation between the circular game chromatic number and the game colouring number, and proves that for any graph $G$, $\chi_{cg}(G) \leq 2\text{col}_g(G) - 2$. By using the upper bounds on the game colouring number for classes of graphs, such as planar graphs, outerplanar graphs, partial $k$-trees, we obtain upper bounds on the circular game chromatic number of these classes of graphs. We shall prove that the upper bound on the circular game chromatic number derived through the game colouring number is sharp for forests. An upper bound for the circular game chromatic number of a graph in terms of its acyclic chromatic number is also given. We also determine the exact value of the circular game chromatic number of some special graphs, including complete graphs, paths and cycles.

In our definition of circular game chromatic number of graphs, we assumed that Alice takes the first move. Sometimes we need to consider games in which Bob has the first move. We shall denote by $\chi_{cg}^B(G)$ the infimum of the real numbers $r$ for which Alice has a winning strategy in the Bob-first circular $r$-colouring game on $G$.

## 2 $\chi_{cg}(G)$ and $\chi_g(G)$

This section explores between the circular game chromatic number $\chi_{cg}(G)$ of $G$ and other parameters, including the game chromatic number, game colouring number and acyclic chromatic number. First we give an upper bound for the game chromatic number of $G$ in terms of $\chi_{cg}(G)$.

**Theorem 1** *For any graph $G$, $\chi_g(G) \leq \lfloor \chi_{cg}(G) \rfloor + 1$.*

**Proof.** Assume $\chi_{cg}(G) = r$. By the definition of the circular game chromatic number, for any real number $\delta > 0$ there is a real number $\varepsilon \in [0, \delta)$ such that Alice has a winning strategy on the circular $(r + \varepsilon)$-colouring game on $G$.

Let $\delta$ be a positive number such that $r + \delta < \lfloor r \rfloor + 1$. And let $\varepsilon \in [0, \delta)$ be the number such that Alice has a winning strategy on the circular $(r+\varepsilon)$-colouring game on $G$. Let $r' = r + \varepsilon$. Identify points on the circle $S(r')$ with points from the interval $[0, r')$ (by viewing $S(r')$ be obtained from $[0, r']$ by identifying 0 and $r'$ into a single point.

Assume Alice and Bob are playing the colouring game on $G$ with colour set $X = \{0, 1, \cdots, k\}$, where $k = \lfloor r' \rfloor$. We shall show that Alice has a winning strategy. Alice will imagine that she is playing a circular $r'$-colouring game on $G$. (Be careful that the argument below involves two games: the colouring game which has colour set $X$, and the circular $r'$-colouring game which has colour set $S(r')$.) All the colours in $X$ are contained in $S(r')$. So each of Bob's move in the colouring game can be viewed as a move in the circular $r'$-colouring game. By our assumption, Alice has a winning strategy for the circular $r'$-colouring game on $G$, which we call Strategy A. Suppose Bob has just finished a move in the colouring game. It is now Alice's turn to play the colouring game. Alice view Bob's last move as a move in the circular $r'$-colouring game (this can be done as $X$ is a subset of $S(r')$). Assume by Strategy A, she should colour vertex $x$ with colour $t$ in the circular $r'$-colouring game. Then in the colouring game, Alice colours $x$ with colour $\lfloor t \rfloor$.

We need to show that Alice's move is a legal move. If this move is not legal, then there is a neighbour $y$ of $x$ coloured with colour $\lfloor t \rfloor$. If $y$ was coloured by Bob, then $\lfloor t \rfloor$ is also the colour for $y$ in the circular $r'$-colouring game. But $|\lfloor t \rfloor - t|_{r'} < 1$, which means that $t$ is not a legal colour for $x$ in the circular $r'$-colouring game on $G$, in contrary to our assumption. If $y$ was coloured by Alice, then in the circular $r'$-colouring game, $y$ is coloured with a colour $s$ such that $\lfloor s \rfloor = \lfloor t \rfloor$. Again $|s - t|_{r'} < 1$, in contrary to our assumption. So Alice's move is indeed a legal move.

Thus whenever Alice needs to colour a vertex, the vertex does have a legal colour. The same is true for Bob: for any uncoloured vertex $x$, if $t$ is a legal colour for $x$ in the $r'$-colouring game on $G$, then $\lfloor t \rfloor$ is a legal colour for $x$ in the colouring game on $G$ with colour set $X$. This completes the proof. ∎

The upper bound for $\chi_g(G)$ in Theorem 1 is usually not tight. As a simple example, consider the path $P = v_1v_2v_3v_4$. It is easy to verify that $\chi_g(P) = 3$, however, $\chi_{cg}(P) = 4$. In Section 4, we shall show that for complete graphs $K_n$, $\chi_{cg}(K_n)$ is approximately $4n/3$, where $\chi_g(K_n) = n$. So the difference $\chi_{cg}(G) - \chi_g(G)$ can be arbitrarily large. The following question is open:

**Question 1** *Is there a function $f : N \to R$ such that $\chi_{cg}(G) \leq f(\chi_g(G))$?*

# 3 $\chi_{cg}(G)$ and $\mathrm{col}_g(G)$

The following theorem shows that $\chi_{cg}(G)$ is bounded in terms of the game colouring number of $G$.

**Theorem 2** *For any graph $G$, $\chi_{cg}(G) \leq 2\mathrm{col}_g(G) - 2$.*

**Proof.** Suppose $\mathrm{col}_g(G) = k + 1$. Let $r = 2k$. We shall show that Alice has a winning strategy when they play the circular $r$-colouring game on the graph $G$ with $S(r)$. According to the definition of $\mathrm{col}_g(G)$, Alice has a strategy for choosing vertices to be coloured in her moves so that each uncoloured vertex has at most $k$ coloured neighbors. Alice use this strategy to choose the vertex to be coloured. Thus at any moment any uncoloured vertex has at most $k$ coloured neighbors. Since $S(r)$ has circumference $2k$, by the pigeon hole principle, for any uncoloured vertex $x$, there is at least one point on $S(r)$ which is legal for $x$. In her each move, Alice just chooses any legal point (colour) on $S(r)$ for the vertex she chooses. In this way, it is easy to see that all vertices will be coloured properly when the game is over. Therefore $\chi_{cg}(G) \leq 2\mathrm{col}_g(G) - 2$. ∎

Given a class $\mathcal{K}$ of graphs, let

$$\begin{aligned}\chi_{cg}(\mathcal{K}) &= \max\{\chi_{cg}(G) : G \in \mathcal{K}\},\\ \chi_g(\mathcal{K}) &= \max\{\chi_g(G) : G \in \mathcal{K}\},\\ \mathrm{col}_g(\mathcal{K}) &= \max\{\mathrm{col}_g(G) : G \in \mathcal{K}\}.\end{aligned}$$

If in the definition of $\mathrm{col}_g(G)$ we let Bob take the first move then we get the "Bob first" version of the game colouring number, denote it by $\mathrm{col}_g^B(G)$. Then similar to Theorem 2, the following theorem holds clearly.

**Theorem 3** *For any graph $G$, $\chi_{cg}^B(G) \le 2\mathrm{col}_g^B(G) - 2$.*

Let $P_n$ be the path on $n$ vertices and let $C_n$ be the cycle on $n$ vertices. Then, by Theorems 2 and 3, $\chi_{cg}(P_n) \le 4$, $\chi_{cg}(C_n) \le 4$; and $\chi_{cg}^B(P_n) \le 4$, $\chi_{cg}^B(C_n) \le 4$. Let $r < 4$ be a positive real number. We shall show that Bob has a winning strategy when play the circular $r$-colouring game on $P_n$ ($n \ge 4$) (respectively, $C_n$ ($n \ge 3$)). Clearly, if $r < 2$ then Alice will never win the game. So we assume $r \ge 2$. Now suppose in her first move Alice chooses a vertex $x$ on $P_n$ (respectively, $C_n$ ($n \ge 4$)) and assigns $x$ the point (colour) $a$ on $S(r)$. Then Bob just chooses a vertex $y$ on $P_n$ (respectively, $C_n$ ($n \ge 4$)) which is distance two apart from $x$ (since $n \ge 4$, such vertex exists) and assigns the colour $[a+r/2]_r$ to $y$. This makes the common neighbor of $x$ and $y$ having no legal colours, hence Bob wins. By Theorem 6, $\chi_{cg}(K_3) = 4$ and $\chi_{cg}^B(K_3) = 3$. Thus we have $\chi_{cg}(P_n) = 4$ for $n \ge 4$, $\chi_{cg}(C_n) = 4$ for $n \ge 3$. It is easy to verify that $\chi_{cg}(K_{1,m}) = \chi_{cg}^B(K_{1,m}) = 2$. It is not difficult to verify that $\chi_{cg}^B(P_4) = \chi_{cg}^B(C_4) = 2$ and $\chi_{cg}^B(P_n) = \chi_{cg}^B(C_n) = 4$ for $n \ge 5$.

**Corollary 1** *For any tree $T$, $\chi_{cg}(T) \le 6$.*

**Theorem 4** *For any positive real number $\varepsilon$, there is a tree $T$ with $\chi_{cg}(T) > 6 - \varepsilon$.*

**Proof.** Let $\varepsilon$ be any positive real number. Let $r = 6 - \varepsilon$. We shall construct a tree $T$ with $\chi_{cg}(T) > r$. Assume the circular $r$-colouring game is played on a tree $T$. A vertex $v$ is called *free* if $v$ and all its neighbours are uncoloured (so $v$ is free to be coloured by any colour).

Let $t = \lceil 6/\varepsilon \rceil + 1$ and let $m = 3t + 2$.

**Claim**: Assume there is a path $P = xv_1v_2v_3 \dots v_{2k}v_{2k+1}y$ of length $2k + 2$, where $k \geq m$, such that the following hold:

- The end vertices $x, y$ are coloured.
- For each even $i$, $v_i$ is free.
- For each odd $i$, $v_i$ has a free neighbour $u_i$ which does not lie on the path $P$.

If it is Bob's turn, then Bob has a strategy to win the game.

**Proof.** Assume $x$ is coloured by $\alpha$ and $y$ is coloured by $\beta$.

Let

$$
\begin{aligned}
\beta' &= [\beta + 2 - \varepsilon/2]_r \\
\ell &= [\alpha - \beta']_r \\
\delta &= \ell/t \\
\beta_j &= [\alpha + j(r + \delta)/3]_r, j = 0, 1, 2, \cdots, 3t.
\end{aligned}
$$

Observe that $\beta_0 = \alpha$, $\beta_{3t} = [\alpha + t(r + \delta)]_r = \beta'$ and $r + \delta < 6$, which implies that $[\beta_j - \beta_{j-1}]_r < 2$ and $[\beta_{j-1} - \beta_j]_r < 4$.

For convenience, we count Bob's next move as the 1st move. In the 1st move, Bob colours $v_2$ with colour $\beta_1$. If after the 2nd move (the 2nd move is Alice's move), vertex $v_1$ still has a free neighbour $u_1$, then Bob will colour $u_1$ with the colour $[\beta_1+(\alpha-\beta_1)/2]_r$ in the 3rd move. Then $v_1$ has three coloured neighbours $x, v_2, u_1$ and the colours of these three vertices partition $S(r)$ into three intervals, each has length less than 2. Therefore $v_1$ has no legal colour and Bob wins the game.

Thus we assume in the 2nd move, Alice colours a vertex in the close neighborhood $N[u_1]$ of $u_1$. Then in the 3rd move, Bob colours $v_4$ with colour $\beta_2$. Similarly in the 4th move, Alice needs to colour a vertex in $N[u_3]$. In general, for $j = 1, 2, \cdots, 3t$, in the $(2j-1)$th move, Bob colours $v_{2j}$ with colour $\beta_j$. In the $2j$th move, Alice needs to colour a vertex in $N[u_{2j-1}]$.

Starting from the $6t+1$st move, Bob will colour $v_{6t+2}, v_{6t+4}, \ldots, v_{2k}$ one by one in this order (each move colours one vertex). The colours Bob uses on these vertices depends on the parity of $k-3t$. If $k-3t$ is even, then Bob colours vertices $v_{6t+2}, v_{6t+6}, \ldots, v_{2k-2}$ by the colour $[\beta'+2-\varepsilon/3]_r$ and vertices $v_{6t+4}, v_{6t+8}, \ldots, v_{2k}$ by the colour $\beta'$. If $k-3t$ is odd, then Bob colours $v_{6t+2}$ by $[\beta'+2-\varepsilon/3]_r$, $v_{6t+4}$ by $[\beta'+2(2-\varepsilon/3)]_r$, and all vertices $v_{6t+6}, v_{6t+10}, \ldots, v_{2k}$ by the colour $\beta'$, and all vertices $v_{6t+8}, v_{6t+12}, \ldots, v_{2k-2}$ by the colour $[\beta'+2-\varepsilon/3]_r$. Similarly, whenever Bob colours $v_{2j}$, Alice needs to colour a vertex in $N[u_{2j-1}]$. In particular, after Bob colours $v_{2k}$ by $\beta'$, Alice needs to colour a vertex in $N[u_{2k-1}]$. Now Bob colours the free neighbour $u_{2k+1}$ of $v_{2k+1}$ by colour $[\beta'+(\beta-\beta')/2]_r$. Then $v_{2k+1}$ has no legal colour and Bob wins the game. ∎

Let $T_q$ be the tree with root $x_0$, in which each non-leaf vertex has degree 4 and each leaf is at distance $q$ from $x_0$, where $q \geq 16m+16$. For two vertices $x, y$ of $T_q$, denote by $P_{xy}$ the unique $x$-$y$-path, and denote by $\ell(P_{xy})$ the length of $P_{xy}$.

Assume in the 1st move, Alice colours a vertex $u$. In the 2nd move, Bob colours a vertex $v$ so that the distance between $v$ and $u$ is even and is at least $q$. To avoid the configuration described in the claim above, Alice needs to colour a vertex $w$, which is either on $P_{uv}$ or adjacent to a vertex on $P_{uv}$, so that $P_{uw}$ and $P_{wv}$ both are of odd length. Without loss of generality, assume $P_{uw}$ has length at least $q/2$. Let $z$ be the middle vertex on $P_{uw}$ such that $\ell(P_{uz}) = \ell(P_{zw}) - 1$. In the 4th move, Bob colours a vertex $z'$ such that $P_{zz'} \cap P_{uw} = \{z\}$ and $\ell(P_{zz'}) \geq q/4$. It is easy to verify that no matter which vertex is coloured by Alice in the next move, the configuration described in the claim cannot be avoided. Thus Bob has a winning strategy for the circular $r$-colouring game on $T_q$. ∎

## 4 $\chi_{cg}(G)$ and $\chi_a(G)$

A colouring of a graph $G$ is called *acyclic* if it is a proper colouring such that no cycle of $G$ is 2-coloured. In other words, the union of any two colour classes induces a forest. The minimum number of colours needed is the *acyclic chromatic number* of a graph $G$, denoted by $\chi_a(G)$. The proof of the following theorem uses a similar argument as in the proof of the result $\chi_g(G) \leq \chi_a(G)(\chi_a(G)+1)$ [3].

**Theorem 5** *For any graph $G$, $\chi_{cg}(G) \leq \chi_a(G)(2\chi_a(G)+2)$ and $\chi_{cg}^B(G) \leq \chi_a(G)(2\chi_a(G)+2)$.*

**Proof.** Let $\chi_a(G) = t$. Suppose an optimal acyclic colouring $f$ of $G$ uses the colour set $\{c_0, c_1, \ldots, c_{t-1}\}$. Let $V_0, V_1, \ldots, V_{t-1}$ be the corresponding colour classes. So for any $i \neq j$, the subgraph $F_{ij}$ induced by the set $V_i \cup V_j$ is a forest. Fix an orientation of each $F_{ij}$ so that every vertex has out-degree at most one. This clearly gives an orientation of $G$ in which each vertex has out-degree at most $t-1$.

Let $r = t(2t+2)$. We shall prove that $\chi^B_{cg}(G) \leq r$. We shall show that Alice has a winning strategy in the (Bob-first) circular $r$-colouring game played on $G$. The conclusion is easily seen to hold for the Alice-first game.

For $j = 0, 1, \cdots, t-1$, let $p_j = j(2t+2)$. For $j = 0, 1, \cdots, t-1$, the colours in the interval $[p_j, p_{j+1} - 2]_r$ are called *correct colours* for vertices in $V_j$.

A vertex $x \in V_j$ is *in danger* if it is uncoloured but there is an in-neighbor $y$ of $x$ coloured with a colour in $(p_j - 1, p_{j+1} - 1)_r$.

Alice's strategy is as follows: If Bob's last move makes a vertex $x$ in danger, then Alice colours $x$ with a correct colour. Otherwise, Alice arbitrarily chooses an uncoloured vertex $x$ and colours it with a correct colour.

To prove that this is a winning strategy for Alice, it suffices to prove that at any moment of the game, any uncoloured vertex $x$ has a correct colour which is legal for $x$. For a coloured vertex $w$, let $c(w)$ be the colour of $w$.

**Claim** If Bob has just finished a move, then there is at most one vertex in danger. If Alice has just finished a move, then there is no vertex in danger.

We prove this claim by induction on the number of moves. At the beginning of the game, this is certainly true. Assume Bob has just finished a move, by colouring a vertex $y \in V_i$. By induction hypothesis, before Bob's last move (at that time Alice has just finished a move), there was no vertex in danger. If Bob's last move made one vertex $x \in V_j$ in danger, then we must have $c(y) \in (p_j - 1, p_{j+1} - 1)_r$. Thus every vertex in danger now belong to the set $V_j$ and is an out-neighbour of $y$ in $F_{ij}$. But each vertex in $F_{ij}$ has at most one out-neighbour. Thus Bob's last move made at most one vertex in danger. After Alice's move, the vertex in danger is coloured, and hence no vertex is in danger. Observe that since Alice always colours a vertex with correct colour, her own move will never make any vertex in danger.

Assume $x \in V_i$ is an uncoloured vertex. We shall show that there is a colour $c \in [p_j, p_{j+1} - 2]_r$ which is a legal colour for $x$. Let $Z = \{w : w \sim x, c(w) \in (p_j - 1, p_{j+1} - 1)_r\}$. Because $x$ is coloured immediately after it becomes in danger, we conclude that $x$ has at most one in-neighbour coloured with colour from the interval $(p_j - 1, p_{j+1} - 1)_r$. On the other hand, $x$ has out-degree at most $t - 1$. So $x$ has at most $t - 1$ out-neighbours that are coloured with colours from the interval $(p_j - 1, p_{j+1} - 1)_r$. Therefore $|Z| \leq t$. Since the interval $[p_j, p_{j+1} - 2]_r$ has length $2t$, it follows that $[p_j, p_{j+1} - 2]_r - \cup_{w \in Z}(c(w) - 1, c(w) + 1)_r \neq \emptyset$. It is obvious that any colour from the set $[p_j, p_{j+1} - 2]_r - \cup_{w \in Z}(c(w) - 1, c(w) + 1)_r$ is a legal colour for $x$. ∎

# 5 The complete graphs

This section determines the circular game chromatic number of complete graphs.

**Theorem 6** *For a positive integer $n$, let*

$$\varphi(n) = \begin{cases} 4k+1, & \text{if } n = 3k+1, \\ 4k+2, & \text{if } n = 3k+2, \\ 4k+4, & \text{if } n = 3k+3; \end{cases} \quad \text{and} \quad \psi(n) = \begin{cases} 4k, & \text{if } n = 3k+1, \\ 4k+2, & \text{if } n = 3k+2, \\ 4k+3, & \text{if } n = 3k+3. \end{cases}$$

*Then $\chi_{cg}(K_n) = \varphi(n)$ and $\chi_{cg}^B(K_n) = \psi(n)$.*

**Proof.** First we show that $\chi_{cg}(K_n) \leq \varphi(n)$. We shall prove that if $r \geq \varphi(n)$, then Alice has a winning strategy for playing the circular $r$-colouring game on $K_n$.

At any time of the game, there is no difference as to which uncoloured vertex of $K_n$ Alice or Bob wants to colour. What matters is which colour is used to colour that vertex. In other words, the circular $r$-colouring game on $K_n$ is equivalent to the following game: Alice and Bob take turns to play a game, and at each turn, the player chooses a point from $S(r)$ (as the colour to be assigned to an uncoloured vertex) such that any two chosen points must be at distance at least 1. The game ends when no more point in $S(r)$ can be chosen. If $n$ points are chosen at the end of the game, then Alice wins the circular $r$-colouring game on $K_n$. If less than $n$ points are chosen at the end of the game, Bob wins the game.

Initially Alice arbitrarily chooses a point in $S(r)$. Suppose $i$ points, $a_1, a_2, \cdots, a_i$, have already been chosen, and it is Alice's turn. Assume the points $a_1, a_2, \cdots, a_i$ occurs in $S(r)$ in this cyclic order. Then these points divide $S(r)$ into $i$ intervals $(a_1, a_2)_r, (a_2, a_3)_r, \cdots, (a_i, a_1)_r$. If there is an interval, say $(a_j, a_{j+1})_r$, of length at least 3, then Alice choose the point $a_j + 2$.

Suppose in Alice's $t$-th move, she can find an interval $(a_j, a_{j+1})_r$ of length at least 3, but in her $(t+1)$-th move, each of the intervals $(a_j, a_{j+1})_r$ has length less than 3.

Then after Alice finished her $t$-th move (so Bob made $(t-1)$ moves by that time), the circle $S(r)$ is divided into $2t-1$ intervals $(a_1, a_2)_r, (a_2, a_3)_r, \cdots, (a_{2t-1}, a_1)_r$. We may assume that Bob's next move choose a point $a_{2t}$ from the interval $(a_{2t-1}, a_1)_r$. By the definition of $t$, after Bob's next move, each of the intervals $(a_1, a_2)_r, (a_2, a_3)_r, \cdots, (a_{2t-1}, a_{2t})_r, (a_{2t}, a_1)_r$ has length less than 3. This implies that $(a_{2t-1}, a_1)_r$ has length at least 2 but less than 6.

Once there is no interval $(a_j, a_{j+1})_r$ of length at least 3, the later moves of the game are trivial for both Alice and Bob. Each of their remaining moves chooses an interval $(a_j, a_{j+1})_r$ which has length at least 2 and chooses a point from that interval. No matter which interval $(a_j, a_{j+1})_r$ they choose, the outcome of the game is the same: *If $p$ points are chosen to reach the configuration that each interval $(a_j, a_{j+1})_r$ has length less than 3, and $q$ of the intervals $(a_j, a_{j+1})_r$ have length at least 2, then Alice wins the game if and only if $p + q \geq n$.*

Depending on the length of the interval $(a_{2t-1}, a_1)_r$, we divide the remaining discussion into a few cases.

First we consider the case that the interval $(a_{2t-1}, a_1)_r$ has length less than 4. Observe that except for the first move, each time Alice colours a vertex, she creates an interval $(a_j, a_{j+1})_r$ of length 2, i.e., with $a_{j+1} = a_j + 2$. So altogether, Alice created $t-1$ intervals $(a_j, a_{j+1})_r$ of length 2. We call these $t-1$ intervals of length 2 the *A-intervals*. It is possible that Bob may colour a vertex in a later move with colour $a_j + 1$. If Bob does so, then an A-interval is broken into two intervals of length 1. Assume Bob broked $s$ A-intervals. Then $2s$ of the intervals $(a_1, a_2)_r, (a_2, a_3)_r, \cdots, (a_{2t-2}, a_{2t-1})_r$ have length 1, and the remaining $t-1-s$ A-intervals have length 2. The other intervals (not including the interval $(a_{2t-1}, a_1)_r$, there are $t-1-s$ other intervals) have lengths at least 1 but less than 3. Assume $q$ of these $t-s-1$ other intervals have length at least 2, and $t-s-q-1$ of them have length less than 2. Then after Alice coloured $t$ vertices (so Alice and Bob together have chosen $2t-1$ points), in the remaining moves, $t-s-1+q+1 = t+q-s$ more points can be chosen. If $2t-1+t+q-s = 3t+q-s-1 \geq n$, then at least $n$ points can be chosen, and hence all vertices of $K_n$ can be coloured and Alice wins the game. Now we prove that indeed we have $3t+q-s-1 \geq n$.

Since the sum of the lengths of all the intervals $(a_1, a_2)_r, (a_2, a_3)_r, \cdots, (a_{2t-1}, a_1)_r$ is equal to $r \geq \varphi(n)$, we conclude that

$$2s + 2(t-1-s) + 3q + 2(t-s-q-1) + 4 = 4t + q - 2s > r \geq \varphi(n).$$

If $n = 3k+1$, then

$$\begin{aligned} & 4t + q - 2s > 4k + 1 \Rightarrow 4t + q - 2s \geq 4k + 2 \\ \Rightarrow \quad & 3t + 3q/4 - 3s/2 \geq 3k + 3/2 \Rightarrow 3t + q - s - 1 \geq 3k + 1. \end{aligned}$$

If $n = 3k + 2$, then

$$4t + q - 2s > 4k + 2 \Rightarrow 4t + q - 2s \geq 4k + 3$$
$$\Rightarrow \quad 3t + 3q/4 - 3s/2 \geq 3k + 9/4 \Rightarrow 3t + q - s - 1 \geq 3k + 2.$$

If $n = 3k + 3$, then

$$4t + q - 2s > 4k + 4 \Rightarrow 4t + q - 2s \geq 4k + 5$$
$$\Rightarrow \quad 3t + 3q/4 - 3s/2 \geq 3k + 15/4 \Rightarrow 3t + q - s - 1 \geq 3k + 3.$$

Next we consider the case that the interval $(a_{2t-1}, a_1)_r$ has length at least 4 but less than 5. In this case, two more points can be chosen from the interval $(a_{2t-1}, a_1)_r$. Then after Alice has chosen $t$ points, in the remaining moves, $t - s - 1 + q + 2 = t + q - s + 1$ more points can be chosen. To prove that Alice wins the game, it suffices to show that $2t - 1 + t + q - s + 1 = 3t + q - s \geq n$.

As the sum of the lengths of all the intervals $(a_1, a_2)_r, (a_2, a_3)_r, \cdots, (a_{2t-1}, a_1)_r$ is equal to $r \geq \varphi(n)$, we conclude that

$$2s + 2(t - 1 - s) + 3q + 2(t - s - q - 1) + 5 = 4t + q - 2s + 1 > r \geq \varphi(n).$$

By considering the three cases $n = 3k+1, 3k+2, 3k+3$ separately, a similar calculation as above shows that the inequality does hold.

Finally we consider the case that the interval $(a_{2t-1}, a_1)_r$ has length at least 5. In this case, three more points can be chosen from the interval $(a_{2t-1}, a_1)_r$. To prove that Alice wins the game, it suffices to show that $2t - 1 + t + q - s + 2 = 3t + q - s + 1 \geq n$. Again a similar calculation as above shows that this inequality does hold.

In the Bob-first circular $r$-colouring game on $K_n$, Alice uses the same strategy. The only difference is that on her first move, Alice chooses the point $a_1 + 2$ (where

$a_1$ is the point chosen by Bob on his first move). Then the same argument as above shows that if $r \geq \psi(n)$, then Alice wins the game. Hence $\chi^B_{cg}(K_n) \leq \psi(n)$.

Next we prove that $\chi_{cg}(K_n) \geq \varphi(n)$. Assume $r = \varphi(n) - \epsilon < \varphi(n)$. We shall show that Bob has a winning strategy for the circular $r$-colouring game on $K_n$.

Again we consider the game that Alice and Bob take their turns to choose points from $S(r)$ such that any two chosen points have distance at least 1. We shall show that Bob has a strategy to ensure that at most $n-1$ points can be chosen.

Suppose it is Bob's turn and $i$ points, $a_1, a_2, \cdots, a_i$, have already been chosen. Assume the points $a_1, a_2, \cdots, a_i$ occurs in $S(r)$ in this cyclic order. Then these points divide $S(r)$ into $i$ intervals $(a_1, a_2)_r, (a_2, a_3)_r, \cdots, (a_i, a_1)_r$. If there is an interval, say $(a_j, a_{j+1})_r$, of length at least 3, then Bob chooses the point $a_j + 2 - \epsilon/n$.

Suppose in Bob's $t$-th move, he can still find an interval $(a_j, a_{j+1})_r$ of length at least 3, but in his $(t+1)$-th move, each of the intervals $(a_j, a_{j+1})_r$ has length less than 3.

Then after Bob's finished his $t$-th move (so Alice has also made $t$ move by that time), the circle $S(r)$ is divided into $2t$ intervals $(a_1, a_2)_r, (a_2, a_3)_r, \cdots, (a_{2t}, a_1)_r$. We may assume that Alice's next move chooses a point $a_{2t+1}$ from the interval $(a_{2t}, a_1)_r$. By the definition of $t$, after Alice's next move, each of the intervals $(a_1, a_2)_r, (a_2, a_3)_r, \cdots, (a_{2t}, a_{2t})_r, (a_{2t+1}, a_1)_r$ has length less than 3. It follows that $(a_{2t}, a_1)_r$ has length at least 2 but less than 6.

Similarly as above, if $p$ points are chosen to form a configuration in which each interval $(a_j, a_{j+1})_r$ has length less than 3, and $q$ of the intervals $(a_j, a_{j+1})_r$ have length at least 2, then Bob wins the game if and only if $p + q < n$.

Depending on the length of the interval $(a_{2t}, a_1)_r$, we divide the remaining discussion into a few cases.

First we consider the case that the interval $(a_{2t}, a_1)_r$ has length less than 3. Assume among the intervals $(a_1, a_2)_r, (a_2, a_3)_r, \cdots, (a_{2t}, a_1)_r$, there are $q$ intervals of length at least 2. For the reason described in the proof of the upper bound, if $2t + q < n$, then Bob wins the game. Now we prove that indeed we have $2t + q < n$.

Since the sum of the lengths of all the intervals $(a_1, a_2)_r, (a_2, a_3)_r, \cdots, (a_{2t}, a_1)_r$ is equal to $r = \varphi(n) - \epsilon$, we conclude that

$$t(2 - \epsilon/n) + 2q + (t - q) = 3t + q - t\epsilon/n < r = \varphi(n) - \epsilon.$$

Hence $3t + q < \varphi(n)$. If $t \leq k$, then since $q \leq t$, we have $2t + q \leq 3t \leq 3k < n$. If $t \geq k + 1$, we have $2t + q < \varphi(n) - t \leq \varphi(n) - k - 1 \leq n$. So we are done.

Assume the interval $(a_{2t-1}, a_1)_r$ has length at least 3 but less than 4. In this case, two more points can be chosen from the interval $(a_{2t-1}, a_1)_r$. Let $q$ be the number of intervals among $(a_1, a_2)_r, (a_2, a_3)_r, \cdots, (a_{2t-1}, a_{2t})_r$, which has length at least 2. To prove that Bob wins the game, it suffices to show that $2t + q + 2 < n$.

Since the sum of the lengths of all the intervals $(a_1, a_2)_r, (a_2, a_3)_r, \cdots, (a_{2t}, a_1)_r$ is equal to $r = \varphi(n) - \epsilon$, we conclude that

$$t(2 - \epsilon/n) + 2q + (t - 1 - q) + 3 = 3t + q + 2 - t\epsilon/n < r = \varphi(n) - \epsilon.$$

Hence $3t + q + 2 < \varphi(n)$.

If $t \leq k - 1$, then since $q \leq t - 1$, we have $2t + q + 2 \leq 3t + 1 \leq 3k - 2 < n$. If $t \geq k + 1$, we have $2t + q + 2 < \varphi(n) - t \leq \varphi(n) - k - 1 \leq n$. Assume $t = k$. If $n \neq 3k + 1$, then $2t + q + 2 \leq 3t + 1 \leq 3k + 1 < n$. If $n = 3k + 1$, then $2t + q + 2 < 4k + 1 - t \leq 3k + 1 = n$. So in any case $2t + q + 2 < n$ and we are done.

Assume the interval $(a_{2t-1}, a_1)_r$ has length at least 4. In this case, three more points can be chosen from the interval $(a_{2t-1}, a_1)_r$. Let $q$ be the number of intervals among $(a_1, a_2)_r, (a_2, a_3)_r, \cdots, (a_{2t-1}, a_{2t})_r$, which has length at least 2. To prove that Bob wins the game, it suffices to show that $2t + q + 3 < n$.

Since the sum of the lengths of all the intervals $(a_1, a_2)_r, (a_2, a_3)_r, \cdots, (a_{2t}, a_1)_r$ is equal to $r = \varphi(n) - \epsilon$, we conclude that

$$t(2 - \epsilon/n) + 2q + (t - 1 - q) + 4 = 3t + q + 3 - t\epsilon/n < r = \varphi(n) - \epsilon.$$

Hence $3t + q + 3 < \varphi(n)$. The same calculation as above shows that this implies $2t + q + 3 < n$.

In the Bob-first circular $r$-colouring game on $K_n$, Bob uses the same strategy, except that the first point chosen by Bob is arbitrary. The same argument as above shows that if $r < \psi(n)$, then Bob has a winning strategy. ∎

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