Refined activation strategy for the marking game

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Abstract

This paper introduces a new strategy for playing the marking game on graphs. Using this strategy, we prove that if G is a planar graph, then the game colouring number of G, and hence the game chromatic number of G, is at most 17.

1 Introduction

Suppose $G = (V, E)$ is a graph. The game colouring number of G is defined through a two-person game: the *marking game*. Alice and Bob, with Alice playing first, take turns in playing the game. Each play by either player consists of marking an unmarked vertex of G. The game ends when all vertices are marked. For a vertex x of G, let $b(x)$ be the number of neighbours of x that are marked before x is marked. The *score* of the game is

$$s = 1 + \max_{x \in V(G)} b(x).$$

Alice's goal is to minimize the score, while Bob's goal is to maximize it. The *game colouring number* $\mathrm{col}_g(G)$ of G is the least s such that Alice has a strategy that results in a score at most s.

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The game colouring number of a graph was first formally introduced in [22] as a tool in the study of the game chromatic number. The *game chromatic number* $\chi_{\rm g}(G)$ of a graph G is also defined through a two person game. Let G be a finite graph and let X be a set of colours. Alice and Bob, with Alice moving first, take turns in playing the game. Each play by either player consists of colouring an uncoloured vertex of G with a colour from the colour set X so that no two adjacent vertices receive the same colour. Alice wins the game if all the vertices of G are coloured. Otherwise, Bob wins the game. The game chromatic number $\chi_{\rm g}(G)$ of G is the least number of colours in a colour set X for which Alice has a winning strategy.

It is easy to see that for any graph G, $\chi_{\rm g}(G) \le {\rm col_g}(G)$. For many natural classes of graphs, the best known upper bounds for their game chromatic number are obtained by finding upper bounds for their game colouring number. Game colouring number of graphs and its generalization to oriented graphs are also of independent interests, and have been studied extensively in the literature [1, 3, 4, 7-9, 11-13, 15-23].

Suppose $\mathcal{H}$ is a family of graphs. We define the game chromatic number and the game colouring number of $\mathcal{H}$ as

$$\chi_{\rm g}(\mathcal{H}) = \max\{\chi_{\rm g}(G) : G \in \mathcal{H}\},$$

and

$${\rm col_g}(\mathcal{H}) = \max\{{\rm col_g}(G) : G \in \mathcal{H}\}.$$

We denote by $\mathcal{F}$ the family of forests, by $\mathcal{I}_k$ the family of interval graphs with clique number k, by $\mathcal{P}$ the family of planar graphs, by $\mathcal{Q}$ the family of outer planar graphs, by $\mathcal{PT}_k$ the family of partial k-trees. The exact value of the game colouring numbers of $\mathcal{F}$, $\mathcal{I}_k$, $\mathcal{Q}$ and $\mathcal{PT}_k$ are known. It is proved by Faigle, Kern, Kierstead and Trotter [10] that ${\rm col_g}(\mathcal{F}) = 4$, proved by Faigle, Kern, Kierstead and Trotter [10] and Kierstead and Yang [17] that ${\rm col_g}(\mathcal{I}_k) = 3k-2$, proved by Guan and Zhu [11] and Kierstead and Yang [17] that ${\rm col_g}(\mathcal{Q}) = 7$, and proved by Zhu [23] and Wu and Zhu [20] that ${\rm col_g}(\mathcal{PT}_k) = 3k+2$ for $k \ge 2$.

Although there are relatively rich results concerning the game chromatic number and game colouring number of graphs, there are very few strategies for either Alice or Bob to play the colouring game or marking game. It is proved in [13] that there is a single strategy, *the activation strategy*, such that if Alice uses this strategy to play the marking game then she achieves the sharp upper bounds on the game colouring numbers of $\mathcal{F}, \mathcal{I}_k, \mathcal{Q}, \mathcal{PK}_k$ as well as the best known upper bounds for many other classes of graphs, including $\mathcal{P}$.

In this paper, we introduce a new strategy, the refined activation strategy, for playing the marking game (it can also be used as a strategy for playing the colouring game). It is quite similar to the activation strategy, however, there are two new ingredients in the recipe. The key idea in the activation strategy is to use a special linear ordering of

$V(G)$ as Alice's preference in activating and marking vertices. In the refined activation strategy, Alice still uses orderings of the vertices as her preference in activating and marking vertices. However, there are two features that are different from the activation strategy. (1): The ordering is a 'dynamic rough ordering'. The vertex set is partitioned into small blocks. Within a block, there maybe non-comparable vertices, the order relation is not transitive and moreover, the order relation between vertices may change from time to time. (2): Each vertex has a preference of its own. If Alice moves from a vertex v to her next target, the preference of v will affect Alice's choice as well.

We shall prove an upper bound for the game colouring number of a graph G in terms of a dynamic rough ordering and a preference function, by applying the refined activation strategy. Then we estimate the bound for planar graphs, which yields a better upper bound for $\mathrm{col_g}(\mathcal{P})$. The game chromatic number and game colouring number of planar graphs are benchmark problems in the study of the colouring game and marking game. It was conjectured by Bodlaender [2] that $\chi_{\mathrm{g}}(\mathcal{P}) < \infty$. This conjecture is confirmed by Kierstead and Trotter [14], who proved that $\chi_{\mathrm{g}}(\mathcal{P}) \leq 33$. This bound is reduced to 30 by Dinski and Zhu [6]. Then by introducing the game colouring number, Zhu [22] proved that $\chi_{\mathrm{g}}(\mathcal{P}) \leq \mathrm{col_g}(\mathcal{P}) \leq 19$, and this bound is reduced to 18 by Kierstead [13]. Recently, Wu and Zhu [20] proved that $\mathrm{col_g}(\mathcal{P}) \geq 11$. By using the refined activation strategy, this paper proves that $\mathrm{col_g}(\mathcal{P}) \leq 17$.

Theorem 1 *If $G = (V, E)$ is a planar graph, then $\chi_{\mathrm{g}}(G) \leq \mathrm{col_g}(G) \leq 17$.*

2 A review of the activation strategy and a sketch of the refinement

Suppose a marking game is played on a graph G. For Alice to apply the activation strategy, we need a linear ordering $v_1, v_2, \cdots, v_n$ of $V(G)$. We write $v_i < v_j$ if $i < j$. If $x \sim y$ and $x < y$, then x is an *out-neighbour* of y, and y is an *in-neighbour* of x. In her first move, Alice activates vertex v_1 and marks it. Suppose Bob has just marked a vertex v. Then Alice starts with activating v (provided it was not activated so far) and jumps to its least-indexed unmarked out-neighbour x. If x is already active, then Alice stops and marks x. Otherwise she repeats the activation step for x, that is, activates x and jumps to its least-indexed unmarked out-neighbour y. And so on, until she stops at some vertex u, either because u is already active, or because u has no unmarked out-neighbour. In both cases she activates and marks u. If it happens that the vertex v marked by Bob has no unmarked out-neighbour, then she picks the least-indexed unmarked vertex and activates and marks it.

Assume that Alice has just finished a move. We denote by A the set of active vertices. Note that each marked vertex is active. To obtain an upper bound for the score of the game, it suffices to find an upper bound for the number of active neighbours of any unmarked vertex. Let $N(u)$ be the set of neighbours of u. Then the score of this game is at most $2 + \max |A \cap N(u)|$, where the maximum is taken over all unmarked vertices u at the end of all Alice's moves. This is so because it may be the case that u is marked by Alice, and in Bob's last move before Alice marks u, he marked a neighbour of u. So u has at most $1 + |A \cap N(u)|$ marked neighbours, where A is the set of active vertices after Alice's previous move.

The method used in the literature for obtaining an upper bound on the game colouring number of a graph is to prove an upper bound for $|A \cap N(u)|$ for any unmarked vertex u. This bound is determined by the linear ordering of the vertices of G. For a vertex u, let $V^+(u) = \{x : x < u\}$, $V^-(u) = \{x : u < x\}$, $N^+(u) = N(u) \cap V^+(u)$ and $N^-(u) = N(u) \cap V^-(u)$. Note that $|A \cap N(u)| = |A \cap N^+(u)| + |A \cap N^-(u)|$. The method used in [13] to bound $|A \cap N(u)|$ for planar graphs is to find a linear ordering of G so that the following is true:

For each vertex u, there is a set $A(u) \subseteq V^+(u)$ and a set $D(u) \subseteq N^-(u)$ such that $N^+(u) \subseteq A(u)$ and for every vertex $x \in N^-(u) \setminus D(u)$, $N(x) \cap V^+(u) \subseteq A(u)$. Moreover, $3|A(u)| + |D(u)| \leq 16$ for each vertex u.

To see that $3|A(u)|+|D(u)|$ is an upper bound for $|A \cap N(u)|$, we partition $A \cap N(u)$ into three parts:

$$A \cap N(u) = (A \cap N^+(u)) \cup (A \cap (N^-(u) \setminus D(u))) \cup (A \cap D(u)).$$

By our assumption, $|A \cap N^+(u)| \leq |A(u)|$, and each marked neighbour in $(A \cap N^-(u)) \setminus D(u)$ contributes one jump to $A(u)$. The latter implies that $|N^-(u) \setminus D(u)| \leq 2|A(u)|$ as each vertex in $A(u)$ can receive at most 2 jumps (the first jump activates it, and the second jump marks it). Therefore $|A \cap N(u)| \leq 3|A(u)| + |D(u)|$. Thus if G has an ordering in which $3|A(u)| + |D(u)| \leq 16$, by the argument in the third previous paragraph, we have $\chi_g(G) \leq 18$.

In an attempt to improve this bound, we found that for general planar graphs, the bound $3|A(u)| + |D(u)| \leq 16$ cannot be improved. However, by modifying the strategy at two places, one can improve the bound $|A \cap N(u)|$. We use examples to explain these two modifications.

Suppose the vertices of G are linearly ordered, except that the order relation among three vertices x, y, z are not determined. These three vertices will be consecutive in the linear ordering, and the set $V(G) \setminus \{x, y, z\}$ is divided into two parts $U \cup C$ such that for any $u \in U$, $u < x, y, z$ and for any $v \in C$, $x, y, z < v$.

Suppose $x \sim y$ and $y \sim z$. For $w \in \{x, y, z\}$, the sets $A(w) \cap U$ and $D(w) \cap C$ have been determined. However, the sets $A(w)$ and $D(w)$ still depend on the order relation among x, y, z. Assume $|A(x) \cap U| = |A(z) \cap U| = 4$ and $|A(y) \cap U| = 3$ and

$|D(x) \cap C| = |D(y) \cap C| = |D(z) \cap C| = 1$. By applying the original activation strategy, we need to fix an ordering among x, y, z. If we choose the ordering among x, y, z as $x < y < z$, then $x \in A(y)$, $y \in A(z)$. Hence $|A(z)| = 5$, and hence $3|A(z)| + |D(z)| = 16$. If we order the three vertices as $x < z < y$, then we have $x, z \in A(y)$ and hence $|A(y)| = 5$, implying that $3|A(y)| + |D(y)| = 16$.

In the refined activation strategy, instead of fixing a preference all the time, we allow the preference to change during the game. Suppose the rule is as follows: among those common in-neighbours of x and y that jump to $\{x, y\}$, the first and the third jump to y, the second and the fourth jump to x. (Compare to the original activation strategy: if $x < y$, then the first and second jump to x, the third and the fourth jump to y). Similarly, among those common in-neighbours of z and y that jump to $\{z, y\}$, the first and the third jump to y, the second and the fourth jump to z.

Let us analyze the set $|A \cap N(y)|$. We have $|A(y) \cap U| = 3$ and $|D(y)| = 1$. By applying the rules described in the previous paragraph, before y is marked, at most one of x and z can receive two jumps from $N^-(y) \cap A$, and hence x and z together can received at most 3 jumps from $N^-(y) \cap A$. Plus x and z themselves, they contribute at most 5 active neighbours of y. Therefore $|A \cap N(y)| \leq 3|A(y) \cap U| + |D(y)| + 5 \leq 15$. Similarly, one can show that by using the modified strategy, $|A \cap N(z)| \leq 15$ and $|A \cap N(x)| \leq 15$.

In general, the refined activation strategy works as follows: We partition the vertex set of G into blocks $B_1, B_2, \cdots, B_m$. The preference in the jumping process is that vertices in B_i are preferred to vertices in B_j if $i < j$. Within a block B_i, the preference may change during the game, as explained in the above example (of course, there are more configurations to be considered).

Now we explain the second modification of the strategy. Suppose the blocks $B_1, B_2, \cdots, B_m$ are determined. Suppose there are two vertices $x, y \in B_i$ that have a common neighbour z in B_j for some $j > i$. Suppose z is activated and jumps to the set $\{x, y\}$. The vertex z makes a contribution to each of $A \cap N^-(x)$ and $A \cap N^-(y)$. In the activation strategy described above, there are two different ways to count this contribution: Either put z into $D(x)$ and $D(y)$ and count this vertex directly, or count the jump from z to $A(x)$ and $A(y)$. In the latter case, we need to put x into $A(y)$ or put y into $A(x)$, depending on whether x or y is the preferred vertex among the two. In our second modification, we allow, for example, that z be put into $D(x)$, but not into $D(y)$, and when jumping from z to $\{x, y\}$, y is the preferred vertex among the two. Thus when we estimate $A \cap N^-(x)$, the vertex z is counted directly. Hence we do not need to put y into $A(x)$, although y is the preferred vertex (by z) among the two vertices x, y.

In general, each vertex z will be associated a preference set $\rho(z)$, which determines for each block B_i, which vertices are preferred by z. When Alice jumps from a vertex z, among the vertices of the same block B_i, the preference set affects the destination of the jump. However, between vertices of different blocks, the preference is still determined by the ordering of the blocks.

3 Refined activation strategy

In this section, we describe the refined activation strategy in detail.

Definition 1 *Suppose $G = (V, E)$ is a graph. A* dynamic rough ordering *of G is a pair $(L_0, \mathcal{P})$ such that L_0 is a digraph on V without opposite arcs, and $\mathcal{P}$ is a partition of V. Each $B \in \mathcal{P}$ is called a* block. *The blocks are ordered as $B_1, B_2, \cdots, B_m$ such that for any $i < j$, if $x \in B_i$ and $y \in B_j$ then $\overrightarrow{yx} \in L_0$, i.e., $\overrightarrow{yx}$ is an arc of L_0.*

The digraph L_0 is viewed as a rough ordering. In the remaining of this paper, we write $x <_{L_0} y$ if $\overrightarrow{yx}$ is an arc of L_0. We say two vertices x, y are *comparable* in L_0 if and only if either $x <_{L_0} y$ or $y <_{L_0} x$. The digraph L_0 is not really an ordering, because inside a block B_i, there may be non-comparable vertices, the relation $<_{L_0}$ may not be transitive, and there may be directed cycles. However, if we ignore what happens inside the blocks, it becomes a linear ordering.

In the definition, there is nothing which is really dynamic. What we have here are simply a fixed rough ordering L_0 and a partition $\mathcal{P}$. However, we use the adjective 'dynamic' to suggest that the rough ordering used in the strategy will change from time to time, and L_0 is just the initial state of the 'real' dynamic rough ordering.

We write $x \approx y$ if x and y are in the same block of $\mathcal{P}$, and write $x \not\approx y$ otherwise.

Let

$$
\begin{array}{rclrcl}
V^+_{L_0}(x) & = & \{y : y <_{L_0} x\}, & V^-_{L_0}(x) & = & \{y : x <_{L_0} y\}, \\
V^+_{L_0,\not\approx}(x) & = & \{y \in V^+_{L_0}(x) : x \not\approx y\}, & V^-_{L_0,\not\approx}(x) & = & \{y \in V^-_{L_0}(x) : x \not\approx y\}, \\
V^+_{L_0,\approx}(x) & = & \{y \in V^+_{L_0}(x) : x \approx y\}, & V^-_{L_0,\approx}(x) & = & \{y \in V^-_{L_0}(x) : x \approx y\}, \\
V_{L_0,\approx}(x) & = & V^+_{L_0,\approx}(x) \cup V^-_{L_0,\approx}(x), & V^\times_{L_0}(x) & = & \{y : x \not<_{L_0} y, y \not<_{L_0} x\}.
\end{array}
$$

Let $V^+_{L_0}[x] = V^+_{L_0}(x) \cup \{x\}$ and $V^-_{L_0}[x] = V^-_{L_0}(x) \cup \{x\}$. If $x \approx y$, then $V^+_{L_0,\not\approx}(x) = V^+_{L_0,\not\approx}(y)$ and $V^-_{L_0,\not\approx}(x) = V^-_{L_0,\not\approx}(y)$. We let $V^+_{L_0,\not\approx}(B_i) = V^+_{L_0,\not\approx}(x)$ and $V^-_{L_0,\not\approx}(B_i) = V^-_{L_0,\not\approx}(x)$ for some (and hence for all) $x \in B_i$.

Given a digraph Q, we denote by $\overline{Q}$ the graph obtained from Q by omitting the orientation of the arcs, i.e., an arc $\overrightarrow{xy}$ of Q becomes an edge xy of $\overline{Q}$.

A *preference function* of $(L_0, \mathcal{P})$ is a mapping ρ which assigns to each vertex $y \in V(G)$ a subset $\rho(y)$ of $N_G(y) \cap V^+_{L_0,\not\approx}(y)$ such that the following holds:

[P1]: For any index i, if $y \in V^-_{L_0,\not\approx}(B_i)$, then $\rho(y) \cap B_i$ contains at most one edge of $\overline{L_0}$.

The set $\rho(y) \cap B_i$ (which could be empty) is called the *y-preferred subset of B_i*. If $\rho(y) \cap B_i$ does contain an edge uv of $\overline{L_0}$, then we call the edge uv a *y-affected edge of B_i*.

For any vertex x, let $\rho^{-1}(x) = \{y : x \in \rho(y)\}$. Note that $\rho^{-1}(x) \subset N_G(x) \cap V^-_{L_0,\not\approx}(x)$. Let $D(x) = (N_G(x) \cap V^-_{L_0,\not\approx}(x)) \setminus \rho^{-1}(x)$.

In the description of the refined activation strategy, we need to refer to a digraph L (a rough ordering), which is the 'real' dynamic rough ordering obtained from L_0 by possibly reversing the orientations of some arcs. So L is a living creature, and the letter L always stands for the current digraph L. By *reversing the arc $\overrightarrow{uv}$ of L*, we mean let $L := (L \setminus \{\overrightarrow{uv}\}) \cup \{\overrightarrow{vu}\}$. The strategy will give reversal rules that describe how the arcs of L will be reversed. Here we just note the following properties of L, which follow easily from the reversal rules (which will be given later):

(1): Although L and L_0 may have different arcs, we always have $\overline{L} = \overline{L_0}$. So two vertices x, y are comparable in L if and only if they are comparable in L_0.

(2): Arcs of L_0 between vertices of different blocks will not be reversed at any time. All the reversals of arcs take place inside the blocks only.

The sets $V^+_L(x), V^-_L(x)$, etc. will be defined similarly as $V^+_{L_0}(x), V^-_{L_0}(x)$, etc., except that in place of L_0 we use the digraph L.

Suppose X is a subset of V. A *minimal element of X with respect to L* is an element $x \in X$ such that for any $y \in X$, $y \not<_L x$. As L may contain directed cycles, for an arbitrary subset X of V, X may not have a minimal element. In case a minimal element exists, it may not be unique. The following definition of $\min_v X$ combines the rough ordering and the preference function together in finding a (more or less minimal) element $\min_v X$ of X.

Definition 2 *Suppose v is a vertex of V and X is a nonempty subset of V. Then $\min_v X$ is an element of X defined as follows:*

Let i be the smallest index such that $B_i \cap X \neq \emptyset$. If $X \cap B_i \cap \rho(v) \neq \emptyset$, then $\min_v X$ is an arbitrary (but fixed) minimal element of $X \cap B_i \cap \rho(v)$ with respect to L. Note that by our definition of the preference function, $\rho(v) \cap B_i$ contains at most one arc of L, and hence the minimal element exists. If $X \cap B_i \cap \rho(v) = \emptyset$, then $\min_v X$ is an arbitrary (but fixed) element of $X \cap B_i$.

Note that if each block B_i is a singleton, then L_0 is a linear order and $\min_v X$ is simply the minimum element of X. Indeed, in this case, the refined activation strategy (which we will describe soon) is the same as the activation strategy. Also note that $\min_v X$ depends on the current rough ordering L. So at different times of the game, $\min_v X$ may refer to different vertices.

In the play of the game, Alice will maintain a subset A of *active vertices.* We say a vertex v is *activated* to mean that v is added to A. Once a vertex is activated, it remains active afterwards. Let U be the set of unmarked vertices. To unify the description we consider an equivalent version of the marking game in which Bob plays first by marking a new vertex x_0, which is an isolated vertex, and $x_0 <_{L_0} y$ for all $y \in V$.

Initialization: $A := \emptyset$, $U := V(G)$ and $L := L_0$.

Suppose Bob has just marked a vertex b and now it is Alice's turn. If all the vertices are marked, then the game is over. Otherwise, let u be an arbitrary unmarked vertex.

- if $N_G(b) \cap V_L^+(b) \cap U \neq \emptyset$ then $x := b$, else $x := u$ end if;

- while $x = b$ or $x \notin A$ do
  $A := A \cup \{x\}$;
  $w := \min_x N_G[x] \cap V_L^+[x] \cap U$;
  if there is an arc $\overrightarrow{uw}$ of L incident to w such that uw is an x-affected edge, then reverse the arc $\overrightarrow{uw}$ end if;
  $x := w$ end do;

- Mark x (i.e., $U := U \setminus \{x\}$) end do;

This strategy is similar to the activation strategy in [13]. Starting from the vertex b which has just been marked by Bob (or starting from any unmarked vertex, if $N_G(b) \cap V_L^+(b) \cap U = \emptyset$), Alice starts to activate vertices. After Alice activated a vertex x, she 'jumps' to the least unmarked 'forward' neighbour w of x, which she will either activate if it is not active yet, or mark if it is already active. The difference between this strategy and the original activation strategy is that the 'least element' refers to a dynamic rough ordering L. Moreover, this dynamic ordering L is 'modified' by the preference of x.

If there is a jump from x to w, we say x *made a contribution* to w, and say w *received a contribution* from x. If X, Y are subsets of V, then we say Y received a contribution from X if a vertex $y \in Y$ received a contribution from a vertex $x \in X$. Observe that only unmarked vertex can receive contributions. If a vertex receives the first contribution, it becomes active. After receiving the second contribution, it becomes marked. So each vertex can receive at most 2 contributions. At the time a vertex x is activated, it will make a contribution to a least unmarked vertex (according the current order with modification through ρ) in $V_L^+(x)$, unless $N_{G_L}^+(x) \cap U$ is empty, in which case x will make a contribution to itself, and be marked.

Similarly as in the activation strategy, we shall find an upper bound for the number of active neighbours of an unmarked vertex. Assume Alice has just finished a move and x is an unmarked vertex. Let X be the set of active neighbours of x, i.e., $X = A \cap N_G(x)$. We shall determine the maximum possible value of $|X|$.

Assume $x \in B_i$. We partition the set X into three parts.

$$\begin{aligned} X_1 &= X \cap (V_{L_0}^+(x) \cup B_i), \\ X_2 &= X \cap \rho^{-1}(x), \\ X_3 &= X \cap (V_{L_0,\not\approx}^-(x) \setminus \rho^{-1}(x)) = X \cap D(x). \end{aligned}$$

Then $|X_1| \leq |N_G(x) \cap (V_{L_0}^+(x) \cup B_i)|$ and $|X_3| \leq |D(x)|$. The difficult part is to find an upper bound for $|X_2|$. For the purpose of finding an upper bound for $|X_2|$, we introduce the concept of a bound graph for $(G, L_0, \mathcal{P}, \rho)$.

Suppose $(L_0, \mathcal{P})$ is a dynamic rough ordering of G and ρ is a preference function of $(L_0, \mathcal{P})$. Let H be a graph with vertex set $V(H) = V(G)$. We say H is a *bound graph* for $(G, L_0, \mathcal{P}, \rho)$ if the following hold:

[B1] G is a subgraph of H.

[B2] If $x \approx y$ and $\rho^{-1}(x) \cap \rho^{-1}(y) \neq \emptyset$, then $x \sim_H y$.

[B3] If $x \in V_{L_0,\not\approx}^-(y)$ and there is a vertex $z \in \rho^{-1}(x)$ such that $y \sim_G z$, then $x \sim_H y$.

Suppose H is a bound graph for $(G, L_0, \mathcal{P}, \rho)$. Suppose $x \in B_i$. Let

$$\begin{aligned} A(x) &= \{y \in V_{L_0,\not\approx}^+(B_i) \cup V_{L_0}^\times(x) : x \sim_H y\}, \\ B(x) &= \{y \in V_{L_0,\approx}^+(x) : x \sim_H y\}. \\ C(x) &= \{y \in V_{L_0,\approx}^-(x) : x \sim_H y\}. \end{aligned}$$

Let

$$\tau(x) = \begin{cases} 0, & \text{if } B(x) = C(x) = \emptyset \\ 1, & \text{otherwise} \end{cases}.$$

Recall that $D(x) = (N_G(x) \cap V_{L_0,\not\approx}^-(x)) \setminus \rho^{-1}(x)$.

Lemma 1 *Suppose $(L_0, \mathcal{P})$ is a dynamic rough ordering of G, ρ is a preference function of $(L_0, \mathcal{P})$, and H is a bound graph for $(G, L_0, \mathcal{P}, \rho)$. Let $x \in B_i$ and let X_2 be the set defined as above. Then*

$$|X_2| \leq 2|A(x)| + |B(x)| + \tau(x).$$

Proof. Since x is unmarked, it follows from the refined activation strategy that each vertex $z \in X_2$ makes a contribution to a vertex in $A(x) \cup B(x) \cup C(x)$ or makes a contribution to x. However, if z makes a contribution to x, then x is activated and it makes a contribution to $A(x) \cup B(x) \cup C(x)$.

Each vertex in $A(x)$ can receive at most 2 contributions from X_2. In case $B(x) \cup C(x) = \emptyset$, we have $|X_2| \leq 2|A(x)|$. Assume $B(x) \cup C(x) \neq \emptyset$. If a vertex y in $B(x)$ receives one contribution from $z \in X_2$, then since the edge xy is z-affected, the arc $\overrightarrow{xy}$ is changed to $\overrightarrow{yx}$. Hence y cannot receive the second contribution from X_2 before x itself receives one contribution from X_2 and becomes activated. Also for a vertex $y \in C(x)$, we have $\overrightarrow{yx} \in L_0$. Before y receives a contribution from X_2, x must receives a contribution from X_2 and becomes active. Since x has received at most one contribution (as x is unmarked yet), the total number of contributions received by vertices in $B(x) \cup C(x)$ from X_2 is at most $|B(x)| + 1$. Thus we conclude that $|X_2| \leq 2|A(x)| + |B(x)| + \tau(x)$. ∎

By the definition of bound graph H and the sets $A(x), B(x), C(x)$, we have $|X_1| \leq |A(x)| + |B(x)| + |C(x)|$. For a bound graph H for $(G, L_0, \mathcal{P}, \rho)$, for $x \in V(G)$, let

$$\phi_H(x) = 3|A(x)| + 2|B(x)| + |C(x)| + |D(x)| + \tau(x).$$

Then we have the following theorem.

Theorem 2 *Suppose $(L_0, \mathcal{P})$ is a dynamic rough ordering of a graph G and ρ is a preference function of $(L_0, \mathcal{P})$. Suppose H is a bound graph for $(G, L_0, \mathcal{P}, \rho)$. Let $\phi_H(x)$ be defined as above. Then*

$$\mathrm{col_g}(G) \leq \max_{x \in V(G)} \phi_H(x) + 2.$$

Proof. By the argument above, if Alice uses the refined activation strategy, at any time after Alice finished a move, an unmarked vertex x has at most $\phi_H(x)$ marked neighbours. In Bob's next move, he may mark one more neighbour of x. So before x is marked, it has at most $\phi_H(x) + 1$ marked neighbours. ∎

4 Proof of Theorem 1

We shall prove Theorem 1, by finding, for any planar graph G, a dynamic rough ordering $(L_0, \mathcal{P})$, a preference function ρ of $(L_0, \mathcal{P})$, and a bound graph H for $(G, L_0, \mathcal{P}, \rho)$, such that for each vertex $x \in V$, $\phi_H(x) \leq 15$. For this purpose, we need a lemma about the structure of plane triangulations.

Suppose R is a plane triangulation and $V(R)$ is partitioned into two sets $C \cup U$, where C (could be an empty) is an independent set of R, and each vertex of C has degree 4 or 5. A *candidate* for (R, C, U) is a triple (B, ρ, Q) such that B is a non-empty subset of U, Q is a digraph with vertex set B, and ρ is a mapping which assigns to each vertex $y \in C$ a subset $\rho(y)$ of B. Moreover, the following hold:

[C1] If $\{v_1, v_2, v_3\} \subset B$ is a face of R, then $\{v_1, v_2, v_3\}$ contains at most one arc of Q.

[C2] For any $y \in C$, $|\rho(y) \cap B| \leq 2$.

[C3] If $x, x' \in \rho(y)$ for some $y \in C$, then $x \sim_R x'$.

[C4] If there is a $y \in C$ such that $x \in \rho(y)$ and $x' \in N_R(y) \cap (U \setminus B)$, then $x \sim_R x'$.

Suppose (B, ρ, Q) is a candidate for (R, C, U) and $x \in B$. Let

$$A(x) = (N_R(x) \cap U) \setminus N_Q(x), B(x) = N_Q^+(x), C(x) = N_Q^-(x).$$

Let $D(x) = (N_R(x) \cap C) \setminus \rho^{-1}(x)$.

Let

$$\tau(x) = \begin{cases} 0, & \text{if } C(x) = B(x) = \emptyset \\ 1, & \text{otherwise.} \end{cases}$$

Let

$$\phi(x) = 3|A(x)| + 2|B(x)| + |C(x)| + |D(x)| + \tau(x).$$

We call the candidate (B, ρ, Q) a *valid candidate* if the following holds:

[C5] For all $x \in B$, $\phi(x) \leq 15$.

Theorem 3 *Suppose R is a plane triangulation, $C \cup U$ is a partition of $V(R)$, C is an independent set of R and each vertex of C has degree 4 or 5. If $U \neq \emptyset$, then (R, C, U) has a valid candidate.*

We shall leave the proof of Theorem 3 to the next section. Now we use Theorem 3 to prove Theorem 1. It suffices to prove Theorem 1 for plane triangulations.

Suppose G is a plane triangulation. We shall construct a dynamic rough ordering $(L_0, \mathcal{P})$, a preference function ρ of $(L_0, \mathcal{P})$, and a bound graph H for $(G, L_0, \mathcal{P}, \rho)$ as follows.

The blocks of $\mathcal{P}$ are constructed one by one. First we construct B_m, then B_{m-1}, and so on. At the time we construct B_i, we shall construct simultaneously the restriction of the digraph L_0 to B_i, the intersection $\rho(y) \cap B_i$ for each $y \in V^-_{L_0,\not\approx}(B_i)$, and the edges of $H \setminus (B_m \cup B_{m-1} \cup \cdots \cup B_{i+1})$ incident to vertices of B_i. Initially, B_m consists of a single vertex of degree at most 5 in G. The edges of H incident to the vertex of B_m are exactly the edges of G incident to it.

Suppose we have constructed $B_m, B_{m-1}, \cdots, B_{i+1}$. Let $C' = \cup_{j=i+1}^m B_j$, and let $U = V \setminus C'$. By our construction of $B_m, B_{m-1}, \cdots, B_{i+1}$, each vertex of C' is adjacent to at most 5 vertices of U. First we delete all edges of G joining two vertices of C'. If $z \in C'$ is adjacent to at most three vertices of U, then delete z, and add edges between each pair of non-adjacent neighbours of z in U. Let $C = C' \setminus \{z : |N_G(z) \cap U| \leq 3\}$. If z is adjacent to 4 or 5 vertices of G, then add edges between each pair of non-adjacent 'consecutive' neighbours of z in U. Here consecutive refers to the particular plane embedding of $G \setminus E(C')$. Now the resulting graph is a plane triangulation R. Obviously $C \cup U$ is a partition of $V(R)$, and C is an independent set of R.

By Theorem 3, (R, C, U) has a valid candidate (B, ρ', Q). Let $B_i = B$. Let the restriction of L_0 to B_i be Q. For each vertex $y \in C'$, if $y \in C' \setminus C$, then let $\rho(y) \cap B_i = N_G(y) \cap B_i$; if $y \in C$, then let $\rho(y) \cap B_i = \rho'(y)$. Let the edges of $H \setminus (B_m \cup B_{m-1} \cup \cdots \cup B_{i+1})$ incident to vertices of B_i be exactly the edges of $R \setminus C$ incident to vertices of B_i. Note that by definition of valid candidate, $\phi(x) \leq 15$ for each $x \in B_i$, which implies that x is adjacent to at most 5 vertices of $U \setminus B_i$.

We claim that this process constructs a dynamic rough ordering $(L_0, \mathcal{P})$, a preference function ρ of $(L_0, \mathcal{P})$, and a bound graph H for $(G, L_0, \mathcal{P}, \rho)$, such that for each vertex $x \in V$, $\phi_H(x) \leq 15$. By Theorem 2, we have $\mathrm{col_g}(G) \leq 17$.

It follows from the definition that $(L_0, \mathcal{P})$ is a dynamic rough ordering of G. To prove that ρ is a preference function of $(L_0, \mathcal{P})$, we need to show that for any index i, if $y \in V^-_{L_0,\not\approx}(B_i)$, then $B_i \cap \rho(y)$ contains at most one edge of $\overline{L_0}$.

Let C', C, U, B and R be the sets and graph defined as above at the time B_i is constructed. Then $y \in C'$. If $y \in C' \setminus C$ then $|\rho(y)| = |N_G(y) \cap B| \leq |N_R(y) \cap U| \leq 3$. If $|N_R(y) \cap U| \leq 2$, then of course $\rho(y) \cap B$ contains at most one arc of L_0. If $|N_R(y) \cap U| = 3$, then $N_R(y) \cap U$ is a facial triangle of R. By [C1], the facial triangle contains at most one arc of Q. Hence $B_i \cap \rho(y)$ contains at most one arc of L_0.

Assume $y \in C$. Then it follows from [C2] that $|B_i \cap \rho(y)| \leq 2$, and $B_i \cap \rho(y)$ contains at most one arc of L_0. So ρ is a preference function of $(L_0, \mathcal{P})$.

Now we prove that H is a bound graph for $(G, L_0, \mathcal{P}, \rho)$. It is obvious that G is a subgraph of H, i.e., [B1] is satisfied. Assume $x, y \in B_i$ and $\rho^{-1}(x) \cap \rho^{-1}(y) \neq \emptyset$. Let C', C, U, B and R be the sets and graph defined as above at the time B_i is constructed. Let $z \in \rho^{-1}(x) \cap \rho^{-1}(y)$. Then $z \in C'$. If $z \in C' \setminus C$, then by definition of R, we have $x \sim_R y$, hence $x \sim_H y$. Assume $z \in C$. Since $x, y \in \rho(z)$, By [C3], we have $x \sim_R y$ and hence $x \sim_H y$, i.e., [B2] is satisfied. Assume $x \in B_i$ and $y \in V^+_{L_0,\not\approx}(B_i)$ and there is a vertex $z \in \rho^{-1}(x)$ such that $y \sim_G z$. If $z \in C' \setminus C$, then by definition of R, we have $x \sim_R y$ and hence $x \sim_H y$. If $z \in C$, then by [C4], we have $x \sim_R y$ and hence $x \sim_H y$. Thus [B3] is satisfied, and hence H is indeed a bound graph for $(G, L_0, \mathcal{P}, \rho)$.

It remains to show that for each x, we have $\phi_H(x) \leq 15$. This follows from the construction, because if $x \in B_i$ and $B_i = B$, where (B, Q, ρ) is the corresponding valid candidate, then $\phi_H(x) = \phi(x) \leq 15$. ∎

5 Proof of Theorem 3

The definition of a valid candidate is a little bit technical, and the proof of Theorem 3 is quite long. To help the readers to have a rough idea of this concept before we get to the proof, we first prove a weaker result: under the assumption of Theorem 3, (R, C, U) has a candidate (B, ρ, Q) such that for all $x \in B$, $\phi(x) \leq 16$. Indeed, we shall find such a candidate with B being a single element set and $\rho(y) = \emptyset$ for all $y \in C$. (So there is no need to introduce ρ and Q for this result). The proof is from [13]: For $x \in C$, let $c(x) = \frac{3}{2}d_R(x)$, and for $x \in U$, let $c(x) = d_R(x) - \frac{1}{2}|N_R(x) \cap C|$. Then each edge of R contributes 2 to the summation $\sum_{x \in C \cup U} c(x)$. By Euler's formula $\sum_{x \in C \cup U} c(x) < 6|C \cup U|$. So there is a vertex x^* with $c(x^*) \leq 5.5$. For $x \in C$, we have $d_R(x) \geq 4$ implying that $c(x) = \frac{3}{2}d_R(x) \geq 6$. So $x^* \in U$. Let $B = \{x^*\}$. Then $A(x^*) = N_R(x^*) \cap U$ and $D(x^*) = N_R(x^*) \cap C$, and $B(x^*) = C(x^*) = \emptyset$. As $|A(x^*)| + \frac{1}{2}|D(x^*)| \leq 5.5$, it easily follows that $\phi(x^*) = 3|A(x^*)| + |D(x^*)| \leq 16$. The complicated notion introduced in this paper is to reduce $\phi(x) \leq 16$ to $\phi(x) \leq 15$ for all $x \in B$, which then reduces the upper bound for $\mathrm{col_g}(\mathcal{P})$ from 18 to 17.

The remaining of this section is devoted to the proof of Theorem 3. For each vertex $x \in V(R)$, let

$$p(x) = |N_R(x) \cap U|$$

$$q(x) = |N_R(x) \cap C|.$$

As C is an independent set, if $x \in C$, then $q(x) = 0$, if $x \in U$, then $p(x) \geq q(x)$.

Assume $u \in U$. If $p(u) \leq 3$ or $p(u) = 4$ and $q(u) \leq 3$ or $p(u) = 5$ and $q(u) = 0$, then $3p(u) + q(u) \leq 15$. Let $B = \{u\}$, Q is the trivial digraph containing only one vertex, and $\rho(y) = \emptyset$ for all $y \in C$. Then it is straightforward to verify that (B, ρ, Q) is a valid candidate for (R, C, U).

In the following, we assume the following

Assumption A *For every $u \in U$, $p(u) \geq 4$. Moreover, if $p(u) = 4$, then $q(u) = 4$; if $p(u) = 5$, then $q(u) \geq 1$.*

Definition 3 *Suppose $x \in U$ and $z \in C$ and $x \sim_R z$. We say x and z are* minor neighbours *of each other if $p(x) = 5$ and $q(x) = 1$.*

Definition 4 *Suppose $x \in U$, $z \in C$, $x \sim_R z$ and $p(z) = 4$. Let the other three neighbours of z be u_1, u_2, u_3. We say x and z are* major neighbours *of each other if one of the following holds:*

1. *$p(x) \geq 6$.*
2. *$p(x) = 5$, $3 \leq q(x) \leq 5$ and two of the u_i's are minor neighbours of z.*
3. *$p(x) = 5$, $4 \leq q(x) \leq 5$, one of the u_i's, say u_1, is a minor neighbour of z and moreover, $p(u_2), p(u_3) \leq 5$ and $q(u_2), q(u_3) \leq q(x)$.*

We denote by $n_{\text{minor}}(x)$ and $n_{\text{major}}(x)$ the number of minor neighbours of x and the number of major neighbours of x, respectively.

Lemma 2 *Let (R, U, C) be a plane triangulation as in Theorem 3. Then one of the following holds:*

1. *There is a vertex $x \in U$ with $p(x) = 5$, $q(x) \geq 3$ and $n_{\text{major}}(x) \geq q(x) - 1$.*
2. *There is a vertex $x \in C$ with $p(x) = 4$, and $n_{\text{minor}}(x) > n_{\text{major}}(x)$.*
3. *There is a vertex $x \in C$ with $p(x) = 5$ and $n_{\text{minor}}(x) \geq 4$.*

Proof. Charge each vertex $v \in V(R)$ with a charge $c_0(v) = d_R(v)$. We redistribute the charges according to the following rules:

Suppose $x \in U$ and $z \in C$ and $x \sim_R z$. If x, z are major neighbours of each other, then move a charge of 1 from x to z. If x, z are neither major neighbours nor minor neighbours of each other, then move a charge of $1/2$ from x to z. If x is a minor neighbour of z, then no charge is moved from x to z.

Denote by c^* the new charge assignment. Since $\sum_{x\in V(R)} c^*(x) = \sum_{x\in V(R)} c_0(x) = 6|V(R)| - 12$, there is a vertex x^* with $c^*(x^*) < 6$.

It follows easily from the discharging rule that if $x \in U$, then

$$c^*(x) = p(x) + \frac{1}{2}(q(x) + n_{\text{minor}}(x) - n_{\text{major}}(x)).$$

If $x \in C$, then

$$c^*(x) = \frac{3}{2}p(x) + \frac{1}{2}(n_{\text{major}}(x) - n_{\text{minor}}(x)).$$

First we consider the case that $x^* \in U$. Since $c^*(x^*) \geq p(x^*)$, we have $p(x^*) \leq 5$. If $p(x^*) = 4$, then by Assumption A, $q(x) = 4$ and hence $c_0(x) = 8$. Each neighbour of x in C receives a charge of $1/2$ from x. So the total charge sent out from x is 2. Hence $c^*(x) = 6$, contrary to our assumption.

Thus we have $p(x^*) = 5$. By Assumption A, $q(x^*) \geq 1$. If $q(x^*) = 1$, then x^* has only one minor neighbour in C, and hence $c^*(x^*) = 6$, contrary to our assumption. If $q(x) = 2$, then x has no major neighbour in C, and hence $c^*(x^*) = 7 - 1 = 6$, contrary to our assumption. Thus we assume $q(x^*) \geq 3$. Then we have

$$6 > c^*(x^*) = p(x^*) + \frac{1}{2}(q(x^*) + n_{\text{minor}}(x^*) - n_{\text{major}}(x^*)) \geq 5 + \frac{1}{2}(q(x^*) - n_{\text{major}}(x^*))$$

which implies that $n_{\text{major}}(x^*) \geq q(x^*) - 1$. So (1) holds.

Next we consider the case that $x^* \in C$. If $p(x^*) = 4$, then

$$6 > c^*(x^*) = 6 + \frac{1}{2}(n_{\text{major}}(x^*) - n_{\text{minor}}(x^*)).$$

Hence $n_{\text{minor}}(x^*) > n_{\text{major}}(x^*)$, and (2) holds.

If $p(x^*) = 5$, then

$$6 > c^*(x^*) \geq \frac{3}{2}p(x^*) - \frac{1}{2}n_{\text{minor}}(x^*).$$

Hence $n_{\text{minor}}(x^*) \geq 4$, and (3) holds. ∎

In the remainder of the paper, we shall explicitly construct a valid candidate in each of the cases stated in Lemma 2. As the argument is still long, we divide it into a few lemmas.

Lemma 3 *If there is a vertex $z \in C$ with $p(z) = 4$ and with $n_{\text{minor}}(z) > n_{\text{major}}(z)$, then there exists a valid candidate.*

Proof. Case 1 $n_{\text{minor}}(z) = 2$ and $n_{\text{major}}(z) \leq 1$.

Assume y_1, y_2 are two minor neighbours of z. Let u_1, u_2 be the other two neighbours of z and assume u_1 is not a major neighbour of z. Then $p(u_1) \leq 5$. Depending on whether y_1 and y_2 are adjacent or not, we have two cases as depicted in Figure 1. Since y_1 is a minor neighbour of z, all the other neighbours of y_1 are in U. In y_1 and u_1 have a common neighour in U. Hence $q(u_1) \leq p(u_1) - 1$. By Assumption A and the definition of major neighbour, we conclude that $p(u_1) = 5$ and $q(u_1) \leq 2$.

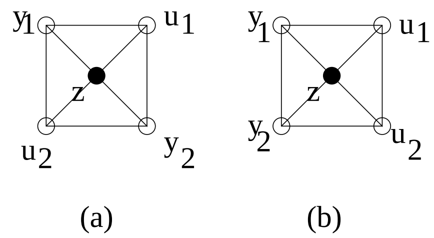

Figure 1: A vertex $z \in C$ with two minor neighbours

(In all the figures of this paper, a filled circle is a vertex of C, and an unfilled circle is a vertex of U.)

First we consider the case that $y_1 \not\sim_R y_2$, as depicted in Figure 1 (a). Let $B = \{u_1, y_1, y_2\}$, let Q be the digraph which consists of arcs $\overrightarrow{y_2u_1}, \overrightarrow{u_1y_1}$, and let $\rho(z) = \{y_2\}$ and $\rho(y) = \emptyset$ for $y \in C \setminus \{z\}$. The digraph Q and the mapping ρ are as depicted in Figure 2 (a). Note that z is not a vertex of Q. We put a dotted line from z to y_2 to indicate that $\rho(z) = \{y_2\}$. We claim that (B, ρ, Q) is a valid candidate.

[C1]: We need to show that no two arcs of Q are contained in a facial triangle of R. Assume $\overrightarrow{y_2u_1}, \overrightarrow{u_1y_1}$ is contained in a facial triangle. Then $N_R(u_1) \cap U = \{y_1, y_2\}$, i.e., $p(u_1) = 2$, contrary to Assumption A.

For [C2], [C3], [C4], it suffices to consider $z \in C$ and its neighbours (as $\rho(y) = \emptyset$ for $y \in C \setminus \{z\}$). The verification is straightforward (by referring to Figure 2 (a)) and is left to the readers. The following table verifies [C5] for each vertex v of B.

v	$3\|A(v)\|$	$2\|B(v)\|$	$\|C(v)\|$	$\|D(v)\|$	$\tau(v)$	$\phi(v)$
y_1	12	0	1	1	1	15
y_2	12	2	0	0	1	15
u_1	9	2	1	2	1	15

The numbers in the table are upper bounds for the corresponding parameters. For example, the number 1 at row y_1 and column $|D(v)|$ means that $|D(y_1)| \leq 1$.

We verify this table for u_1 and y_2. We have $|A(u_1)| = p(u_1) - |N_Q(u_1)| = 5 - 2 = 3$, so $3|A(u_1)| = 9$. As $N_Q^+(u_1) = B(u_1) = \{y_1\}$, we have $2|B(u_1)| = 2$. As $N_Q^-(u_1) = C(u_1) = \{y_2\}$, we have $|C(u_1)| = 1$. By definition, as $\rho^{-1}(u_1) = \emptyset$, $|D(u_1)| = q(u_1) \leq 2$.

As $C(u_1) \neq \emptyset$ we have $\tau(u_1) = 1$. Therefore $\phi(u_1) \leq 9 + 2 + 1 + 2 + 1 = 15$. Now we consider y_2. Similarly $|A(y_2)| = p(y_2) - |N_Q(y_2)| = 5 - 1 = 4$. From Figure 2 (a), we see that $B(y_2) = N_Q^+(y_2) = \{u_1\}$ and $C(y_2) = N_Q^-(y_2) = \emptyset$. So $2|B(y_2)| = 2$ and $|C(y_2)| = 0$. Since $\rho^{-1}(y_2) = \{z\} = N_R(y_2) \cap C$, we have $D(y_2) = \emptyset$. Therefore $|D(y_2)| = 0$. As $B(y_2) \neq \emptyset$, we have $\tau(y_2) = 1$. Therefore $\phi(y_2) = 12+2+0+0+1 = 15$.

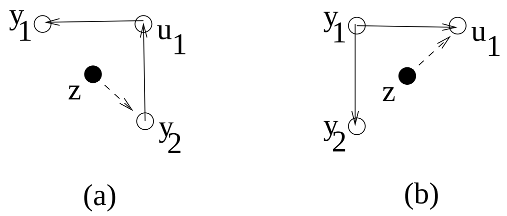

Figure 2: Digraphs in the proof of Case 1 of Lemma 3

Next assume that y_1 and y_2 are adjacent, as depicted in Figure 1 (b). Let $B = \{u_1, y_1, y_2\}$, let Q be the digraph which consists of arcs $\overrightarrow{y_1u_1}, \overrightarrow{y_1y_2}$, and let $\rho(z) = \{u_1\}$ and $\rho(y) = \emptyset$ for $y \in C \setminus \{z\}$. The digraph Q and the mapping ρ are as depicted in Figure 2 (b). We claim that (B, ρ, Q) is a valid candidate.

[C1] [C2], [C3], [C4] are easily verified as in the previous case. The following table verifies [C5].

v	$3\|A(v)\|$	$2\|B(v)\|$	$\|C(v)\|$	$\|D(v)\|$	$\tau(v)$	$\phi(v)$
y_1	9	4	0	1	1	15
y_2	12	0	1	1	1	15
u_1	12	0	1	1	1	15

Case 2 $n_{\text{minor}}(z) = 1$ and $n_{\text{major}}(z) = 0$.

Assume z has one minor neighbour u_1. Let the other neighbours of z be u_2, u_3, u_4 so that (u_1, u_2, u_3, u_4) is a 4-cycle in R. For each $i \in \{2, 3, 4\}$, as u_i is not a major neighbour, we have $p(u_i) \leq 5$, and if $p(u_i) = 5$ then $q(u_i) \leq 3$. Let $B = \{u_1, u_2, u_3, u_4\}$, let Q be the digraph with arcs $\overrightarrow{u_1u_2}, \overrightarrow{u_1u_4}, \overrightarrow{u_2u_3}, \overrightarrow{u_3u_4}$. Let $\rho(z) = \{u_2, u_3\}$ and $\rho(y) = \emptyset$ for $y \in C \setminus \{z\}$. The digraph Q and the mapping ρ are as depicted in Figure 3.

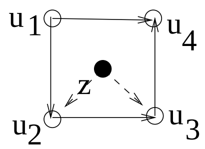

Figure 3: Digraph in the proof of Case 2 of Lemma 3

We claim that (B, ρ, Q) is a valid candidate. Similarly, [C1]-[C4] are easily verified by referring to Figure 3. The following table verifies [C5].

v	$3\|A(v)\|$	$2\|B(v)\|$	$\|C(v)\|$	$\|D(v)\|$	$\tau(v)$	$\phi(v)$
u_1	9	4	0	1	1	15
u_2	9	2	1	2	1	15
u_3	9	2	1	2	1	15
u_4	9	0	2	3	1	15

∎

Lemma 4 *If there is a vertex $z \in C$ with $p(z) = 5$ and with $n_{\text{minor}}(z) \geq 4$, then there exists a valid candidate.*

Proof. The four minor neighbours of z form a path, say $P = (u_1, u_2, u_3, u_4)$, of R. Let $B = \{u_1, u_2, u_3\}$, let Q be the digraph with arcs $\overrightarrow{u_2u_1}, \overrightarrow{u_2u_3}$. Let $\rho(y) = \emptyset$ for $y \in C$. We claim that (B, ρ, Q) is a valid candidate. Similarly, we just list a table to verify [C5].

v	$3\|A(v)\|$	$2\|B(v)\|$	$\|C(v)\|$	$\|D(v)\|$	$\tau(v)$	$\phi(v)$
u_1	12	0	1	1	1	15
u_2	9	4	0	1	1	15
u_3	12	0	1	1	1	15

∎

Lemma 5 *If there is a vertex $x \in U$ with $p(x) = 5$, $q(x) \geq 3$ and $n_{\text{major}}(x) \geq q(x) - 1$, then then there exists a valid candidate.*

Proof. Case 1 $q(x) = 5$.

Assume z_1, z_2, z_3, z_4 are four major neighbours of x, and the neighbours of x and z_i's be as depicted in Figure 4. Since z_i is a major neighbour of x, by definition, $p(z_i) = 4$ and z_i has a minor neighbour.

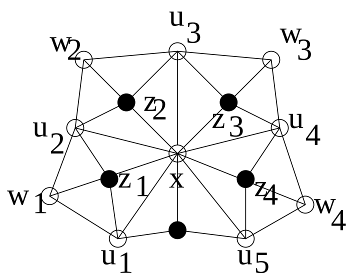

Figure 4: A vertex $x \in U$ with four major neighbours

As $q(u_i) \geq 2$, for $i = 1, 2, 3, 4$, w_i is the only minor neighbour of z_i. Therefore x, z_i is a pair of vertices satisfying Condition 3 of Definition 4, implying that $p(u_i) \leq 5$ for $i = 1, 2, \cdots, 5$. Also because w_i is a minor neighbour of z_i, we have $q(w_i) = 1$ for $i = 1, 2, 3, 4$. This implies that for $j = 2, 3, 4$, u_j and w_j have a common neighbour in U, and u_j and w_{j+1} have a common neighour in U. As a consequence, $q(u_j) \leq p(u_j)-2 \leq 3$ for $j = 2, 3, 4$. By Assumption A, $p(u_j) = 5$ for $j = 2, 3, 4$.

Let $B = \{w_1, u_2, w_2, u_3, w_3, x\}$, let Q be the digraph with arcs $\overrightarrow{u_2w_1}, \overrightarrow{w_2u_2}, \overrightarrow{w_2u_3}, \overrightarrow{u_3w_3}, \overrightarrow{u_2x}, \overrightarrow{u_3x}$. Let $\rho(z_1) = \rho(z_2) = \rho(z_3) = \{x\}$ and $\rho(y) = \emptyset$ for $y \in C \setminus \{z_1, z_2, z_3\}$. The digraph Q and the mapping ρ are as depicted in Figure 5. We claim that (B, ρ, Q) is a valid candidate. Similarly, we just list a table to verify

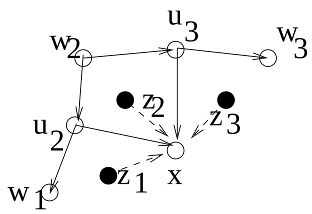

Figure 5: Digraph in the proof of Case 1 of Lemma 5

[C5].

v	$3\lvert A(v)\rvert$	$2\lvert B(v)\rvert$	$\lvert C(v)\rvert$	$\lvert D(v)\rvert$	$\tau(v)$	$\phi(v)$
w_1	12	0	1	1	1	15
u_2	6	4	1	3	1	15
w_2	9	4	0	1	1	15
u_3	6	4	1	3	1	15
w_3	12	0	1	1	1	15
x	9	0	2	2	1	14

Case 2 $q(x) = 4$.

As $n_{\text{major}}(x) \geq 3$, two of the major neighbours, say z_1, z_2, are as depicted in Figure 6. By definition, each of z_1, z_2 has at least one minor neighbour. Since $q(u_2) \geq 2$, u_2 is not a minor neighbour of z_1 or z_2. Since $q(x) = 4$, for some $j \in \{1, 3\}$, the common neighbour of u_j and x not shown in Figure 6 is a vertex in C. Without loss of generality, assume the common neighbour of u_3 and x not shown in Figure 6 is a vertex in C. Hence $q(u_3) \geq 2$ and u_3 is not a minor neighbour of z_2. Thus z_2 has a unique minor neighbour w_2. By Definition 4, this implies that $p(u_2) \leq 5$ and $q(u_2) \leq q(x) = 4$.

Assume first that w_1 is a minor neighbour of z_1.

By Definition 4, $p(u_i) \leq 5$ and $q(u_i) \leq q(x) = 4$ for $i = 1, 3$.

Let $B = \{u_1, w_1, u_2, w_2, u_3, x\}$, let Q be the digraphs with arcs $\overrightarrow{w_1u_1}, \overrightarrow{w_1u_2}, \overrightarrow{xu_1}, \overrightarrow{w_2u_2}, \overrightarrow{w_2u_3}, \overrightarrow{xu_3}, \overrightarrow{xu_2}$. Let $\rho(z_1) = \{x, u_1\}$, $\rho(z_2) = \{x, u_3\}$ and $\rho(y) = \emptyset$ for $y \in C \setminus \{z_1, z_2\}$. The digraph Q and the mapping ρ are as depicted in Figure 7 (a). We claim that (B, ρ, Q) is a valid candidate. [C1]-[C4] can be verified easily, by referring to Figure 7 (a). The following table verifies [C5].

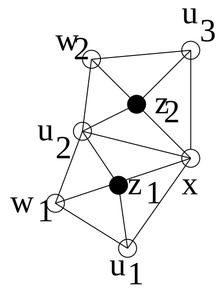

Figure 6: A vertex $x \in U$ with major neighbours z_1, z_2

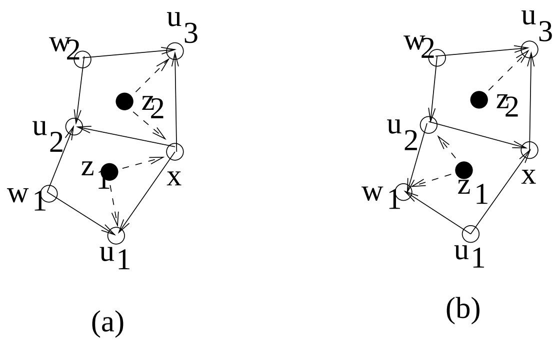

Figure 7: Digraph in the proof of Case 2 of Lemma 5

v	$3\|A(v)\|$	$2\|B(v)\|$	$\|C(v)\|$	$\|D(v)\|$	$\tau(v)$	$\phi(v)$
u_1	9	0	2	3	1	15
w_1	9	4	0	1	1	15
u_2	6	0	3	4	1	14
w_2	9	4	0	1	1	15
u_3	9	0	2	3	1	15
x	6	6	0	2	1	15

Assume w_1 is not a minor neighbour of z_1. Then u_1 is a minor neighbour of z_1. By Definition 4, $p(u_2), p(u_3), p(w_1) \leq 5$, $q(u_2), q(u_3), q(w_1) \leq 4$.

Let $B = \{u_1, w_1, u_2, w_2, u_3, x\}$, let Q be the digraphs with arcs $\overrightarrow{u_1w_1}, \overrightarrow{u_2w_1}, \overrightarrow{u_1x}, \overrightarrow{w_2u_2}, \overrightarrow{w_2u_3}, \overrightarrow{xu_3}, \overrightarrow{u_2x}$. Let $\rho(z_1) = \{w_1, u_2\}$, $\rho(z_2) = \{u_3\}$ and $\rho(y) = \emptyset$ for $y \in C \setminus \{z_1, z_2\}$. The digraph Q and the mapping ρ are as depicted in Figure 7 (b).

We claim that (B, ρ, Q) is a valid candidate. [C1]-[C4] can be verified easily, by referring to Figure 7 (b). The following table verifies [C5].

v	$3\|A(v)\|$	$2\|B(v)\|$	$\|C(v)\|$	$\|D(v)\|$	$\tau(v)$	$\phi(v)$
u_1	9	4	0	1	1	15
w_1	9	0	2	3	1	15
u_2	6	4	1	3	1	15
w_2	9	4	0	1	1	15
u_3	9	0	2	3	1	15
x	6	2	2	4	1	15

Case 3 $q(x) = 3$.

Assume z_1, z_2 are two major neighbours of x. By Definition 4, each z_i has $p(z_i) = 4$ and has two minor neighbours. If z_1 and z_2 are as depicted in Figure 6, then u_1, w_1 are minor neighbours of z_1 and w_2, u_3 are minor neighbours of z_2. Let $B = \{w_1, u_1, x, u_3, w_2\}$, let Q be the digraphs with arcs $\overrightarrow{u_1w_1}, \overrightarrow{u_1x}, \overrightarrow{u_3w_2}, \overrightarrow{u_3x}$, and let $\rho(y) = \emptyset$ for all $y \in C$. It is easy to verify that (B, ρ, Q) is a valid candidate.

Otherwise, let z_3 be the other neighbour of x in C. The relative position of z_1, z_2, z_3 are as depicted in Figure 8 (a) or (b).

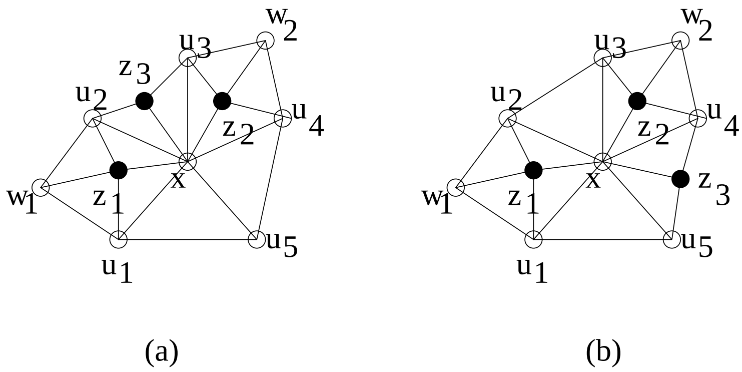

Figure 8: A vertex $x \in U$ with major neighbours z_1, z_2

Subcase 1 This case is as depicted in Figure 8 (a). As $q(u_2) \geq 2$ and $q(u_3) \geq 2$, we conclude that u_1, w_1 are minor neighbours of z_1 and w_2, u_4 are minor neighbours of z_2. Let $B = \{u_1, w_1, w_2, u_4, x\}$, let Q be the digraph with arcs $\overrightarrow{u_1w_1}, \overrightarrow{u_1x}, \overrightarrow{u_4w_2}, \overrightarrow{u_4x}$. Let $\rho(y) = \emptyset$ for $y \in C$. The digraph Q and the mapping ρ are as depicted in Figure 9 (a). We claim that (B, ρ, Q) is a valid candidate. Below is a table to verify [C5].

v	$3\|A(v)\|$	$2\|B(v)\|$	$\|C(v)\|$	$\|D(v)\|$	$\tau(v)$	$\phi(v)$
w_1	12	0	1	1	1	15
u_1	9	4	0	1	1	15
w_2	12	0	1	1	1	15
u_4	9	4	0	1	1	15
x	9	0	2	3	1	15

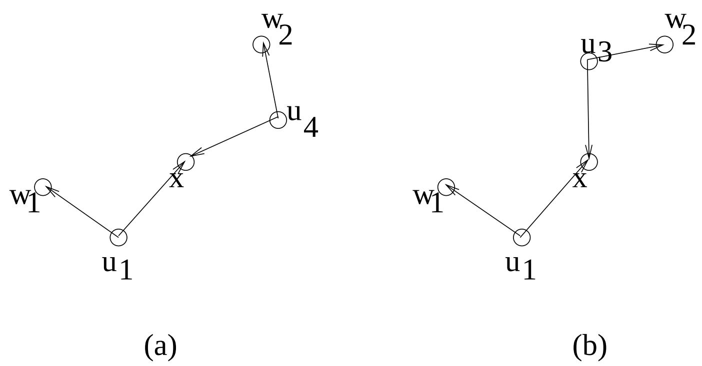

Figure 9: Digraph in the proof of Case 3 of Lemma 5

Subcase 2 This case is as depicted in Figure 8 (b). As $q(u_4) \geq 2$, we conclude that u_3, w_2 are minor neighbours of z_2. If u_2 is a minor neighbour of z_1, then let $B = \{u_2, u_3, w_2\}$, let Q be the digraph with arcs $\overrightarrow{u_3u_2}, \overrightarrow{u_3w_2}$. Let $\rho(y) = \emptyset$ for $y \in C$. Then (B, ρ, Q) is a valid candidate. Otherwise, w_1, u_1 are minor neighbours of z_1. Let $B = \{u_1, w_1, u_3, w_2, x\}$, let Q be the digraph with arcs $\overrightarrow{u_1w_1}, \overrightarrow{u_1x}, \overrightarrow{u_3w_2}, \overrightarrow{u_3x}$. Let $\rho(y) = \emptyset$ for $y \in C$. The digraph Q and the mapping ρ are as depicted in Figure 9 (b). Then (B, ρ, Q) is a valid candidate. The verifications are similar as above and omitted. ∎

Theorem 3 follows from Lemmas 2, 3, 4 and 5.

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