

Pattern periodic coloring of distance graphs

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Running head: Pattern periodic coloring

Abstract

Suppose D is a subset of Z . The distance graph $G(Z, D)$ with distance set D is the graph with vertex set Z and two vertices x, y are adjacent if $|x - y| \in D$. We introduce a coloring method for distance graphs, the pattern periodic coloring, and we shall compare this method with other general coloring methods of distance graphs.

1 Introduction

Let Z be the set of all integers. For a subset D of Z , the distance graph $G(Z, D)$ is the graph with vertex set Z in which two vertices x, y are adjacent if and only if $|x - y| \in D$. The set D is called the *distance set*. Distance graphs are investigated in [1, 2, 4, 5, 6, 7, 8, 11, 12, 15, 16, 17, 18, 19, 20]. It was motivated by the plane coloring problem: What is the least number of colors which can be used to color all points of the euclidean plane so that vertices of unit distance are colored with distinct colors. This problem is equivalent to determine the chromatic number of the distance graph on the plane R^2 with distance set $D = \{1\}$. It is known that the chromatic number of this distance graph is between 4 and 7 [10, 13], but there is no substantial progress on this problem in the last three decades. Distance graphs are also related to the channel assignment problem (or the T -coloring problem) and problems in number theory. We refer the readers to [2, 12, 20] for discussion of the relations among these problems.

The problem of determining the chromatic numbers of distance graphs on the integers has received much recent attention [1, 2, 4, 5, 11, 12, 15, 16, 17, 18, 20]. The cases that $|D| = 1$ or 2 are easy [1, 17]. The case that $|D| = 3$ is very complicated, and has been completely settled only very recently [20]. The case that D is a subset of primes were discussed in [5, 15, 16], and the problem is still open, although the case that D consists of 4 primes is solved by Voigt and Walther [16]. The case that D is an interval is solved in [5]. The case that D is of the form $\{1, 2, \dots, m\} - \{k\}$ is discussed in [2, 5, 12, 11], and the chromatic numbers of such distance graphs are now completely determined [2]. For general distance sets D , it seems very difficult to determine the chromatic numbers of the distance graphs $G(Z, D)$.

To prove a distance graph $G(R, D)$ is n -colorable, one usually uses two coloring methods. One of the method is called the *periodic coloring method*, which was introduced by Eggleton, Erdős and Skilton [5]. A coloring c of $G(Z, D)$ is called a *periodic coloring* if the sequence $(\dots c(1)c(2)c(3)\dots)$ of colors is a periodic sequence, where $c(i)$ is the color of vertex i . It was shown in [5] that if D is a finite set of integers, and $G(Z, D)$ is n -colorable, then $G(Z, D)$ has an n -coloring which is periodic. Indeed, as shown by Eggleton, Erdős and Skilton [Theorem 2 and its corollary of [8]] the smallest period of such a periodic coloring is at most dn^d , where $d = \max\{i : i \in D\}$. As a consequence, the chromatic number of any distance graph whose distance set contains finitely many integers can be determined by a bounded search.

Another coloring method is called the *regular coloring method* [18, 19, 20]. Instead of coloring the set of integers, the regular coloring method colors the real line R . We first choose a real number $r > 0$, then partition the real line R into intervals of length r , say $I_i = [ir, (i + 1)r)$ ($i = 0, \pm 1, \pm 2 \dots$). Then color the i th interval I_i with color $i(\text{mod}n)$. It is proved in [18] that such a

regular coloring of the real line R is a proper coloring of the distance graph $G(R, D)$ if and only if for every element $x \in D$ there is an integer $k(x)$ such that $k(x)n + 1 \leq x/r \leq k(x)n + n - 1$. Thus to find a regular n -coloring of a distance graph $G(R, D)$ is equivalent to find integer solutions to a system of linear inequalities.

As the regular coloring method colors the real line instead of the set of integers, it is more natural to consider the distance graph $G(R, D)$ which has vertex set R . However, when D is a set of integers then each connected component of $G(R, D)$ is isomorphic to the graph $G(Z, D)$.

Compared to the periodic colorings, a regular coloring is determined by just one parameter: the length of the interval, while for a periodic coloring, one needs to record a complete period of the color sequence which could be very long. Some of the previously designed complicated coloring rules can be replaced by simple regular colorings [18, 20]. The regular coloring method was successfully used in proving a conjecture which was proposed independently by Chen, Chang and Huang [1] and Voigt [17], which determines the chromatic numbers of distance graphs with distance set of cardinality 3 [20]. On the other hand, it is not true that every n -colorable distance graph G has a regular n -coloring, [21], although the results in [20] suggest that most n -colorable distance graphs do have regular n -colorings.

We shall introduce in this paper another general coloring method, the pattern periodic coloring, for distance graphs with integer distance sets. We shall compare this method to the above two coloring methods. Moreover, using the idea of pattern periodic coloring, we shall reduce considerably the upper bound of the period of periodic colorings of distance graphs.

2 Pattern periodic coloring

We only consider distance graphs $G(Z, D)$ for which the distance set D is a finite set of integers.

Definition 2.1 *We say a coloring c of the set Z of integers is a pattern periodic coloring if there is an integer p , and a permutation f of the set of colors such that for every $x \in Z$, we have $c(x) = f(c(x - p))$. We call p the period of the pattern periodic coloring c , and call f the color permutation of c .*

Our first observation is that any periodic coloring is a pattern periodic coloring and any pattern periodic coloring is a periodic coloring.

Theorem 2.1 *Suppose c is a periodic coloring of $G(Z, D)$ with period p . Then c is a pattern periodic coloring of $G(D, Z)$ with period p and with color*

permutation f being the identity permutation. On the other hand, suppose c is a pattern periodic coloring with period p and color permutation f , if f has order k , i.e., f^k is the identity, then c is a periodic coloring of $G(Z, D)$ with period pk .

Proof. The first half of the theorem follows trivially from the definition. For the last half, we note that for any integer x , $c(x) = f(c(x - p)) = f^2(c(x - 2p)) = \dots = f^k(c(x - kp)) = c(x - kp)$, as f^k is the identity. Therefore c is a periodic coloring of $G(Z, D)$ with period kp . ■

This theorem leads to the question of what is the purpose of discussing pattern periodic colorings?

As we noted in the previous section, an important (if not the only) consequence of the existence of a $\chi(G)$ -periodic coloring of any distance graph G with a finite integer distance set is that it reduces an infinite problem (the determination of the chromatic number of an infinite graph) to a finite problem. Let D be a finite set of positive integers and suppose $\chi(G(Z, D)) = n$. We denote by $\alpha(D)$ the smallest period of an n -periodic coloring of the distance graph $G(Z, D)$. If we know an upper bound, say k , of $\alpha(D)$, then it suffices to check all the sequences of colors of length k to determine whether the distance graph $G(Z, D)$ is n -colorable or not. If one of them produces a periodic n -coloring of a segment of $G(Z, D)$, then we find a periodic coloring of the whole graph $G(Z, D)$. Otherwise G is not n -colorable. The cost of such a bounded search depends on $\alpha(D)$. A general upper bound for $\alpha(D)$ is given by Eggleton, Erdős and Skilton. It is shown in [5] that for any finite set D of integers, $\alpha(D) \leq dn^d$, where $d = \max\{x : x \in D\}$ and n is the chromatic number of $G(Z, D)$ which is bounded by $|D| + 1$. By using an argument similar to the proof of Theorem 2.3 below, this bound can be reduced to n^d , which is still a very large number. The actual value $\alpha(D)$ is much smaller for all known examples. It is proved in [17] that if $D = \{d_1, d_2, \dots, d_r\}$ where $r \geq 3$ and $d_1 < d_2 < \dots < d_r$, then it suffices to investigate an interval of length $d_{r-2} + d_{r-1} + d_r$ to determine whether the graph $G(Z, D)$ is 3-colorable or not. However, for general D , the upper bound for $\alpha(D)$ is usually large enough to make the bounded search very expensive, or impractical. A computer search was done in [8], for periodic colorings of distance graphs with distance sets $D = \{2, 3, p\}$, where $p < 1000$ is a prime, and distance sets $D = \{2, 3, p, q\}$, where $p < q < 100$ are primes. It seems impractical to systematically try larger sets.

The purpose of studying pattern periodic coloring is to reduce the bound of period. Let $\beta(D)$ be the smallest period of a pattern periodic n -coloring of the distance graph $G(Z, D)$. It follows trivially from the definition that $\beta(D) \leq \alpha(D)$ for any finite set D of integers. We shall show that $\beta(D)$ is usually much smaller than $\alpha(D)$, and that the ratio $\alpha(D)/\beta(D)$ could be arbitrarily large. Moreover we shall give a general bound for $\beta(D)$ which

is much smaller than the bound we know for $\alpha(D)$. Interestingly, the idea of pattern periodic coloring can be used to reduce considerably the upper bound for $\alpha(D)$.

The following Theorem concerning the relationship between $\alpha(D)$ and $\beta(D)$ follows easily from Theorem 2.1 (note that the maximum order of a permutation f of n colors is equal to the maximum order of a cyclic subgroup of the symmetric group S_n).

Theorem 2.2 *For any finite set D of integers, we have $\beta(D) \leq \alpha(D) \leq g(n)\beta(D)$, where $g(n)$ is the maximum order of a cyclic subgroup of the symmetric group S_n , and n is the chromatic number of $G(Z, D)$.*

Theorem 2.2 can be applied in an effective way if we are able to estimate the maximum order $g(n)$ of an element of the symmetric group S_n . The function $g(n)$, called the Landau's function, is a well-studied function in number theory. It was proved by Landau (cf. [14]) that $\log g(n) \sim \sqrt{n \log n}$. Therefore $g(n) = e^{O(\sqrt{n \log n})}$.

We shall now start establishing upper bounds for $\beta(D)$, and also deduce upper bounds for $\alpha(D)$ from that.

Theorem 2.3 *Suppose $G(Z, D)$ is n -colorable. Then there exists a pattern periodic coloring c of G with period $p \leq n^d/n!$, where $d = \max\{x : x \in D\}$.*

Proof. Consider the set S of all sequences s of n colors of length d . We define an equivalence relation on S as follows: We say two sequences of colors $s = (x_1, x_2, \dots, x_d)$ and $s' = (y_1, y_2, \dots, y_d)$ are equivalent, denoted by $s \sim s'$, if there is a permutation f of the n colors such that $y_i = f(x_i)$ for $i = 1, 2, \dots, d$. Each equivalent class contains $n!$ sequences, so there are $n^d/n!$ equivalent classes. Let $m = n^d/n!$. We consider an n -coloring c of the graph $G(Z, D)$, which is a two-way infinite sequence of colors. Each segment of this sequence of length d is an element of S . Let $s_i = (c(i), c(i+1), \dots, c(i+d-1))$. Consider the sequences s_0, s_1, \dots, s_m . By the pigeonhole principle, there are two integers $i < j$, such that $s_i \sim s_j$. Hence there is a permutation f of the colors such that $s_j = f(s_i)$, i.e., $c(j+k) = f(c(i+k))$ for $k = 0, 1, \dots, d-1$. We now define a partial pattern periodic coloring c' of the distance G with period $j-i$ as follows:

For $i \leq x \leq j+d-1$, let $c'(x) = c(x)$; for $x \geq j+d$, let $c'(x) = f(c'(x-(j-i)))$.

It is obvious that by recursively applying the above rule, c' is defined on the set of all integers $x \geq i$. Moreover, for any $x \geq j$, $c'(x) = f(c'(x-(j-i)))$. (For $x \leq i-1$, $c'(x)$ will be defined later.)

Now we shall show that c' is a proper coloring of the subgraph of G induced by the integers $x \geq i$. Assume to the contrary that c' is not a proper coloring. Let y be the smallest integer such that there exists an integer $x < y$ such that $c'(x) = c'(y)$ and $y - x \in D$. Since c is a proper coloring of $G(Z, D)$ and c' agrees with c on the segment $\{i, i + 1, \dots, j + d - 1\}$, it follows that $y \geq j + d$. As $d = \max\{t : t \in D\}$, it follows that $x \geq j$, hence $x - (j - i) \geq i$. Therefore $c'(y - (j - i)), c'(x - (j - i))$ are defined, and $c'(y) = f(c'(y - (j - i)))$ and $c'(x) = f(c'(x - (j - i)))$. However, by the choice of y , we know that $c'(y - (j - i)) \neq c'(x - (j - i))$ because $(y - (j - i)) - (x - (j - i)) = y - x \in D$. This is a contradiction, as f is a permutation of the colors.

Now we extend c' to the whole set Z of integers as follows:

For $x \leq i - 1$, let $c'(x) = f^{-1}(c'(x + (j - i)))$.

The same argument shows that c' is indeed a proper coloring of G . Obviously, $j - i \leq n^d/n!$. Therefore $\beta(D) \leq n^d/n!$. This completes the proof of Theorem 2.3. ■

Corollary 2.1 *Suppose D is a finite set of integers with $d = \max\{t : t \in D\}$. If the subgraph of G induced by the set $\{0, 1, \dots, n^d/n! + d - 1\}$ is n -colorable, then the graph G is also n -colorable.*

Proof. The proof of Theorem 2.3 shows that if c is a proper n -coloring of the segment $\{0, 1, \dots, n^d/n! + d - 1\}$, then there exist integer $0 \leq i < j \leq n^d/n!$ such that the restriction of c to the segment $\{i, i + 1, \dots, j + d - 1\}$ can be extended to a pattern periodic n -coloring of the whole graph. ■

We note that the proof of Theorem 2.3 applies to periodic colorings as well, because every periodic coloring is a pattern periodic coloring with identity permutation of the colors. This leads to an improvement of the general upper bound for $\alpha(D)$ obtained in [5]. Namely we can show that $\alpha(D) \leq n^d$, where n is the chromatic number of $G(Z, D)$ and $d = \max\{t : t \in D\}$. However, by applying Theorem 2.2, we obtain a better upper bound.

Corollary 2.2 *For any finite set D of integers, we have $\alpha(D) \leq g(n)n^d/n!$, where $d = \max\{x : x \in D\}$.*

It follows from Corollary 2.2 that to determine whether a distance graph $G(Z, D)$ is n -colorable or not by using the periodic coloring method, it suffices to do a comprehensive search for the subgraph of order $g(n)n^d/n! + d$. This improves the corresponding result of [5]. However, Corollary 2.1 gives a much better result by using pattern periodic coloring. Especially when one needs to try some small examples by hand, or by a computer search, the difference is quite noticeable, or even from impractical to practical.

Compare the pattern periodic coloring method to the regular coloring method, the most significant difference is that every distance graph $G(Z, D)$ has a $\chi(G)$ -coloring which is pattern periodic, while G may have no regular $\chi(G)$ -colorings. Thus by doing a comprehensive search, one obtains a definite answer, which is not the case for the regular coloring method. However the regular coloring method does have its advantages, and for most distance graphs $G(Z, D)$ there does exist a $\chi(G)$ -regular coloring. For small examples, it is usually worth to try the regular coloring method. Furthermore the regular coloring method is a good tool to solve a class of problems systematically.

Another interesting relation between the regular coloring method and the pattern periodic coloring method will be explored in detail in Section 4. We shall show that if D is a subset of integers, and $G(R, D)$ is n -regular colorable with parameter r then $G(R, D)$ is n -regular colorable with parameter r' for some rational number r' . Then we show that if c is a regular coloring with a rational parameter r' , then the restriction of c to Z is a pattern periodic coloring (and hence a periodic coloring). In this sense, all the three methods result in a periodic coloring of the distance graph $G(Z, D)$. However, the process of obtaining such a coloring is quite different depends on which method is used.

3 Better bounds for $\beta(D)$

The idea of pattern periodic coloring makes the comprehensive search method applicable to more examples. But we have to admit that the applicable examples are still very limited. To make the method works better, we need to improve significantly the upper bound for $\beta(D)$. For those examples we tried, we find that $\beta(D)$ are actually very small. In this section, we shall improve the upper bound for $\beta(D)$.

Let $d = \max\{t : t \in D\}$, let $\omega(G)$ be the clique number of $G(Z, D)$. Our next result improves the bound for $\beta(D)$ given in Theorem 2.3.

Theorem 3.1 *Suppose $d, \omega = \omega(G(Z, D))$ and $n = \chi(G(Z, D))$ are defined as above. Then $\beta(D) \leq n^{d-\omega+1}/(n - \omega + 1)!$.*

Proof. Let c be an n -coloring of the graph $G(Z, D)$. Consider the subgraph of $G(Z, D)$ induced by the set $\{0, 1, \dots, d-1\}$. Since $G(Z, D)$ has clique size ω and each edge ij has $|i - j| \leq d$, it follows that the subgraph of $G(Z, D)$ induced by the set $\{0, 1, \dots, d-1\}$ has a clique, say $X = \{j_1, j_2, \dots, j_{\omega-1}\}$, of size $\omega - 1$. For any integer i , let $X_i = \{x + i : x \in X\}$. Then X_i is a clique contained in the subgraph of G induced by $\{i, i+1, \dots, i+d-1\}$. Therefore for any i , $c(x) \neq c(y)$ for all $x, y \in X_i$. We define a partial permutation p_i of the colors as follows: $p_i(c(j_t)) = c(j_t + i)$ for $t = 1, 2, \dots, \omega - 1$. As

$c(x) \neq c(y)$ for all $x, y \in X_i$, p_i is well-defined, and one to one. Hence p_i is indeed a partial permutation of the colors. Let F_i be the set of permutations of the colors which are extensions of p_i . Then $|F_i| = (n - \omega + 1)!$.

For each integer i , let $s_i = (c(i), c(i+1), \dots, c(i+d-1))$ be a sequence of colors of length d . Let $k = n^{d-\omega+1}/(n-\omega+1)!$, and we consider the sequences s_0, s_1, \dots, s_k . Let S be the set of all sequences of colors of length d , and we define the equivalence relation \sim on S as before. It is straightforward to verify that if there are integers $j < j'$ and $f_j \in F_j, f_{j'} \in F_{j'}$ such that for all $x \in \{j', j'+1, \dots, j'+d-1\}$, $f_{j'}^{-1}(c(x)) = f_j^{-1}(c(x - (j' - j)))$, then $s_j \sim s_{j'}$. If $s_j \sim s_{j'}$ then similar to the argument in the proof of Theorem 2.3, we can extend the restriction of c to the segment $\{j, j+1, \dots, j'+d-1\}$ to a pattern periodic coloring with period $j' - j$ and with color permutation $f_{j'} \circ f_j^{-1}$.

Therefore to prove Theorem 3.1, it suffices to find two integers $0 \leq j < j' \leq k$ and $f_j \in F_j, f_{j'} \in F_{j'}$ such that for all $x \in \{j', j'+1, \dots, j'+d-1\}$, $f_{j'}^{-1}(c(x)) = f_j^{-1}(c(x - (j' - j)))$.

For $x \in X_{j'}$, it follows from the definition that $f_{j'}^{-1}(c(x)) = f_j^{-1}(c(x - (j' - j)))$. Therefore, it suffices to show that for $f_{j'}^{-1}(c(x)) = f_j^{-1}(c(x - (j' - j)))$ for all $x \in \{j', j'+1, \dots, j'+d-1\} - X_{j'}$. Consider the sequences $s'_i = (c(x) : x \in \{i, i+1, \dots, i+d-1\} - X_i)$, which are sequences of colors of length $d - \omega + 1$, for $i = 0, 1, \dots, k$. Let $S'_i = \{f_i^{-1}(s'_i) : f_i \in F_i\}$. Each S'_i contains $(n - \omega + 1)!$ distinct color sequences of length $d - \omega + 1$. Since there are exactly $n^{d-\omega+1}$ sequences of colors of length $d - \omega + 1$, and that $k(n - \omega + 1)! = n^{d-\omega+1}$, by the pigeonhole principle, there exist two integers $0 \leq j < j' \leq k$ and $f_j \in F_j, f_{j'} \in F_{j'}$ such that for all $x \in \{j', j'+1, \dots, j'+d-1\}$, $f_{j'}^{-1}(c(x)) = f_j^{-1}(c(x - (j' - j)))$. This completes the proof of Theorem 3.1. \blacksquare

We note that it is possible that the subgraph of $G(Z, D)$ induced by the set $\{0, 1, \dots, d-1\}$ contains a clique of size ω . If this is the case, then the above proof shows that $\beta(D) \leq n^{d-\omega}/(n-\omega)!$.

Corollary 3.1 *Suppose D is a finite set of integers with $d = \max\{t : t \in D\}$. Let ω be clique number of the distance graph $G(Z, D)$. If the subgraph of $G(Z, D)$ induced by the set $\{1, 2, \dots, n^{d-\omega+1}/(n-\omega+1)! + d\}$ is n -colorable, then the graph G is also n -colorable.*

Corollary 3.1 is proved the same way as that of Corollary 2.1, and we omit the details.

By applying Theorem 3.1 and Theorem 2.2, we have the following upper bound for $\alpha(D)$.

Corollary 3.2 *Suppose $d, \omega = \omega(G(Z, D))$ and $n = \chi(G(Z, D))$ are defined as above. Then $\alpha(D) \leq g(n)n^{d-\omega+1}/(n-\omega+1)!$.*

Theorem 3.1 and Corollary 3.1 improve the bound given in Theorem 2.3 and Corollary 2.1. However, for all the known examples, this bound is still much larger than $\beta(D)$. Usually, the application of the method of pattern periodic coloring for small examples gives much better results.

4 Pattern periodic colorings induced by regular colorings

We shall prove in this section that if D is a set of integers, and that $G(R, D)$ has a regular n -coloring, then $G(R, D)$ has a regular n -coloring, say c' , with parameter r such that r is a rational number, and the restriction of such a coloring c' to the set Z of integers is indeed a pattern periodic coloring (and hence a periodic coloring). We shall further investigate the period of such a pattern periodic coloring. Interestingly, in case $|D| = 3$, the bound deduced this way improves the bound given in the previous section.

Lemma 4.1 *Suppose D is a finite set of integers. If there is a regular n -coloring of $G(R, D)$ with parameter r , then there is a regular n -coloring of $G(R, D)$ with parameter r' such that r' is rational. Moreover, we can choose r' so that $r' = x/t$, where $x \in D$ and t is an integer such that $0 < t \leq n(x - 1) + 1$.*

Proof. Recall that a regular coloring c of the real line R with parameter r is to partition the real line R into half open intervals, say $I_i = [ir, (i + 1)r)$, and then color the interval I_i with color $i(\bmod n)$. It is straightforward to verify that such a coloring is a proper coloring of the distance graph $G(R, D)$ if and only if for each $x \in D$, there is an integer $k(x)$ such that $1/r \in [(nk(x) + 1)/x, (nk(x) + n - 1)/x]$, (cf. [20]). Therefore the graph $G(R, D)$ has a regular n -coloring if and only if there are integers $k(x), x \in D$ such that

$$\cap_{x \in D} [(nk(x) + 1)/x, (nk(x) + n - 1)/x] \neq \emptyset.$$

Moreover for any r' such that $1/r' \in \cap_{x \in D} [(nk(x) + 1)/x, (nk(x) + n - 1)/x]$, the regular n -coloring of R with parameter r' is an n -coloring of $G(R, D)$. Since the element of D are integers, the intersection $\cap_{x \in D} [(nk(x) + 1)/x, (nk(x) + n - 1)/x]$ is either empty or contains a rational number. Indeed, if $\cap_{x \in D} [(nk(x) + 1)/x, (nk(x) + n - 1)/x] \neq \emptyset$, then the maximum of the numbers $\{(nk(x) + 1)/x : x \in D\}$, as well as the minimum of $\{(nk(x) + n - 1)/x : x \in D\}$, is contained in the intersection. Therefore r' can be chosen so that $r' = x'/(nk(x') + 1)$ for some $x' \in D$. To finish the proof of the lemma, we need to show that we can choose $k(x')$ in such a way

that $0 < nk(x') + 1 \leq n(x' - 1) + 1$. For this purpose, it suffices to note that if

$$\cap_{x \in D} [(nk(x) + 1)/x, (nk(x) + n - 1)/x] \neq \emptyset,$$

then

$$\cap_{x \in D} [(n(k(x) - x) + 1)/x, (n(k(x) - x) + n - 1)/x] \neq \emptyset.$$

This completes the proof of Lemma 4.1. ■

Lemma 4.2 *If c is a regular n -coloring of $G(R, D)$ with parameter r , and $r = p/q$ is a rational number. Then the restriction of c to Z is a pattern periodic coloring with period at most p .*

Proof. Let f be the permutation of the colors defined as $f(i) = i + q \pmod{n}$. Then it is straightforward to verify that $c(x) = f(c(x - p))$ for any $x \in Z$. ■

The following theorem follows easily from Lemmas 4.1 and 4.2.

Theorem 4.1 *Suppose D is a finite set of integers and the graph $G(R, D)$ has a regular n -coloring, where $n = \chi(G(R, D))$. Let $d = \max\{t : t \in D\}$. Then $\beta(D) \leq d$.*

Corollary 4.1 *If $D = \{a, b, c\}$, where $a < b < c$ are integers, then $\beta(D) \leq c$.*

Proof. By Theorem 4.1, it suffices to show that if $\chi(G(R, D)) = n$, then G has a regular n -coloring. We may assume that $\gcd(a, b, c) = 1$. It is well known (and also easy to see) that $2 \leq \chi(G(R, D)) \leq 4$, [1, 17]. Moreover $\chi(G) = 2$ if and only if a, b, c are odd, and in this case the coloring $c(x) = 0$ for odd x and $c(x) = 1$ for even x is a periodic coloring of G with period 2, which implies that $\beta(D) = 1, \alpha(D) = 2$.

If $\chi(G(R, D)) = 3$ then it follows from results in [18, 20] that there is a regular 3-coloring of $G(R, D)$.

Suppose $\chi(G(Z, D)) = 4$. Then it follows from results in [18, 20] that either $\{a, b, c\} = \{1, 2, 3m\}$ or $c = a + b$ and $a \not\equiv b \pmod{3}$. To show that G has a regular 4-coloring, it suffices to show that there are integers i, j, k such that

$$[(4i + 1)/a, (4i + 3)/a] \cap [(4j + 1)/b, (4j + 3)/b] \cap [(4k + 1)/c, (4k + 3)/c] \neq \emptyset.$$

If $\{a, b, c\} = \{1, 2, 3m\}$ and $m \leq 3$, then let $i = j = 0$ and let $k = \lfloor (3m - 1)/4 \rfloor$. It is straightforward to verify that

$$1 \in [(4i + 1)/a, (4i + 3)/a] \cap [(4j + 1)/b, (4j + 3)/b] \cap [(4k + 1)/c, (4k + 3)/c].$$

If $\{a, b, c\} = \{1, 2, 3m\}$ and $m \geq 4$, then let $i = j = 0$ and let $k = \lfloor (3m - 1)/4 \rfloor + 1$. It is straightforward to verify that

$$(4k+1)/c \in [(4i+1), (4i+3)] \cap [(4j+1)/2, (4j+3)/2] \cap [(4k+1)/c, (4k+3)/c].$$

Assume now that $c = a + b$ and $a \not\equiv b \pmod{3}$. If $b \leq 2a$, then $c \leq 3a$. Let $i = j = k = 0$. Then

$$1/a \in [(4i+1)/a, (4i+3)/a] \cap [(4j+1)/b, (4j+3)/b] \cap [(4k+1)/c, (4k+3)/c].$$

If $2a < b \leq 7a/3$ then $3a < c \leq 10a/3$. Let $i = 0, j = 1, k = 2$. Then

$$9/c \in [(4i+1)/a, (4i+3)/a] \cap [(4j+1)/b, (4j+3)/b] \cap [(4k+1)/c, (4k+3)/c].$$

If $7a/3 < b \leq 5a$, then $[5/b, 7/b] \subset [1/a, 3/a]$. As $c \geq b$, it is easy to see that there is an integer k_0 such that $[5/b, 7/b] \cap [(4k_0+1)/c, (4k_0+3)/c] \neq \emptyset$. Thus we may let $i = 0, j = 1$ and $k = k_0$. If $b > 5a$, then let j_0 be the smallest integer such that $(4j_0+1)/b \geq 1/a$, then $[(4j_0+1)/b, (4j_0+3)/b] \subset [1/a, 3/a]$. Again, because $c \geq b$, there is an integer k_0 such that $[(4j_0+1)/b, (4j_0+3)/b] \cap [(4k_0+1)/c, (4k_0+3)/c] \neq \emptyset$. Hence we may let $i = 0, j = j_0$ and $k = k_0$. This completes the proof of Corollary 4.1. \blacksquare

Applying Theorem 4.1 and Theorem 2.2, we have the following corollary:

Corollary 4.2 *Suppose D is a set of integers and that the graph $G(R, D)$ has a regular n -coloring, where $n = \chi(G(R, D))$. Let $d = \max\{t : t \in D\}$. Then $\alpha(D) \leq dg(n)$.*

The bound for $\beta(D)$ given in Theorem 4.1 is a very good and practical bound. It is unknown whether this bound applies to those distance graphs whose chromatic numbers are not realized by a regular coloring. As we noted before, there are distance graphs with integer distance sets and whose chromatic numbers are not realized by regular colorings. In [20], we defined the *regular chromatic number* of a distance graph $G(R, D)$ to be the least integer n such that there is a regular n -coloring for $G(R, D)$. The proof of Theorem 4.1 shows that if the regular chromatic number of $G(R, D)$ is n , then $G(Z, D)$ has a pattern periodic coloring with period d .

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