Activation strategy on asymmetric marking games

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Abstract

This paper extends the widely used activation strategy of the marking game on graphs to asymmetric marking games. The extended activation strategy is then applied to asymmetric marking games on chordal graphs, (s, t)-pseudo partial k-trees and interval graphs. Our results improve earlier upper bounds on $(a, 1)$-$\mathrm{gcol}\,(\mathbb{I}_k)$ and $(a, 1)$-$\mathrm{gcol}\,(\mathbb{C}_k)$, where $\mathbb{I}_k$ and $\mathbb{C}_k$ denote the classes of interval and chordal graphs with maximum clique size $k + 1$ respectively. Moreover, the upper bound of $(a, 1)$-$\mathrm{gcol}\,(\mathbb{I}_k)$ is tight when k is a multiple of a.

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1 Introduction

For a graph $G = (V, E)$, the game coloring number of G is defined through a *marking game*. The marking game is played by two players, Alice and Bob, with Alice playing first. At the start of the game all vertices are unmarked. A move by either player consists of marking an unmarked vertex. The game ends when all the vertices have been marked. For each vertex x of G, let $b(x)$ be the number of neighbours of x that are marked before x is marked. The score of the game is

$$s = 1 + \max_{v \in V(G)} b(v).$$

Alice's goal is to minimize the score, while Bob's goal is to maximize the score. The *game coloring number* of G, denoted by $\operatorname{gcol}(G)$, is the least s such that Alice has a strategy that results in a score of at most s. If $\mathbb{C}$ is a class of graphs then $\operatorname{gcol}(\mathbb{C}) = \max_{G \in \mathbb{C}} \operatorname{gcol}(G)$.

The game coloring number was first explicitly introduced in [14] as a tool to study the game chromatic number, which is defined through a coloring game. The *coloring game* is played like the marking game, except that instead of marking vertices the players color them from a set X of colors so that no two adjacent vertices receive the same color. Alice wins if eventually the whole graph is properly colored. Bob wins if there comes a time when all the colors have been used on the neighborhood of some uncolored vertex u. The *game chromatic number* of G, denoted $\chi_g(G)$, is the least t such that Alice has a winning strategy in the coloring game on G using a set of t colors. It is easy to see that for any graph G, $\chi_g(G) \leq \operatorname{gcol}(G)$. For the classes of interval, chordal, planar, and outerplanar graphs, the best known upper bounds for their game chromatic number are obtained by finding upper bounds for their game coloring number. Game coloring number of a graph and its generalization to oriented graphs are also of independent interest, and have been studied extensively.

In this paper, we are interested in an *asymmetric* variant of the marking game called the (a, b)-marking game. This game is played and scored like the marking game, except that on each turn Alice marks a vertices and Bob marks b vertices. (If the last vertex is marked during a player's turn, then this completes the turn.) The (a, b)-game coloring number, denoted by (a, b)-$\operatorname{gcol}(G)$, of G is the least s such that Alice has a strategy that results in a score of at most s. If $\mathbb{C}$ is a class of graphs then (a, b)-$\operatorname{gcol}(\mathbb{C}) = \max_{G \in \mathbb{C}} (a, b)$-$\operatorname{gcol}(G)$. This game was introduced by Kierstead in [4] where the (a, b)-game coloring number of the class of trees is determined for all positive integers a and b. In particular it was shown that if $a < b$ then the (a, b)-game coloring number is unbounded even for the class of trees.

For a graph G, let $O(G)$ be the set of all orientations of G. For an orientation $\vec{G}$ of G and for a vertex x of $\vec{G}$, let $N^+_{\vec{G}}(x)$ denote the set of out-neighbors of x, i.e., $N^+_{\vec{G}}(x) = \{y : x \to y\}$, let $d^+_{\vec{G}}(x)$ be the out-degree of x, i.e., $d^+_{\vec{G}}(x) = |N^+_{\vec{G}}(x)|$. Let $\Delta^+\left(\vec{G}\right) = \max_{v \in V} d^+_{\vec{G}}(v)$ and let $\Delta^*(G) = \min_{\vec{G} \in O(G)} \Delta^+\left(\vec{G}\right)$. In [9] Kierstead and

Yang showed the following theorem by introducing the so-called *Harmonious Strategy* for Alice:

Theorem 1.1 (Kierstead and Yang, [9]) *Let a be an integer and G be a graph with $\Delta^*(G) = k \leq a$. Then $(a, 1)$-$\mathrm{gcol}(G) \leq 2k + 2$.*

It is also shown in [9] that (a, b)-$\mathrm{gcol}(G)$ is bounded for the class of graphs G with $\Delta^*(G) \leq k$ if and only if $k \leq \frac{a}{b}$. The classes of interval, chordal, planar and outerplanar graphs are studied in [9] with respect to the (a, b)-marking game. The asymmetric marking games of the interval graphs, chordal graphs and line graphs are further explored by Kierstead and Yang in [11], [13]. Wu and Zhu in [10] showed for any $k \geq 2$, there are chordal graphs G with $\mathrm{gcol}(G) = 3k + 2$. The following theorem summarizes the best known results for the asymmetric coloring number on the classes of interval and chordal graphs.

Theorem 1.2 *Let $\mathbb{I}_k$ and $\mathbb{C}_k$ be, respectively, the class of interval graphs and the class of chordal graphs with maximum clique size $k + 1$.*

1. *If $a > k$, then $2k + 1 \leq (a, 1)$-$\mathrm{gcol}(\mathbb{I}_k) \leq (a, 1)$-$\mathrm{gcol}(\mathbb{C}_k) \leq 2k + 2$.*
2. $(k, 1)$-$\mathrm{gcol}(\mathbb{I}_k) = (k, 1)$-$\mathrm{gcol}(\mathbb{C}_k) = 2k + 2$.
3. *If $1 \leq a < k$, then $2k + \lfloor \frac{k}{a} \rfloor + 1 \leq (a, 1)$-$\mathrm{gcol}(\mathbb{I}_k) \leq (a, 1)$-$\mathrm{gcol}(\mathbb{C}_k) \leq 2k + \lfloor \frac{k}{a} + \frac{a+1}{2} \rfloor + 1$.*
4. $\mathrm{gcol}(\mathbb{I}_k) = 3k + 1$.
5. $\mathrm{gcol}(\mathbb{C}_k) = 3k + 2$ *for $k \geq 2$.*

In this paper, we extend the activation strategy to asymmetric marking games. Using this strategy, we prove that if G has an orientation $\vec{G}$ with $\Delta^+(\vec{G}) = k > a$ and with rank r (see definition in Section 2), then $(a, 1)$-$\mathrm{gcol}(G) \leq k + \lfloor (1 + \frac{1}{a})r \rfloor + 2$. This implies that if G is a chordal graph with $\omega(G) = k + 1$ and $a < k$, then $(a, 1)$-$\mathrm{gcol}(G) \leq 2k + \lfloor \frac{k}{a} \rfloor + 2$, improving the bound listed in Theorem 1.2 above. Then we apply the strategy to asymmetric marking games on (s, t)-pseudo partial k-trees. It is proved that if G is an (s, t)-pseudo partial k-tree and $1 \leq a < k$, then $(a, 1)$-$\mathrm{gcol}(G) \leq 2k + s + t + \lfloor \frac{k+s}{a} \rfloor + 2$. In Section 4, we prove that if G is an interval graph with $\omega(G) = k + 1$ and $1 \leq a < k$, then $(a, 1)$-$\mathrm{gcol}(G) \leq 2k + \lceil \frac{k}{a} \rceil + 1$. On the other hand, by Theorem 1.2, there is an interval graph G with $\omega(G) = k + 1$ for which $(a, 1)$-$\mathrm{gcol}(G) \geq 2k + \lfloor \frac{k}{a} \rfloor + 1$. Thus when $k = aq$ is a multiple of a, we have $(a, 1)$-$\mathrm{gcol}(\mathbb{I}_k) = 2k + q + 1$.

2 Activation strategy on asymmetric marking games

Although there are relatively rich results concerning the game chromatic number and game coloring number of graphs, there are few strategies for either Alice or Bob to play the coloring game or marking game. It follows from results in [3] that there is a single strategy, *the Activation Strategy*, such that if Alice uses this strategy to play the marking game then she achieves the best known upper bounds on the game coloring numbers of the classes of forests, interval graphs, chordal graphs, partial k-trees and outerplanar graphs. For the class of planar graphs the best known upper bound on their game coloring number is obtained by using a refinement of the activation strategy [16].

In the study of (a, b)-marking games of a graph G, if the game is *very asymmetric*, i.e. if $\Delta^*(G) \leq \frac{a}{b}$, then Alice can apply the Harmonious Strategy to achieve very good upper bound in the game. When $\Delta^*(G) > \frac{a}{b}$, Kierstead and Yang [9] introduced *the Limited Harmonious Strategy* for Alice to use. However, the upper bounds that Alice achieves by using the Limited Harmonious Strategy are often not as tight.

In this section we study the $(a, 1)$-game coloring number of graphs G with $\Delta^*(G) = k$ and $1 \leq a < k$. For $a = 1$, i.e., for the original coloring game, a widely used and well studied strategy for Alice is the *Activation Strategy.* We will extend the Activation Strategy of Alice to the general $(a, 1)$-marking games when $k > a$. This is a natural extension of both the Activation and Harmonious Strategies.

An example in [9] shows that when $k > a$, the $(a, 1)$-game coloring number of G is not bounded solely in terms of $k = \Delta^*(G)$. To bound the $(a, 1)$-game coloring number of G, we need another parameter defined through orientations and orderings of the vertices of G. Let L be a linear order of on $V(G)$, and let $\vec{G}$ be the corresponding orientation of G. For each vertex v of G, let L_v be a linear order on $N^+_{\vec{G}}(v)$. Let $\Sigma = \{L_v : v \in V(G)\}$. We say that z prefers v to u if $v <_{L_z} u$. We call v a *loose out-neighbor* of u if $v \in N^+_{\vec{G}}(u)$ or there exists a vertex z such that $u, v \in N^+_{\vec{G}}(z)$ and z prefers v to u, and denote by $R_{\vec{G}}(\Sigma, u)$ the set of loose out-neighbors of u. Let

$$r_{\vec{G}}(\Sigma) = \max_{u \in V(G)} \left| R_{\vec{G}}(\Sigma, u) \right|.$$

The *rank* of $\vec{G}$ is defined as

$$r_{\vec{G}} = \min_{\Sigma} r_{\vec{G}}(\Sigma).$$

Alice's Activation Strategy on asymmetric marking games is as follows. To unify the description we consider an equivalent version of the marking game in which Bob plays first by marking a new vertex x_0 with no neighbors in $V(G)$. We use U denote the set of unmarked vertices. At each move, one vertex is marked and is removed from U.
Initialization: $U := V(G)$; for $v \in V(G)$ do $t_v := a$ end do;

Suppose that Bob has just marked a vertex x.
Alice's play: for i from 1 to a while $U \neq \emptyset$ do

1. if $N^+_{\vec{G}}(x) \cap U \neq \emptyset$ and $t_x > 0$ then $y := L_x\text{-}\min N^+_{\vec{G}}(x) \cap U$; $t_x := t_x - 1$; else $y := L\text{-}\min U$ end if;

2. while $N^+_{\vec{G}}(y) \cap U \neq \emptyset$ and $t_y > 0$ do

   $z := L_y\text{-}\min N^+_{\vec{G}}(y) \cap U$; $t_y := t_y - 1$; $y := z$ end do;

3. $U := U - \{y\}$ end do;

Notation 2.1 *For an unmarked vertex u, we say u* receives a contribution *from v and v* made a contribution *to u, if: in Line 1, we have $u = y := L_x\text{-}\min N^+_{\vec{G}}(x) \cap U$ and $v = x$; or in Line 2, we have $u = z := L_y\text{-}\min N^+_{\vec{G}}(y) \cap U$ and $v = y$.*

Informally the above strategy is rephrased as follows: Suppose that Bob has just marked a vertex x. Then Alice starts by *activating* (through making contributions to) unmarked out-neighbors y of x according to its preference. The parameter t_y records the total number of contributions made by y. Once a vertex y receives a contribution, then y passes a contribution to its unmarked out-neighbors according to its preference, provided that y has made less than a contributions in total. In case y has already made a contributions or y has no unmarked out-neighbors, then y will be marked when it receives another contribution. Alice repeats the above procedure a times, each time marks one vertex. The difference between this strategy and the original activation strategy is that in this strategy, an unmarked vertex can be activated a times (i.e., receive a contributions) before it is marked.

Theorem 2.2 *Let $\vec{G}$ be an oriented graph with $\Delta^+\left(\vec{G}\right) = k > a$ and let $r_{\vec{G}}(\Sigma) = r$. If Alice uses the Activation Strategy then the score of the $(a,1)$-marking game is at most $k + \lfloor (1 + \frac{1}{a})r \rfloor + 2$.*

Proof. First we show that as long as there exist unmarked vertices, the Activation Strategy on the asymmetric marking game terminates with Alice marking a vertex. To see this, let $t = |U| + \sum_{v \in V(G)} t_v$. Note that each term in the sum is always nonnegative. Since U is nonempty, $t \geq 1$. At each iteration in Step 2 of the algorithm t decreases by 1, so eventually Step 2 must end at a vertex y. Then Alice marks y in Step 3. If she has not yet completed her turn then she returns to Step 1 and repeats the process.

Consider any time when a vertex v has just been marked by Alice. If Alice has not yet completed her turn, let x be the last vertex marked by Bob. Otherwise x is undefined.

Assume u is an unmarked vertex. We shall show that u has at most $k + \lfloor (1 + \frac{1}{a})r \rfloor$ marked neighbors other than x. Note here that the following fact is considered in the above bound: if x is defined, then it may be adjacent to u; otherwise it is Bob's turn and he may be about to mark a vertex adjacent to u. In the former case, we treat x separately because it may have not yet contributed to a of its out-neighbors. We shall need the following observations.

1. An unmarked vertex has received the same number of contributions as it has made.

2. A marked in-neighbor y of u other than x has made a contributions to its out-neighbors.

3. Any vertex has received at most $a+1$ contributions.

Observation (2) holds because y has an unmarked out-neighbor u, and hence y stops making contribution only if t_y is decreased to 0. Observation (3) holds because if a vertex z receives the $(a+1)$th contribution, z will be marked, and marked vertices receive no contributions.

Let $S = N(u) \cap M - \{x\}$ and let $\{Q, C\}$ be the partition of S, where Q is the set of out-neighbors of u in S, C is the set of in-neighbors of u in S. Then $|Q| \leq k$. For any vertex $y \in C$, since u is an out-neighbor of y, and u is unmarked by the time y is marked, by observation (2) y has made a contributions to its out-neighbors. According to the activation strategy described above, when a contribution is made by y to its out-neighbor, the contribution goes to the vertex $z := L_y\text{-min}\, N^+_{\vec{G}}(y) \cap U$. Since $u \in N^+_{\vec{G}}(y) \cap U$, we concluded that $z \in \bar{R} = R_{\vec{G}}(\Sigma, u) \cup \{u\}$, i.e., either $z = u$ or z is a loose out-neighbor of u. Note that if the contribution goes to u, since u is unmarked, this contribution is passed on to some vertex in $N^+_{\vec{G}}(u) \subseteq R_{\vec{G}}(\Sigma, u)$ immediately. By observation (3) any vertex in $R_{\vec{G}}(\Sigma, u)$ can receive at most $a+1$ contributions. So we have

$$a\,|C| \leq (a+1)\,|R_{\vec{G}}(\Sigma, u)| \leq (a+1)\,r.$$

It follows that

$$|S| = |Q| + |C| \leq k + |C| \leq k + (1 + \frac{1}{a})r.$$

This finishes the proof. ■

A graph $G = (V, E)$ is a *chordal graph* if every cycle of G of length ≥ 4 has a chord. An equivalent definition of a chordal graph $G = (V, E)$ is that there is a linear order, say $v_1, v_2, \cdots, v_n$, on the vertex set V, such that for each i, the set $\{v_j : j < i, v_j v_i \in E\}$ induces a complete subgraph of G. We call such an order L a simplicial ordering of G. By orienting the edges of G in such a way that an edge $v_i v_j$ is directed from v_i to v_j if and only if $i > j$, we obtain an oriented graph $G_L = (V, \vec{E})$ which is acyclic and for each vertex v_i, its out-neighbors induce a transitive tournament. The converse is also true, i.e., a graph $G = (V, E)$ is a chordal graph if and only if G has an orientation $\vec{G} = (V, \vec{E})$ which is acyclic and the out-neighbors of each vertex induce a transitive tournament.

Corollary 2.3 *If G is a chordal graph with $\omega(G) = k+1$ and $a < k$, then* $(a,1)$-$\mathrm{gcol}(G) \leq 2k + \lfloor \frac{k}{a} \rfloor + 2$.

Proof. Let L be a simplicial ordering of G and L_v be L restricted to $N^+_{G_L}(v)$ for all $v \in V(G)$. Then $\Delta^+(G_L) = k$ and $r_{G_L} = k$. So we are done by Theorem 2.2. ■

3 Asymmetric marking games of (s,t)-pseudo partial k-trees

The class of (s,t)-pseudo chordal graphs and (s,t)-pseudo partial k-trees is introduced in [15] as a generalization of partial k-trees. For example, it is proved in [15] that planar graphs are $(3,8)$-pseudo partial 2-trees, although planar graphs can have arbitrarily large treewidth.

Definition 3.1 *Suppose s, t are integers such that $0 \leq s \leq t$. A connected graph $G = (V,E)$ is called an (s,t)-*pseudo chordal graph *if there are two oriented graphs $\vec{G}_1 = (V, \vec{E}_1)$ and $\vec{G}_2 = (V, \vec{E}_2)$ such that the following is true:*

1. *$E_1 \cap E_2 = \emptyset$ and $E = E_1 \cup E_2$. Where E_i is the set of edges obtained from $\vec{E}_i$ by omitting the orientations.*
2. *$\vec{G}_1$ is acyclic.*
3. *$\Delta^+\left(\vec{G}_2\right) \leq s$, and $\Delta\left(\vec{G}_2\right) \leq t$.*
4. *Let $N^+(x) = N^+_{\vec{G}_1}(x)$ be the set of out-neighbors of x in $\vec{G}_1$. Let $\vec{G}^* = (V, \vec{E}_1 \cup \vec{E}_2)$. Then $N^+(x)$ induces a transitive tournament in $\vec{G}^*$.*

Definition 3.2 *A graph is called an (s,t)-*pseudo partial k-tree *if it is a subgraph of an (s,t)-pseudo chordal graph in which the directed graph $\vec{G}_1$ in the definition has maximum outdegree at most k.*

Note that any induced subgraph of an (s,t)-pseudo chordal graph is still an (s,t)-pseudo chordal graph. Therefore an (s,t)-pseudo partial k-tree can be equivalently defined as a spanning subgraph of an (s,t)-pseudo chordal graph in which the directed graph $\vec{G}_1$ in the definition has maximum outdegree at most k. It follows from the definition that if $s = 0$ (hence $t = 0$), then a $(0,0)$-pseudo chordal graph is simply a chordal graph, and a $(0,0)$-pseudo partial k-tree is simply a partial k-tree. In this section, we consider the $(a,1)$-marking game of the classes of (s,t)-pseudo chordal graphs and (s,t)-pseudo partial k-trees. First we will prove the following upper bound for the $(a,1)$-game coloring number of the class of (s,t)-pseudo chordal graphs.

Theorem 3.3 *Suppose G is an (s,t)-pseudo chordal graph, and k is the maximum outdegree of all the vertices of $\vec{G}_1$ in the definition above. If $1 \leq a < k$, then $(a,1)$-*$\mathrm{gcol}(G) \leq 2k + s + t + \lfloor \frac{1}{a}(k+s) \rfloor + 2$.

Proof. Suppose $G = (V,E)$ is an (s,t)-pseudo chordal graph, $\vec{G}_1 = (V, \vec{E}_1)$, $\vec{G}_2 = (V, \vec{E}_2)$ and $\vec{G}^* = (V, \vec{E}_1 \cup \vec{E}_2)$ are oriented graphs as in Definition 3.1. When playing the game, Alice will only take the oriented graph $\vec{G}_1$ into consideration. Let L be a linear order on $V(G) = V(\vec{G}_1)$. For each vertex $v \in V(\vec{G}_1)$, let L_v be the linear order on $N^+_{\vec{G}_1}(v)$, such that for $x, y \in N^+_{\vec{G}_1}(v)$, $x <_{L_v} y$ if and only if $yx \in E(\vec{G}^*) = \vec{E}_1 \cup \vec{E}_2$. Note that the linear order L_v is well defined, since $N^+_{\vec{G}_1}(v)$ induces a transitive tournament in $\vec{G}^*$ (this transitive tournament is a subdigraph of $\vec{G}^* = (V, \vec{E}_1 \cup \vec{E}_2)$). Then let $\Sigma = \left\{L_v : v \in V(\vec{G}_1)\right\}$.

Next we will show $r_{\vec{G}_1}(\Sigma) = \max_{u \in V(\vec{G}_1)} \left|R_{\vec{G}_1}(\Sigma, u)\right| \le k + s$, where $R_{\vec{G}_1}(\Sigma, u) = \left\{v \in V\left(\vec{G}_1\right) : v \leftarrow u \ \vee \ \exists z \in V\left(\vec{G}_1\right) \ (u \leftarrow z \rightarrow v \ \wedge \ v <_{L_z} u)\right\}$ (as defined at the beginning of Alice's Activation Strategy), this will give that $r_{\vec{G}_1} = \min_\Sigma r_{\vec{G}_1}(\Sigma) \le k + s$.

Note that if $v \in R_{\vec{G}_1}(\Sigma, u)$, then either $v \in N^+_{\vec{G}_1}(u)$ or there exists a vertex z such that $u, v \in N^+_{\vec{G}_1}(z)$ and z prefers v to u in $L_z \in \Sigma$. According to the definition of L_z, we know that if z prefers v to u in L_z then $uv \in E(\vec{G}^*) = \vec{E}_1 \cup \vec{E}_2$. Therefore in both cases of $v \in R_{\vec{G}_1}(\Sigma, u)$, we all have $v \in N^+_{\vec{G}^*}(u)$. Then combining $|N^+_{\vec{G}_1}(u)| \le k$ and $\Delta^+\left(\vec{G}_2\right) \le s$, we have $\left|R_{\vec{G}_1}(\Sigma, u)\right| \le k + s$. This shows that $r_{\vec{G}_1}(\Sigma) \le k + s$.

Now applying Theorem 2.2 on the marking game of $\vec{G}_1$, we know the score of the game of $\vec{G}_1$ will be at most $k + \lfloor (1 + \frac{1}{a})(k + s) \rfloor + 2$. Finally we note that $\Delta\left(\vec{G}_2\right) \le t$, therefore $(a,1)\text{-gcol}(G) \le k + \lfloor (1 + \frac{1}{a})(k + s) \rfloor + 2 + t = 2k + s + t + \lfloor \frac{1}{a}(k + s) \rfloor + 2$. ■

It is easy to see that if G is a spanning subgraph of H, then $(a,1)\text{-gcol}(G) \le (a,1)\text{-gcol}(H)$. Therefore we have the following corollary:

Corollary 3.4 *If G is an (s,t)-pseudo partial k-tree and $1 \le a < k$, then* $(a,1)\text{-gcol}(G) \le 2k + s + t + \lfloor \frac{1}{a}(k + s) \rfloor + 2$.

When Alice has enough moves in each of her round, by applying Theorem 1.1 to the asymmetric marking games of (s,t)-pseudo partial k-trees, we have the following results:

Corollary 3.5 *Let G be an (s,t)-pseudo partial k-tree. If $a \ge k + s$, then* $(a,1)\text{-gcol}(G) \le 2k + 2s + 2$. *If $a \ge k$, then* $(a,1)\text{-gcol}(G) \le 2k + t + 2$.

Proof. Since G is an (s,t)-pseudo partial k-tree, we may suppose G is a spanning subgraph of $H = (V, E)$, where H is an (s,t)-pseudo chordal graph with $\vec{G}_1$ (as defined in the Definition 3.1) has maximum outdegree at most k. Since $(a,1)\text{-gcol}(G) \le (a,1)\text{-gcol}(H)$, it suffices for us to consider the $(a,1)$-marking game of H. From the Definition 3.1, we know $E(H) = E_1 \cup E_2$, where E_i is the set of edges obtained from $\vec{E}_i$ by omitting the orientations. And $\Delta^+\left(\vec{G}_1\right) \le k$, $\Delta^+\left(\vec{G}_2\right) \le s$. Therefore we have $\Delta^*(H) \le k + s$. Then applying Theorem 1.1, we have if $a \ge k + s$, $(a,1)\text{-gcol}(H) \le 2k + 2s + 2$.

If $a \ge k$, when playing the game, Alice will only take the oriented graph $\vec{G}_1$ into consideration. Now applying Theorem 1.1 on the $(a,1)$-marking game of $\vec{G}_1$, we know the score of the game of $\vec{G}_1$ will be at most $2k+2$. Combining $\Delta\left(\vec{G}_2\right) \leq t$, we have $(a,1)$-gcol$\,(G) \leq (a,1)$-gcol$\,(H) \leq 2k+t+2$. ■

Let S_g be an orientable surface of genus $g \geq 1$, i.e., the sphere with g handles. We consider graphs embeddable on S_g. Let $\mathcal{G}(S_g)$ be the set of graphs of minimum degree at least 2 and embeddable on S_g. The following lemma was proved by Zhu in [15].

Lemma 3.6 (Zhu, [15]) *Given an integer $g \geq 1$, let S_g be the orientable surface of genus g. Then $\mathcal{G}(S_g)$ is a $(\lfloor \frac{1+\sqrt{1+48g}}{2} \rfloor, \lfloor 3+\sqrt{1+48g} \rfloor)$-pseudo partial 2-tree.*

By applying Corollary 3.5 and Lemma 3.6, we have the following results about the scores of the asymmetric marking games of $\mathcal{G}(S_g)$:

Corollary 3.7 *Given an integer $g \geq 1$, let S_g be the orientable surface of genus g. If $a \geq \lfloor \frac{1+\sqrt{1+48g}}{2} \rfloor + 2$, then $(a,1)$-gcol$\,(\mathcal{G}(S_g)) \leq \lfloor \sqrt{1+48g} \rfloor + 7$. If $a \geq 2$, then $(a,1)$-gcol$\,(\mathcal{G}(S_g)) \leq \lfloor \sqrt{1+48g} \rfloor + 9$.*

4 Asymmetric marking games of interval graphs

In this section, we prove that if G is an interval graph with maximum clique size $\omega(G) = k+1$ and $1 \leq a < k$, then $(a,1)$-gcol$\,(G) \leq 2k + \lceil \frac{k}{a} \rceil + 1$. The proof presented here is a natural extension of the proof of Kierstead [3] to the asymmetric case.

Theorem 4.1 *If $G = (V,E)$ is an interval graph with $\omega(G) = k+1$ and $1 \leq a < k$, then $(a,1)$-gcol$\,(G) \leq 2k + \lceil \frac{k}{a} \rceil + 1$.*

Proof. Let L be the ordering of the intervals in V by left endpoints. We define $\vec{G}$ (an orientation of G) as follows: for an edge $e = uv$ in G, we define $v \leftarrow u$ in $\vec{G}$ if and only if $v < u$ in L. Then $N^+_{\vec{G}}(u)$ is a clique for every vertex u.

The strategy that Alice uses for the interval graphs is the same as we described as the *Activation Strategy* in Section 2. Next we show that if she uses the Activation Strategy, she will obtain a score of at most $2k + \lceil \frac{k}{a} \rceil + 1$ in the $(a,1)$-marking game on the interval graph G with $\omega(G) = k+1$ and $1 \leq a < k$. As some arguments needed here already appeared in the proof of Theorem 2.2, we will use them directly to avoid repetition.

Consider any time when a vertex v has just been marked by Alice. If Alice has not yet completed her turn, let x be the last vertex marked by Bob. Otherwise x is undefined. It suffices to show that any unmarked vertex u has at most $2k + \lceil \frac{k}{a} \rceil - 1$ marked neighbors other than x.

Let $S = N(u) \cap M - \{x\}$ and let $\{Q, C\}$ be the partition of S, where Q is the set of out-neighbors of u in S, C is the set of in-neighbors of u in S. Then $|Q| \leq k$. For any vertex $y \in C$, since u is an out-neighbor of y, and u is unmarked by the time y is marked, by observation (2) y has made a contributions to its out-neighbors. According to the activation strategy described above, when a contribution is made by y to its out-neighbor, the contribution goes to the vertex $z = L_y\text{-min}\, N^+_{\vec{G}}(y) \cap U$. Define

$$Z(y) = \{z: \ y \text{ has made contribution(s) to } z\} \text{ and } \ T = \bigcup_{y \in C} Z(y).$$

Choose $z_0 \in T$, such that z_0 has the smallest right endpoint in T. Suppose $z_0 \in Z(w)$, where $w \in C$. Next we claim that $K = T \cup \{w\} \cup \{u\}$ is a clique. Denoting the left endpoint of an interval i by $l(i)$ and the right endpoint by $r(i)$, for any $t \in T$, suppose $t \in Z(y)$ and $y \in C$, we have

$$l(t) \le l(u) \le l(w) \le r(z_0) \le r(t).$$

Thus the left endpoint of w and u are in every t. This proves the claim. Therefore $|T \cup \{u\}| \le \omega(G) - 1 = k$.

By observation (3) any vertex in $T \cup \{u\}$ can receive at most $a + 1$ contributions, and u received at most a contributions, So we have

$$a\,|C| < (a+1)\,k.$$

It follows that

$$|S| = |Q| + |C| \le k + |C| < 2k + \frac{k}{a}.$$

This shows that $|S| \le 2k + \lceil \frac{k}{a} \rceil - 1$, and thus finishes the proof. ■

In [9], the following technical lemma is introduced to present a series of examples for the lower bounds of the marking games on the classes of chordal, interval, planar and outerplanar graphs. For a graph $G = (V, E)$, a vertex $v \in V$ and a set $X \subseteq V$, let $d_X(v)$ denote $|N(v) \cap X|$. We will apply this lemma here to give an example of interval graphs.

Lemma 4.2 *Let a, d and d' be positive integers and let $G = (V, E)$ be a graph whose vertices are partitioned into sets L and S. Let $B \subseteq L$ and $T \subseteq S$. If*

1. $d_L(v) \ge d$ *and* $d_S(v) \ge d'$ *for all* $v \in L - B$,
2. $\operatorname{dist}(x, y) > a + 1$ *for all distinct* $x, y \in T$ *and*
3. $a\,(|B| + |S - T| + 1) < |L - B|$

then $(a, 1)\text{-gcol}\,(G) \ge d + d' + 1$.

Here $\operatorname{dist}(x, y)$ denotes the distance between x and y. The following example gives a lower bound of $(a, 1)$-gcol(G), where G is an interval graph with $\omega(G) = k + 1$ and $a < k$. This example was also presented in [11]. Let $I_{k,t}$ be the interval graph determined by the set of intervals $W_{k,t} = L_{k,t} \cup S_{k,t}$, where $L_{k,t} = \{[i, i + k] : i \in [t]\}$

(where $[t] = \{1, 2, \cdots, t\}$), and $S_{k,t} = \left\{\left[i + \frac{1}{2}, i + \frac{1}{2}\right] : k < i < t\right\}$. We identify $V(I_{k,t})$ with $W_{k,t}$ in the natural way. Set $v_i = [i, i+k]$, $x_i = \left[i + \frac{1}{2}, i + \frac{1}{2}\right]$. Then, for example, $\{v_i, ..., v_{i+k}\}$ is a $(k+1)$-clique in $I_{k,t}$. Notice that $\operatorname{dist}(x_i, x_j) = \left\lfloor \frac{|i-j|}{k} \right\rfloor + 2$. The *border* set $B_{k,t} = \{v_i : i \in [k] \cup ([t] - [t-k])\}$.

Example 4.3 *For every positive integer $1 \le a < k$ there exists an interval graph G such that $\omega(G) = k+1$ and $(a,1)$-gcol$(G) \ge 2k + \lfloor \frac{k}{a} \rfloor + 1$.*

Proof. Given $1 \le a < k$, we will define an interval graph G such that: $d = 2k$ and $d' = \lfloor \frac{k}{a} \rfloor$, where d and d' are referring to Lemma 4.2.

Let $r = 3k + 1$, $s = \lfloor \frac{rk^2+k}{a} \rfloor - \lfloor \frac{k}{a} \rfloor$, $t = rk^2 + 2k$, $L = L_{k,t}$, $B = B_{k,t}$, $S = \left\{x_{ia+1} : i \in \left[\lfloor \frac{rk^2+k}{a} \rfloor\right] - \left[\lfloor \frac{k}{a} \rfloor\right]\right\}$, $T = \{x_{jak+1} : j \in [r]\}$, where L, B, S and T are referring to Lemma 4.2. Let $G = I^{+}_{k,t}$ be the interval graph determined by the set of intervals $W_{k,t} = L_{k,t} \cup S$. Then we have $a < k$, $d = 2k$ and $d' = \lfloor \frac{k}{a} \rfloor$. Note that the distance between any two vertices in T is at least $a + 2$, and

$$\begin{aligned}
a(|B| + |S - T| + 1) &= a(2k + s - r + 1) \\
&= a\left(2k + \lfloor \frac{rk^2+k}{a} \rfloor - \lfloor \frac{k}{a} \rfloor - 3k - 1 + 1\right) \\
&= a\left(\lfloor \frac{rk^2+k}{a} \rfloor - \lfloor \frac{k}{a} \rfloor - k\right) \\
&< rk^2 \\
&= |L - B|.
\end{aligned}$$

So $(a,1)$-gcol$(G) \ge 2k + \lfloor \frac{k}{a} \rfloor + 1$ by Lemma 4.2. ∎

Immediately we have the following corollary:

Corollary 4.4 *For the the class of interval graphs $\mathbb{I}_k$ with maximum clique size $k+1$, if $1 \le a < k$, k mod $a \equiv 0$, then $(a,1)$-gcol$(\mathbb{I}_k) = 2k + \frac{k}{a} + 1$.*

For the classes of chordal graphs with maximum clique size $k+1$, which is denoted by $\mathbb{C}_k$, if $a \in \{1, k\}$ we know exact bounds. If $1 < a < k$, combining Corollary 2.3 and Example 4.3, we know $2k + \lfloor \frac{k}{a} \rfloor + 1 \le (a,1)$-gcol$(\mathbb{C}_k) \le 2k + \lfloor \frac{k}{a} \rfloor + 2$. It would be interesting to determine the exact value of $(a,1)$-gcol$(\mathbb{C}_k)$ for $1 < a < k$.

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