Distinguishing labeling of group actions

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Abstract

Suppose Γ is a group acting on a set X. An r-labeling $f : X \to \{1, 2, \ldots, r\}$ of X is distinguishing (with respect to the action of Γ) if for any $\sigma \in \Gamma, \sigma \neq \mathrm{id}_X$, there exists an element $x \in X$ such that $f(x) \neq f(\sigma(x))$. The distinguishing number, $D_\Gamma(X)$, of the action of Γ on X is the minimum r for which there is an r-labeling which is distinguishing. This paper investigates the relation between the cardinality of a set X and the distinguishing numbers of group actions on X. For a positive integer n, let $D(n) = \{D_\Gamma(X) : |X| = n \text{ and } \Gamma \text{ acts transitively on } X\}$. We prove that $|D(n)| = O(\sqrt{n})$. Then we consider the problem of an arbitrary fixed group Γ acting on a large set. We prove that if each proper normal subgroup H of Γ has $\bar{D}(H) \leq 2$, then there is an integer n such that for any set X with $|X| \geq n$, for any action of Γ on X with no fixed points, $D_\Gamma(X) \leq 2$.

Key words: Distinguishing number; Distinguishing set of group actions; Symmetric groups; Group actions, Graphs.

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1 Introduction

Distinguishing labeling was first defined by Albertson and Collins [1] for graphs. A labeling of a graph G, $f : V(G) \to \{1, 2, \cdots, r\}$, is said to be *r-distinguishing* if no non-trivial automorphism of G preserves all the vertex labels. In other words, f is r-distinguishing if for any $\sigma \in Aut(G)$, $\sigma \neq id$, there is a vertex x such that $f(x) \neq f(\sigma(x))$. The *distinguishing number* of a graph G is defined as

$$D(G) = \min\{r : \text{ there exists an } r\text{-distinguishing labeling of } G\}.$$

Distinguishing labeling can be naturally extended to general group actions [21]. Let Γ be a group acting on a set X. For a positive integer r, an r-labeling $f : X \to \{1, 2, \ldots, r\}$

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of X said to be *r-distinguishing with respect to the action of* Γ if for any $\sigma \in \Gamma$, if σ is not the identity, then there is an element $x \in X$ such that $f(x) \neq f(\sigma(x))$. The *distinguishing number* $D_\Gamma(X)$ of the action of Γ on X is defined as

$$D_\Gamma(X) = \min\{r: \text{ there exists an } r\text{-distinguishing labeling}\}.$$

The distinguishing number of graphs and group actions have been studied in [1, 2, 3, 6, 7, 8, 9, 10, 11, 12, 13, 19, 21]. It was proved in [1, 21] that if Γ is a nontrivial Abelian group then $D_\Gamma(X) = 2$ for any action of Γ on a set X, if Γ is a dihedral group, then $D_\Gamma(X) \leq 3$ for any action of Γ on a set X. The result was generalized in [7], where it was proved that if Γ is a nilpotent of class c or supersolvable of length c then $D_\Gamma(X) \leq c+1$ for any action of Γ on a set X. It was proved in [21] that for any group Γ, if $|\Gamma| < (k+1)!$ then $D_\Gamma(X) \leq k$ for any action of Γ on a set X. It was conjectured in [23] that for any action of S_n on a set X, $D_{S_n}(X) = \lceil n^{1/k} \rceil$ or $\lceil (n-1)^{1/k} \rceil$ for some positive integer k, and the conjecture was proved to be true [23] for almost all n.

All the results above explore the relation between the distinguishing number $D_\Gamma(X)$ and the structure of the group Γ. In this paper, we are interested in a different problem: How can the cardinality of X affects the distinguishing number? Assume that $X = [n] = \{1, 2, \cdots, n\}$. It is obvious that $D_\Gamma([n]) \leq n$, as the labeling which assigns to each element of $[n]$ a distinct label is certainly distinguishing. So for a nontrivial group Γ action on $[n]$, $2 \leq D_\Gamma([n]) \leq n$. Now given any positive integer $2 \leq k \leq n$, is it possible to find an action of a group Γ on $[n]$ with $D_\Gamma([n]) = k$? The answer is yes. For any $2 \leq k \leq n$, the subgroup of S_n which fixes each of $k+1, k+2, \cdots, n$ and whose action on $\{1, 2, \cdots, k\}$ is isomorphic to S_k has distinguishing number k. However, this answer is not very convincing, because such an action is basically an action on the set $\{1, 2, \cdots, k\}$. Is there an action Γ on $[n]$ which is transitive on $[n]$ and has $D_\Gamma([n]) = k$? More precisely, for a positive integer n, let

$$D(n) = \{D_\Gamma([n]) : \Gamma \text{ is a transitive subgroup of } S_n\}.$$

The question is to determine $D(n)$. This question was asked by Chan [6, 7]. In this paper, we shall prove that $|D(n)| = O(\sqrt{n})$, which means that most of the integers in the interval $[2, n]$ do not belong to $D(n)$. On the other hand, for infinitely many n, we have $|D(n)| \geq \log_2 n$.

The next question we are interested in also concerns the relation between the distinguishing number and the cardinality of the set X. Given a group Γ. Is there an integer $n = n(\Gamma)$ such that for any set X with $|X| \geq n$ and for any action of Γ on X, we have $D_\Gamma(X) \leq 2$? For a group Γ, Chan [6] defined the *maximum distinguishing number* of Γ, denoted by $\bar{D}(\Gamma)$, as follows:

$$\bar{D}(\Gamma) = \max\{D_\Gamma(X) : \Gamma \text{ acts on } X\}.$$

If $\bar{D}(\Gamma) \leq 2$, then of course the answer to the above question is yes. Indeed, in this case we have $n(\Gamma) = 1$.

If $\bar{D}(\Gamma) > 2$, then the answer is trivially 'no', as one may take an action of Γ on a set X with $D_\Gamma(X) > 2$, and then add arbitrarily many elements x that are fixed by all the group elements, i.e., $\sigma(x) = x$ for all $\sigma \in \Gamma$. The addition of such elements does not change the distinguishing number. To make the problem interesting, one needs to avoid such 'trivial' actions. Suppose O is an orbit of the action of Γ on X. Let H be the point-wise stabilizer of O. Then H is a normal subgroup of Γ. We say O is a restrictive orbit if for any action of H on a set A which can be extended to Γ, we have $D_H(A) \leq 2$. We shall prove that for any group Γ, there is an integer $n = n(\Gamma)$ such that for any action of Γ on X, if the union of the restrictive orbits has cardinality at least n, then $D_\Gamma(X) = 2$. We say two orbits O_1, O_2 of the action of Γ on X are equivalent, written as $O_1 \approx O_2$, if there exist $x \in O_1$ and $y \in O_2$, we have $stab_x = stab_y$, where $stab_x$ and $stab_y$ are the stabilizers of x and y, respectively. The relation "$\approx$" is an equivalence relation. For an orbit O, let $[O]$ be the equivalence class of orbits containing O. We shall prove that if for each orbit O, $2^{|[O]|} \geq |O|$, then $D_\Gamma(X) = 2$.

2 Bounding the size of $D(n)$

This section discusses the distinguishing number of transitive actions on $[n]$. Assume Γ acts transitively on $[n]$. Then Γ is either a primitive subgroup of S_n or an imprimitive subgroup of S_n. In the latter case, it is known (see [17]) that we can write $[n]$ as the Cartesian product of $[k]$ and $[m]$, and $\Gamma = K \wr H$ is the wreath product of K and H, where K is a primitive subgroup of S_k and H is a transitive subgroup of S_m. Recall that the wreath product $K \wr H$ has elements $\{(f,h) : f$ is a map sending $j \in [m]$ to $f_j \in K$ and $h \in H\}$, and the action on $[k] \times [m]$ is defined as for $(i,j) \in [k] \times [m]$, $(f,h)(i,j) = (f_j(i), h(j))$. The sets $B_j = \{(i,j) : i \in [k]\}$ are called blocks. Thus the element $(f,h) \in K \wr H$ maps the block B_j to the block $B_{h(j)}$.

Lemma 2.1 *Suppose $n = km$ and $k, m \geq 2$. For a subgroup K of S_k and a transitive subgroup H of S_m, we consider the action of $K \wr H$ on $[n]$ which is viewed as the Cartesian product of $[k]$ and $[m]$. If $k > m$, then for any subgroup H of S_m, $D_{S_k \wr H}([n]) = k+1$ and $D_{A_k \wr H}([n]) = k-1$. For any subgroup K of S_k and any subgroup H of S_m, if $D_K([k]) = t$ and $\binom{r}{t} \geq D_H([m])$, then $D_{K \wr H}([n]) \leq r$.*

Proof. The distinguishing number of the action of the wreath product of groups is studied in [6]. Lemma 2.1 follows from results in [6]. However, we include here a short direct proof. Assume $k > m$. If $K = S_k$, then let ϕ be the $(k+1)$-labeling of $[k] \times [m]$ which labels the k elements of block B_j with the k distinct labels in $\{1, 2, \cdots, k+1\} \setminus \{j\}$. We claim that ϕ a distinguishing labeling. Indeed, if (f,h) preserves the labeling ϕ, then since $\phi(B_j) = \phi(B_{h(j)})$ and distinct blocks use different label sets, we conclude that $h(j) = j$. Thus $(f,h)(i,j) = (f_j(i), j)$. But distinct elements in B_j are labeled by distinct labels. So $f_j(i) = i$, and hence $(f,h) = id_{[k]\times[m]}$. On the other hand, since $K = S_k$, a distinguishing labeling of $[k] \times [m]$ must label each block with k distinct labels, and there must be two blocks that use different label sets. Therefore

$D_{S_k \wr H}([n]) = k + 1$. If $K = A_n$, then let ϕ be the $(k-1)$-labeling of $[k] \times [m]$ which labels the k elements of block B_j with the $k-1$ labels in $\{1, 2, \cdots, k-1\}$ such that label j is used twice, and every other label is used once. Similarly as above, it is easy to see that this is a distinguishing labeling. On the hand, a distinguishing labeling of a single block already needs $k-1$ labels. So $D_{A_k \wr H}([n]) = k - 1$.

Suppose K is a subgroup of S_k with $D_K([k]) = t$ and r is an integer with $\binom{r}{t} \geq D_H([m])$. We define an r-labeling of $[k] \times [m]$ as follows: Let $\binom{[r]}{t}$ be the family of t-subsets of the r labels. Let $\phi : [m] \to \binom{[r]}{t}$ be a distinguishing labeling of the action of H on $[m]$. Now for each block B_j, use the t labels in the t-subset $\phi(j)$ to label the elements of B_j in such a way that it is a distinguishing labeling of $[k]$ with respect to the action of K. Now if $(f, h) \in K \wr H$ preserves the labeling ϕ, then since $\phi(B_j) = \phi(B_{h(j)})$, we conclude that $h(j) = j$. Within the block B_j, since the labeling is distinguishing with respect to the action of K, we have f_j is the identity mapping. So ϕ is a distinguishing labeling and hence $D_{K \wr H}([n]) \leq r$.

The main result of this section is the following:

Theorem 2.2 *Let $D(n) = \{D_\Gamma([n]) : \Gamma$ is a transitive subgroup of $S_n\}$. Then*

$$D(n) = O(\sqrt{n}).$$

Proof. Let $\mathcal{F}_1 = \{\Gamma : \Gamma$ is a transitive, primitive subgroup of $S_n\}$, $\mathcal{F}_2 = \{\Gamma : \Gamma$ is a transitive, imprimitive subgroup of $S_n\}$. It suffices to show that $|\{D_\Gamma([n]) : \Gamma \in \mathcal{F}_i\}| = O(\sqrt{n})$ for $i = 1, 2$.

Assume first that $\Gamma \in \mathcal{F}_1$. If $\Gamma \neq S_n, A_n$, then it is known [16] that $|\Gamma| \leq 50n^{\sqrt{n}}$. It is proved in [21] that if $|H| \leq k!$ then $D_H(Z) \leq k$ for any action of H on any set Z. Therefore $D_\Gamma(X) \leq 5\sqrt{n}$. Therefore $|\{D_\Gamma([n]) : \Gamma \in \mathcal{F}_1\}| = O(\sqrt{n})$.

Assume Γ is an imprimitive subgroup of S_n. Let $\mathcal{F}_2' = \{\Gamma \in \mathcal{F}_2 : \Gamma = K \wr H, K = S_k$, or $K = A_k$, and $k > \sqrt{n}$. By Lemma 2.1, if $K = S_k$, then $D_\Gamma([n]) = k + 1$; if $K = A_k$, then $D_\Gamma(X) = k - 1$. Let $Q = \{k : k$ is a factor of n and $k > \sqrt{n}\}$. Then

$$|\{D_\Gamma(X) : \Gamma \in \mathcal{F}_2'\}| = 2|Q| < \sqrt{n}.$$

If $\Gamma = K \wr H \in \mathcal{F}_2''$, then $|K| \leq 50n^{\sqrt{n}}$ and hence $D_K([k]) \leq 5\sqrt{n}$. Suppose $D_K([k]) = t$. Then for $r = t + 2\sqrt{n}$, we have $\binom{r}{t} \geq n > D_H([m])$.

By Lemma 2.1, $D_\Gamma([n]) \leq r \leq 7\sqrt{n}$, and hence

$$|\{D_\Gamma(X) : \Gamma \in \mathcal{F}_2''\}| = O(\sqrt{n}).$$

This completes the proof of Theorem 2.2

We do not know if the upper bound given in Theorem 2.2 is of the correct order. In the following we show that for infinitely many n, $|D(n)| \geq \log_2 n$.

Lemma 2.3 *If k is a factor of n, $k < n$ and $k^2 > n$, then $k - 1, k + 1 \in D(n)$.*

Proof. Suppose $n = km$ and $k > m$. Let $\Gamma_1 = S_k \wr S_m$ and $\Gamma_2 = A_k \wr A_m$. Since S_m and A_m are transitive subgroups of S_n, we have $D_{\Gamma_1}([n]) = k+1$ and $D_{\Gamma_2}([n]) = k-1$ by Lemma 2.1. Therefore $k-1, k+1 \in D(n)$.

If $n = 2^m$ and m is odd, then n has $\frac{m-1}{2}$ factors k with $k < n$ and $k^2 > n$ and hence $|D(n)| \geq m - 1 = \log_2 n - 1$. As $2 \in D(n)$, if $m \geq 5$, we have $|D(n)| \geq m = \log_2 n$.

3 A group acting on a large set

Tucker and Condor [20] have shown that there are only a finite number of 3-connected planar graphs G with $D(G) > 2$. Thus if G is a sufficiently large 3-connected planar graph then $D(G) \leq 2$. This leads to the following question:

Suppose Γ is a group which acts on a sufficiently large set X. Under what condition we have $D_\Gamma(X) \leq 2$?

One necessary condition is that Γ should not have too many fixed points. Indeed, if $x \in X$ is a fixed point of Γ, then let $X' = X \setminus \{x\}$, it is obvious that $D_\Gamma(X) = D_\Gamma(X')$. Thus we may assume that Γ has no fixed points. However, for some groups Γ, having no fixed points and X large enough still do not imply that $D_\Gamma(X) \leq 2$. For example, consider the action of S_n on $[n] = \{1, 2, \cdots, n\}$. We extend the action to a large set $X = [n] \cup_{i=1}^k \{a_{i,0}, a_{i,1}\}$ (where k is a large integer) as follows: If σ is even, then $\sigma(a_{i,j}) = a_{i,j}$; if σ is odd, then $\sigma(a_{i,j}) = a_{i,1-j}$. It is obvious that $D_{S_n}(X) = n-1$. The problem in this example is that the "large part" of the set X are fixed by the subgroup A_n of S_n.

Suppose O is an orbit of the action of Γ on X. Let $H = Stab_x$ be the stabilizer of an element x of O. Then there is a one-to-one correspondence ψ between O and the right cosets of H, defined as $\psi(y) = \sigma H$, where $y \in O$ and $\sigma \in \Gamma$ is a group element with $\sigma(x) = y$. In this sense, we identify each orbit with the right cosets of a subgroup of Γ. We write $\pi(O) = H$ to mean that O is identified with the right cosets of a subgroup of Γ which is conjugate to H. Observe that if $\pi(O_1) = \pi(O_2)$, then there exist $x \in O_1$ and $y \in O_2$ with $stab_x = stab_y$. In this case, we say O_1 and O_2 are equivalent orbits, written as $O_1 \approx O_2$. Let $\mathcal{S}$ be the set of non-conjugate subgroups of Γ. Then an action of Γ on a set X can be characterized by an integer vector $\vec{v} = \{v_H : H \in \mathcal{S}\}$, where v_H is the number of orbits O with $\pi(O) = H$.

For $H \in \mathcal{S}$, consider the action of Γ on the set of right cosets of H, defined as $\sigma(\tau H) = (\sigma\tau)H$. Let $\phi(H)$ be the distinguishing number of this action. Let $K_H = \cap_{\sigma \in \Gamma} \sigma H \sigma^{-1}$. Then K_H is the normal subgroup of Γ consisting of those elements σ that fixes each right coset of H, i.e., $\sigma(\tau H) = \tau H$ for all $\tau \in \Gamma$.

Theorem 3.1 *Suppose H is a subgroup of Γ and d is a positive integer. If $(\phi(H))^{1/v_H} \leq d$, then $D_\Gamma(X) \leq d$ if and only if $D_{K_H}(X) \leq d$.*

Proof. Since K_H is a subgroup of Γ, we have $D_{K_H}(X) \leq D_\Gamma(X)$. Thus to prove this lemma, it suffices to show that if $D_{K_H}(X) \leq d$, then $D_\Gamma(X) \leq d$.

Let $O_1, O_2, \cdots, O_{v_H}$ be the orbits of the action of Γ on X with $\pi(O_i) = H$. Let $\{\sigma_1 H, \sigma_2 H, \cdots, \sigma_m H\}$ be the right cosets of H. Each O_i is identified with the right cosets of H. We may assume that $O_i = \{x_{i1}, x_{i2}, \cdots, x_{im}\}$, where x_{ij} is identified with $\sigma_j H$.

Since $d^{v_H} \geq \phi(H)$, there is a distinguishing labeling g of the cosets of H with labels $\{(i_1, i_2, \cdots, i_{v_H}) : 1 \leq i_j \leq d\}$. Label the vertices of $O_1 \cup O_2 \cup \cdots \cup O_{v_H}$ with labels $\{1, 2, \cdots, d\}$ so that for each $1 \leq j \leq m$, $(f(x_{1j}), f(x_{2j}), \cdots, f(x_{v_H j})) = g(\sigma_j H)$.

Let h be a d-distinguishing labeling of X with respect to the action of K_H. We extend the labeling of f to X so that for $x \in X \setminus (O_1 \cup O_2 \cup \cdots \cup O_{v_H})$, $f(x) = h(x)$. We claim that f is a d-distinguishing labeling of X with respect to Γ. Assume $\sigma \in \Gamma$ is a permutation which preserves the labeling f. Then σ preserves the labeling g. As g is a distinguishing labeling of the cosets of H with respect to the action of Γ, we conclude that $\sigma(\sigma_j H) = \sigma_j H$, i.e., $\sigma \in K_H$. This implies that for $x \in O_1 \cup O_2 \cup \cdots \cup O_{v_H}$, $\sigma(x) = x$, and hence $h(\sigma(x)) = h(x)$. For any $x \in X \setminus (O_1 \cup O_2 \cup \cdots \cup O_{v_H})$, we also have $h(\sigma(x)) = h(x)$, because $h(x) = f(x)$ and σ preserves the labeling f. So $\sigma \in K_H$ preserves the labeling h. As h is a d-distinguishing labeling of X with respect to the action of K_H, we conclude that $\sigma = id$. Thus we have proved that f is a d-distinguishing labeling of X with respect to the action of Γ and hence $D_\Gamma(X) \leq d$.

Suppose O is an orbit of the action of Γ on X with $\pi(O) = H$. If there is an action of K_H on a set A which can be extended to an action of Γ on A and $D_{K_H}(A) \geq 3$, then O is called a *loose orbit*. Otherwise O is called a *restrictive orbit*.

Corollary 3.2 *For an action of Γ on X, if the union of restrictive orbits is sufficiently large, then $D_\Gamma(X) \leq 2$.*

Proof. Since the union of the restrictive orbits is sufficiently large, there is a restrictive orbit O with $\pi(O) = H$ and $2^{v_H} \geq \phi(H)$. Apply Theorem 3.1 to H with $d = 2$, we conclude that $D_\Gamma(X) \leq 2$.

Corollary 3.3 *Suppose Γ is a group such that for any proper normal subgroup H of Γ we have $\bar{D}(H) \leq 2$. Then there is an integer $n = n(\Gamma)$ such that for any set X with $|X| \geq n$, for any action of Γ on X with no fixed points, $D_\Gamma(X) \leq 2$.*

Proof. The assumption implies that all the orbits are restrictive. Thus the conclusion follows from Corollary 3.2.

The following is a special case of Corollary 3.3.

Corollary 3.4 *If Γ is a simple group and X is a sufficiently large set, then for any action of Γ on X without fixed points, $D_\Gamma(X) \leq 2$.*

The condition in Corollary 3.2 is necessary in the following sense: If X is large but the union of restrictive orbits is small, then it is possible that $D_\Gamma(X) > 2$. Let H be a subgroup of Γ and suppose that there is an action of K_H on A which can be extended to an action of Γ on A and $D_{K_H}(A) \geq 3$. Let $O_1, O_2, \cdots, O_m$ be disjoint sets, each O_i is a copy the set of right cosets of H. Extend the action of Γ to $X = A \cup O_1 \cup O_2 \cup \cdots \cup O_m$ so that $\pi(O_i) = H$. Then no matter how big is m, we have $D_\Gamma(X) \geq D_{K_H}(A) \geq 3$.

Theorem 3.5 *If* $(\phi(\pi(O)))^{1/v_{\pi(O)}} \le d$ *for each orbit* O*, then* $D_\Gamma(X) \le d$.

Proof. Suppose O is an orbit of the action of Γ on X. Let $O_1, O_2, \cdots, O_{v_{\pi(O)}}$ be the orbits with $\pi(O_i) = H = \pi(O)$. Let $\{\sigma_1 H, \sigma_2 H, \cdots, \sigma_m H\}$ be the right cosets of H. As in the proof of Theorem 3.1, we may assume that $O_i = \{x_{i1}, x_{i2}, \cdots, x_{im}\}$, where x_{ij} is identified with $\sigma_j H$.

Since $d^{v_H} \ge \phi(H)$, there is a distinguishing labeling g of the cosets of H with labels $\{(i_1, i_2, \cdots, i_{v_H}) : 1 \le i_j \le d\}$. Label the vertices of $O_1 \cup O_2 \cup \cdots \cup O_{v_H}$ with labels $\{1, 2, \cdots, d\}$ so that for each $1 \le j \le m$, $(f(x_{1j}), f(x_{2j}), \cdots, f(x_{v_H j})) = g(\sigma_j H)$. Then any permutation $\sigma \in \Gamma$ which preserves the labeling above must fix each element of the orbits $O_1, O_2, \cdots, O_{v_{\pi(O)}}$. We do the same for each orbit of the action of Γ on X. Then any permutation $\sigma \in \Gamma$ which preserves the labeling above must fix all the elements of X. So the labeling is distinguishing.

Corollary 3.6 *If* $(\phi(\pi(O)))^{1/v_{\pi(O)}} \le 2$ *for each orbit* O*, then* $D_\Gamma(X) = 2$.

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