Distance graphs with missing multiples in the distance sets

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Abstract

Given positive integers m, k and s with $m > ks$, let $D_{m,k,s}$ represent the set $\{1, 2, \cdots, m\} - \{k, 2k, \cdots, sk\}$. The distance graph $G(Z, D_{m,k,s})$ has as vertex set all integers Z and edges connecting i and j whenever $|i - j| \in D_{m,k,s}$. The chromatic number and the fractional chromatic number of $G(Z, D_{m,k,s})$ are denoted by $\chi(Z, D_{m,k,s})$ and $\chi_f(Z, D_{m,k,s})$, respectively. For $s = 1$, $\chi(Z, D_{m,k,1})$ was studied by Eggleton, Erdős and Skilton [6], Kemnitz and Kolberg [12], and Liu [13], and was solved lately by Chang, Liu and Zhu [2] who also determined $\chi_f(Z, D_{m,k,1})$ for any m and k. This article extends the study of $\chi(Z, D_{m,k,s})$ and $\chi_f(Z, D_{m,k,s})$ to general values of s. We prove $\chi_f(Z, D_{m,k,s}) = \chi(Z, D_{m,k,s}) = k$ if $m < (s+1)k$; and $\chi_f(Z, D_{m,k,s}) = (m + sk + 1)/(s + 1)$ otherwise. The latter result provides a good lower bound for $\chi(Z, D_{m,k,s})$. A general upper bound for $\chi(Z, D_{m,k,s})$ is found. We prove the upper bound can be improved to $\lceil (m + sk + 1)/(s + 1) \rceil + 1$ for some values of m, k and s. In particular, when $s + 1$ is prime, $\chi(Z, D_{m,k,s})$ is either $\lceil (m + sk + 1)/(s + 1) \rceil$ or $\lceil (m + sk + 1)/(s + 1) \rceil + 1$. By using a special coloring method called the pre-coloring method, many distance graphs $G(Z, D_{m,k,s})$ are classified into

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these two possible values of $\chi(Z, D_{m,k,s})$. Moreover, complete solutions of $\chi(Z, D_{m,k,s})$ for several families are determined including the case $s = 1$ (solved in [2]), the case $s = 2$, the case $(k, s+1) = 1$, and the case that k is a power of a prime.

Keywords. Distance graph, chromatic number, fractional chromatic number, pre-coloring method.

1 Introduction

Given a set D of positive integers, the *distance graph* $G(Z, D)$ has all integers as vertices; and two vertices are adjacent if and only if their difference falls within D, that is, the vertex set is Z and the edge set is $\{uv : |u - v| \in D\}$. We call D the *distance set*. The chromatic number of $G(Z, D)$ is denoted by $\chi(Z, D)$.

For different types of distance sets D, the problem of determining $\chi(Z, D)$ has been studied extensively. (See [2, 3, 4, 6, 7, 8, 9, 12, 16, 15, 17].) For instance, suppose D is a subset of prime numbers and $\{2, 3\} \subseteq D$, Eggleton, Erdős and Skilton [9] proved that $\chi(Z, D)$ is either 3 or 4. The problem of classifying $G(Z, D)$ with distance sets D of primes into chromatic number 3 or 4 was studied by Eggleton, Erdős and Skilton [9], and by Voigt and Walther [16]. However, a complete classification is not obtained yet.

If D has only one element, it is trivial that $\chi(Z, D) = 2$. When D has two elements, it is known that $\chi(Z, D) = 3$ if the two integers in D are of different parity, and $\chi(Z, D) = 2$ otherwise (assuming that $\gcd D = 1$). The case if D has three elements, which is much more complicated, has been studied by Chen, Chang, and Huang [3], and by Voigt [15], and was solved lately by Zhu [17].

A *fractional coloring* of a graph G is a mapping h from $\mathcal{I}(G)$, the set of all independent sets of G, to the interval $[0, 1]$ such that $\sum_{I \in \mathcal{I}(G), x \in I} h(I) \geq 1$ for each vertex x of G. The *fractional chromatic number* $\chi_f(G)$ of G is the infimum of the value $\sum_{I \in \mathcal{I}(G)} h(I)$ of a fractional coloring h of G. The fractional chromatic number of a distance graph $G(Z, D)$ is denoted by $\chi_f(Z, D)$.

For any graph G, it is well-known and easy to verify that

$$\max\{\omega(G), \frac{|V(G)|}{\alpha(G)}\} \leq \chi_f(G) \leq \chi(G), \tag{*}$$

where $\omega(G)$ is the size (number of vertices) of a maximum complete graph, and $\alpha(G)$ is the size of a maximum independent set in G. (See Chapter 3 of [14].)

Given integers m, k and s with $m > ks$, let $D_{m,k,s}$ denote the distance set $D_{m,k,s} = \{1, 2, 3, \cdots, m\} - \{k, 2k, 3k, \cdots, sk\}$. This article studies the chromatic number and the fractional chromatic number of $G(Z, D_{m,k,s})$. If $s = 1$, the chromatic number of $G(Z, D_{m,k,1})$ was first studied by Eggleton, Erdős and Skilton [6] who determined $\chi(Z, D_{m,k,1})$ completely for $k = 1$, and partially for $k = 2$. The same results for the case $k = 1$ were also obtained in [12] by a different approach. For the cases that k is an odd number, $k = 2$ and $k = 4$, $\chi(Z, D_{m,k,1})$ were determined in [13]. Recently, the exact values of $\chi_f(Z, D_{m,k,1})$ and $\chi(Z, D_{m,k,1})$ for all m and k were settled in [2]. We extend the study to general values of s.

Note that it becomes an easy case if $m < (s+1)k$. Define a coloring f of $G(Z, D_{m,k,s})$ by: For any $x \in Z$, $f(x) = x \bmod k$. Since $D_{m,k,s}$ contains no multiples of k, f is a proper coloring. Thus, $\chi(Z, D_{m,k,s}) \leq k$. As any consecutive k vertices in $G(Z, D_{m,k,s})$ form a complete graph, by (*), $\chi_f(Z, D_{m,k,s}) \geq k$. This implies $\chi(Z, D_{m,k,s}) = \chi_f(Z, D_{m,k,s}) = k$, if $m < (s+1)k$. Therefore, throughout the article, we assume $m \geq (s+1)k$.

Section 2 determines the fractional chromatic number of $G(Z, D_{m,k,s})$ for all values of m, k and s with $m \geq (s+1)k$. This result provides a good lower bound for $\chi(Z, D_{m,k,s})$, namely,

$$\lceil (m + sk + 1)/(s+1) \rceil \leq \chi(Z, D_{m,k,s}), \text{ if } m \geq (s+1)k. \tag{**}$$

This lower bound will be shown to be sharp for some families of $G(Z, D_{m,k,s})$ and strict for some others.

Section 3 introduces the pre-coloring method, one of the main tools used in the article. For such a coloring method, we determine when it produces a proper coloring for $G(Z, D_{m,k,s})$, and then determine the number of colors used by the produced proper coloring. These characterizations are used intensively in Sections 4 and 5.

Section 4 starts with the result of a general upper bound of $\chi(Z, D_{m,k,s})$. For some values of m, k and s, we improve the upper bound to $\lceil (m+sk+1)/(s+1) \rceil + 1$. Combining these results with the lower bound (**) mentioned above, the chromatic numbers for many families of $G(Z, D_{m,k,s})$ are determined.

Section 5 focuses on the study of $\chi(Z, D_{m,k,s})$ when $s+1$ is a prime number. Using the results obtained in earlier sections, we show that when $s+1$ is prime, $\chi(Z, D_{m,k,s})$ is either $\lceil (m+sk+1)/(s+1) \rceil$ or $\lceil (m+sk+1)/(s+1) \rceil + 1$. For many families of $G(Z, D_{m,k,s})$, we classify their chromatic numbers into one of these two values. Moreover, we completely determine the exact values of $\chi(Z, D_{m,k,s})$ for the following cases: If $s = 1$ (which was solved recently in [2]); if $s = 2$; if $(k, s+1) = 1$; and if k is a power of a prime.

2 Lower bounds and fractional chromatic number

In this section, we first determine the fractional chromatic number of $G(Z, D_{m,k,s})$ for all values of m, k and s with $m \geq (s+1)k$. This result immediately leads to (**), a lower bound for $\chi(Z, D_{m,k,s})$. Then we prove that in (**), equality holds for some values of m, k and s; while strict inequality holds for some others.

Theorem 1 *For any given integers m, k and s with $m \geq (s+1)k$,*

$$\chi_f(Z, D_{m,k,s}) = (m+sk+1)/(s+1).$$

Proof. For any i with $0 \leq i \leq m+sk$, let $I_i = \{j \in Z : j - i \equiv xk \pmod{m+sk+1}, 0 \leq x \leq s\}$. It is straightforward to verify that I_i is an independent set in

$G(Z, D_{m,k,s})$. It is also easy to verify that any integer is contained in exactly $s+1$ such independent sets. Define a mapping $h : \mathcal{I}(G(Z, D_{m,k,s})) \to [0,1]$ by

$$h(I) = \begin{cases} \frac{1}{s+1}, & \text{if } I = I_i \text{ for } 0 \le i \le m+sk; \\ 0, & \text{otherwise.} \end{cases}$$

Then h is a fractional coloring of $G(Z, D_{m,k,s})$ which has value $\frac{m+sk+1}{s+1}$. Thus, $\chi_f(Z, D_{m,k,s}) \le \frac{m+sk+1}{s+1}$.

To show $\chi_f(Z, D_{m,k,s}) \ge \frac{m+sk+1}{s+1}$, let G be the subgraph of $G(Z, D_{m,k,s})$ induced by the vertices $\{0, 1, 2, \cdots, m+sk\}$. Then $\chi_f(G) \le \chi_f(Z, D_{m,k,s})$. It is straightforward to verify that $\alpha(G) = s+1$. Hence, by (*), $\chi_f(G) \ge \frac{|V(G)|}{\alpha(G)} = \frac{m+sk+1}{s+1}$. This completes the proof of Theorem 1. Q.E.D.

Since $\chi(G)$ is an integer, by (*), we have $\lceil \chi_f(G) \rceil \le \chi(G)$. Hence, the following is obtained.

Theorem 2 *For any given integers m, k and s with $m \ge (s+1)k$,*

$$\chi(Z, D_{m,k,s}) \ge \lceil (m+sk+1)/(s+1) \rceil.$$

The following result indicates that the lower bound of $\chi(Z, D_{m,k,s})$ in Theorem 2 is attained by some values of m, k and s, but not attained by some others.

Theorem 3 *Suppose $m \ge (s+1)k$, $k = (s+1)^a k'$ and $m + sk + 1 = (s+1)^b m'$, where both k' and m' are not divisible by $s+1$. Then*

$$\chi(Z, D_{m,k,s}) \begin{cases} \ge (m+sk+1)/(s+1) + 1, & \text{if } 0 < b \le a; \\ = (m+sk+1)/(s+1), & \text{if } a < b \text{ and } (s+1, k') = 1. \end{cases}$$

Proof. Let $n = (m+sk+1)/(s+1)$. Because $b > 0$, n is an integer.

Suppose $0 < b \le a$, we shall show that $G(Z, D_{m,k,s})$ is not n-colorable. Assume to the contrary, there exits an n-coloring f of $G(Z, D_{m,k,s})$.

For any two integers i and j, let $G[i, j]$ be the subgraph of $G(Z, D_{m,k,s})$ induced by the vertex set $\{i+1, i+2, \cdots, j\}$. Then for any integer i, the graph $G[i, i+m+sk+1]$

has $m + sk + 1$ vertices and a maximum independent set of size $s + 1$. Since f is an $(m + sk + 1)/(s + 1)$-coloring, exactly $s + 1$ vertices of $G[i, i + m + sk + 1]$ are colored by the same color. It follows that $f(i) = f(i + m + sk + 1)$ for any integer i.

Define a circulant graph G on the set $\{0, 1, \cdots, m + sk\}$ with generating set $D_{m,k,s}$, that is, ij is an edge of G if and only if $(j - i) \bmod (m + sk + 1) \in D_{m,k,s}$ or $(i - j) \bmod (m + sk + 1) \in D_{m,k,s}$. The argument in the previous paragraph shows that f induces a proper n-coloring of G. Moreover, each color class consists of $s + 1$ vertices in G. It is not difficult to verify that all $(s + 1)$-independent sets of G are of the form $\{i, i+k, \cdots, i+sk\}$. (Here each number is calculated by modulo $m+sk+1$.)

Let $d = (k, m + sk + 1)$ and $u = (m + sk + 1)/d$. Divide the vertex set of G into d subsets of the form $\{i, i + k, i + 2k, \cdots, i + (u - 1)k\} \pmod{m + sk + 1}$, each of size u. Then each of these d subsets is the union of some color classes of size $s + 1$, so $(s + 1)$ divides u. Therefore $m + sk + 1$ is a multiple of $(s + 1)^{a+1}$, which is impossible since $b \leq a$.

Suppose $a < b$ and $(s + 1, k') = 1$, then u is a multiple of $s + 1$. One can easily define a proper n-coloring f on G by using $u/(s + 1)$ colors to each of the subsets $\{i, i+k, i+2k, \cdots, i+(u-1)k\} \pmod{m+sk+1}$ as defined in the previous paragraph by: the first $s + 1$ vertices in a subset use one color and the next $s + 1$ vertices use the next, and continue the process until all vertices are colored. It is easy to check that f is a proper coloring of G. Furthermore, f can be extended to a proper coloring of $G(Z, D_{m,k,s})$ by letting $f'(y) = f(x)$, where $x = y \bmod (m + sk + 1)$. Therefore, $G(Z, D_{m,k,s})$ is n-colorable. This completes the proof of Theorem 3. Q.E.D.

3 The pre-coloring method

This section introduces the main tool to be used in the remaining part of this article, namely, the *pre-coloring method*. A simpler version of this method was originally applied in [2] in determining the chromatic number of $G(Z, D_{m,k,1})$. Here we extend the idea to a more complex version and use it extensively throughout this article.

Before introducing the pre-coloring method, we note another fact. Let Z^* denote the set of non-negative integers. It is known and easy to verify that for any distance set D, $\chi(Z,D) = \chi(Z^*,D)$, where $G(Z^*,D)$ is the subgraph of $G(Z,D)$ induced by Z^*. Therefore, to color the graph $G(Z,D_{m,k,s})$, it suffices to color the subgraph of $G(Z,D_{m,k,s})$ induced by Z^*.

There are two steps in the pre-coloring method. First, we partition the set Z^* into $s+1$ parts by a mapping $c : Z^* \to \{0,1,2,\cdots,s\}$. Second, for each non-negative integer x, according to the value of $c(x)$, we assign a color to x by the rule defined as follows.

Definition 4 *Suppose m,k,s are positive integers. For a given mapping $c : Z^* \to \{0,1,2,\cdots,s\}$, define a coloring c' of Z^* recursively by:*

$$c'(j) = \begin{cases} j, & \text{if } j < k; \\ c'(j-k), & \text{if } j \geq k \text{ and } c(j) \neq 0; \\ n, & \text{if } j \geq k \text{ and } c(j) = 0, \end{cases}$$

where n is the smallest non-negative integer (color) not been used in the m vertices preceeding j, that is, $n = \min\{t \in Z^* : c'(j-i) \neq t \text{ for } i = 1,2,\cdots,m\}$.

Note that c' defined above is uniquely determined by c. We call c the *pre-coloring*, and c' the *coloring induced by c*. For any $x \in Z^*$, $c(x)$ and $c'(x)$ are called the *pre-color* and the *color* of x, respectively.

In order to ensure that the coloring c' in Definition 4 be a proper coloring for $G(Z^*,D_{m,k,s})$ as desired, the pre-coloring c needs to satisfy certain conditions specified in the following lemma.

Lemma 5 *Suppose c is a pre-coloring of Z^*. If for any integer $j \geq sk$, $c(j), c(j-k), c(j-2k), \cdots,$ and $c(j-sk)$ are all distinct, then the induced coloring c' is a proper coloring for $G(Z,D_{m,k,s})$.*

Proof. It is enough to show by induction that for any $j \in Z^*$, $c'(j) \neq c'(x)$ for any neighbor x of j and $x < j$. If $j < k$, or $j \geq k$ with $c(j) = 0$, then this is true by Definition 4.

Now, assume $j \geq k$ and $c(j) \neq 0$. By definition, $c'(j) = c'(j-k)$. If $j - k < x < j$, then x is adjacent to $j - k$. By the inductive hypotheses, $c'(x) \neq c'(j-k)$, so $c'(x) \neq c'(j)$. If $x < j - k$ and x is adjacent to j, then either x is a neighbor of $j - k$ or $x = j - (s+1)k$. In the former case, according to the inductive hypotheses, $c'(x) \neq c'(j-k)$, hence $c'(x) \neq c'(j)$. We now consider the case that $x = j - (s+1)k$. Because the pre-colors of $j, j-k, j-2k, \cdots, j-sk$ are all distinct, exactly one of them is 0. Suppose $c(j - uk) = 0$ for some $0 \leq u \leq s$. Then by Definition 4, $c'(j - uk)$ is different from the color of any of the m vertices preceding $j - uk$, hence $c'(j-uk) \neq c'(j-(s+1)k)$. Because $c(j), c(j-k), \cdots, c(j-(u-1)k) \neq 0$, $c'(j) = c'(j-k) = c'(j-2k) = \cdots = c'(j-uk)$. Therefore, $c'(j) \neq c'(j-(s+1)k)$, *i.e.*, $c'(j) \neq c'(x)$. This completes the proof of Lemma 5. Q.E.D.

After getting a necessary condition for the pre-coloring c to produce a proper coloring c' for the distance graph $G(Z^*, D_{m,k,s})$, the next natural question to ask is *how many* colors are used by c'. The answer of this question is shown in the following result.

Lemma 6 *Suppose c is a pre-coloring and c' is the induced coloring. Then the number of colors used by c' is at most $k + \ell$, where ℓ is the maximum number of vertices with pre-color 0, among any $m - k + 1$ consecutive integers greater than k.*

Proof. We prove, by induction on j, that vertices $0, 1, 2, \cdots, j$ are colored by the pre-coloring method with at most $k + \ell$ colors. This is trivial when $j < k$, or $j \geq k$ with $c(j) \neq 0$.

Now we assume $j > k$ and $c(j) = 0$. It suffices to show that the m vertices preceeding j use at most $k + \ell - 1$ colors. For the m vertices preceeding j, the first k

vertices use at most k colors. Among the remaining $m-k$ vertices, only those vertices with pre-color 0 require a new color. Due to the facts that $c(j)=0$, and any set of consecutive $m-k+1$ vertices contains at most ℓ vertices of pre-color 0, we conclude that among the remaining $m-k$ vertices, there are at most $\ell-1$ vertices with pre-color 0. Therefore, the total number of colors used by the m vertices preceeding j is at most $k+\ell-1$, and hence there is a color for the vertex j. Q.E.D.

Combining Lemmas 5 and 6, we arrive at the following useful conclusion.

Corollary 7 *Given integers m, k and s, $\chi(Z, D_{m,k,s}) \leq n$ if there exists a pre-coloring c such that the following two conditions are satisfied:*

(1) *for any integer $j \geq sk$, $c(j), c(j-k), c(j-2k), \cdots, c(j-sk)$ are all distinct, and*

(2) *among any consecutive non-negative $m-k+1$ integers, there are at most $n-k$ vertices with pre-color 0.*

Corollary 7 will be used in many of the proofs in the rest of the article. Instead of finding a proper coloring for the distance graph $G(Z, D_{m,k,s})$ with n colors, it is enough to present a pre-coloring c that satisfies (1) and (2) of Corollary 7.

4 Upper bounds

This section shows upper bounds of $\chi(Z, D_{m,k,s})$ for different values of m, k and s. Combining these upper bounds with the lower bounds obtained in Section 2 gives the exact value of $\chi(Z, D_{m,k,s})$ for some families of $G(Z, D_{m,k,s})$. In particular, we prove for many different combinations of m, k and s, $\chi(Z, D_{m,k,s})$ is either $\lceil (m+sk+1)/(s+1) \rceil$ or $\lceil (m+sk+1)/(s+1) \rceil + 1$.

We start with a general upper bound in the following. For any two integers a and b, let (a, b) denote the greatest common divisor of a and b.

Theorem 8 *Suppose* $m \geq (s+1)k$ *and* $(k, m+sk+1) = d$*, then* $\chi(Z, D_{m,k,s}) \leq d\lceil (m+sk+1)/d(s+1) \rceil$.

Proof. Define a circulant graph G on the set $\{0, 1, \cdots, m+sk\}$ with generating set $D_{m,k,s}$, that is, ij is an edge of G if and only if $(j-i) \bmod (m+sk+1) \in D_{m,k,s}$ or $(i-j) \bmod (m+sk+1) \in D_{m,k,s}$. It is easy to verify that any proper coloring f of G can be extended to a proper coloring f' of $G(Z, D_{m,k,s})$ by letting $f'(y) = f(x)$, where $x = y \bmod (m+sk+1)$. Therefore, it is enough to find a proper n-coloring of G, where $n = d\lceil (m+sk+1)/d(s+1) \rceil$.

Let $u = (m+sk+1)/d$. Divide the vertex set of G into d subsets such that each subset has u vertices and is of the form $\{i, i+k, i+2k, \cdots, i+(u-1)k\} \pmod{m+sk+1}$. Any consecutive $s+1$ vertices in a subset are independent, so each subset can be partitioned into $\lceil u/(s+1) \rceil = \lceil (m+sk+1)/d(s+1) \rceil$ independent sets of size $s+1$, except the last one whose size might be smaller than $s+1$. Therefore the vertex set of G can be partitioned into $d\lceil (m+sk+1)/d(s+1) \rceil$ independent sets. Hence $\chi(Z, D_{m,k,s}) \leq d\lceil (m+sk+1)/d(s+1) \rceil$. Q.E.D.

Combining the upper bound above with the lower bound in Theorem 2, the following two results emerge.

Corollary 9 *Suppose* $m \geq (s+1)k$ *and* $(k, m+sk+1) = d$*, then*

$$\lceil (m+sk+1)/(s+1) \rceil \leq \chi(Z, D_{m,k,s}) \leq d\lceil (m+sk+1)/d(s+1) \rceil.$$

Corollary 10 *If* $m \geq (s+1)k$ *and* $(k, m+sk+1) = 1$*, then* $\chi(Z, D_{m,k,s}) = \lceil (m+sk+1)/(s+1) \rceil$.

We note that in Corollary 9, there may exist big gaps between the upper and the lower bounds, depending on the values of $d = (k, m+sk+1)$. However, so far we do not have any example of distance graph $G(Z, D_{m,k,s})$ with chromatic number exceeding $\lceil (m+sk+1)/(s+1) \rceil + 1$. The next theorem provides a better upper bound for some families of $G(Z, D_{m,k,s})$.

Theorem 11 *If $m \geq (s+1)k$ and $s+1$ is a divisor of k, then $\chi(Z, D_{m,k,s}) \leq \lceil (m+sk+1)/(s+1) \rceil + 1$.*

Proof. For any $j \in Z^*$, we can write j uniquely in the form $j = uk + v(s+1) + w$, where u, v and w are integers such that $0 \leq v < k/(s+1)$ and $0 \leq w \leq s$. Then define a pre-coloring c by $c(j) = u + w \pmod{s+1}$. We only need to show that c satisfies (1) and (2) in Corollary 7, with $n = \lceil (m+sk+1)/(s+1) \rceil + 1$.

First we show that for any vertex j, the $s+1$ vertices, $j, j-k, j-2k, \cdots, j-sk$ have distinct pre-colors. Assume $j = uk + v(s+1) + w$ with $0 \leq v < k/(s+1)$ and $0 \leq w \leq s$. Then $j - ik = (u-i)k + v(s+1) + w$, $0 \leq i \leq s$. It follows that $c(j-ik) = (u-i+w) \bmod (s+1)$ which give distinct colors for $0 \leq i \leq s$.

Next we show that among any consecutive $m-k+1$ vertices, there are at most $n-k = \lceil (m-k+1)/(s+1) \rceil + 1$ vertices with pre-color 0. Divide the set of non-negative integers into segments of length $s+1$ by $A_0 = \{0, 1, \cdots, s\}, A_1 = \{s+1, s+2, \cdots, 2s+1\}, \cdots, A_i = \{i(s+1), i(s+1)+1, \cdots, (i+1)(s+1)-1\}, \cdots$. Then each segment A_i contains exactly one vertex of each pre-color. Indeed, it is straightforward to verify that the pre-colors of A_i are $\{j, j+1, \cdots, s, 0, 1, \cdots, j-1\}$, where $i = uk/(s+1) + v$, $0 \leq v < k/(s+1)$ and $j = u \bmod (s+1)$. Any set of consecutive $m-k+1$ vertices intersects at most $\lceil (m-k+1)/(s+1) \rceil + 1$ segments, so it contains at most $\lceil (m-k+1)/(s+1) \rceil + 1$ vertices of pre-color 0. This completes the proof. Q.E.D.

The following corollary follows from Theorems 3 and 11.

Corollary 12 *Suppose $m \geq (s+1)k$, $k = (s+1)^a k'$ and $m + sk + 1 = (s+1)^b m'$, where both k' and m' are not divisible by $s+1$. If $0 < b \leq a$, then $\chi(Z, D_{m,k,s}) = (m+sk+1)/(s+1) + 1$.*

The next result shows another family of $G(Z, D_{m,k,s})$ such that the chromatic number reaches the lower bound.

Theorem 13 *If* $(k, s+1) = 1$, *then* $\chi(Z, D_{m,k,s}) = \lceil (m + sk + 1)/(s+1) \rceil$ *for all* $m \geq (s+1)k$.

Proof. Define a pre-coloring c by $c(j) = j \bmod (s+1)$. We prove that c satisfies (1) and (2) of Corollary 7, with $n = \lceil (m + sk + 1)/(s+1) \rceil$.

To show that for any vertex j, $c(j), c(j-k), c(j-2k), \cdots$, and $c(j-sk)$ are all distinct, we assume to the contrary that $c(j-tk) = c(j-t'k)$ for some $0 \leq t < t' \leq s$. Then $j - tk \equiv j - t'k \pmod{s+1}$, so $(t'-t)k \equiv 0 \pmod{s+1}$. This is impossible, because $(k, s+1) = 1$ and $0 < t' - t \leq s$.

Next we show that among any consecutive $m - k + 1$ vertices, there are at most $\lceil (m - k + 1)/(s+1) \rceil$ vertices with pre-color 0. This is trivial, because the vertices of pre-color 0 are those vertices j for which $j \equiv 0 \pmod{s+1}$, so any two vertices with pre-color 0 are exactly $s+1$ vertices apart. This completes the proof. Q.E.D.

5 The case $s + 1$ is prime

This section focuses on the study of $\chi_f(Z, D_{m,k,s})$ when $s+1$ is a prime number. If $s+1$ is prime, then either $s+1$ is a divisor of k or $(k, s+1) = 1$. Hence by Theorems 11 and 13, $\chi(Z, D_{m,k,s})$ is either $\lceil (m+sk+1)/(s+1) \rceil$ or $\lceil (m+sk+1)/(s+1) \rceil + 1$. In this section, assuming $s+1$ is prime, we classify the chromatic number for most of the families of the distance graphs $G(Z, D_{m,k,s})$ into one of those two possible values.

Similarly to Theorem 3, we let $k = (s+1)^a k'$ and $m + sk + 1 = (s+1)^b m'$, where k' and m' are not divisible by $(s+1)$. As $s+1$ is prime, $(s+1, k') = 1$. Therefore, the following result can be derived immediately from Theorems 3 and 13, and Corollary 12.

Theorem 14 *Suppose $m \geq (s+1)k$, $s+1$ is prime, and m, k, a, b are defined as above. Then*

$$\chi(Z, D_{m,k,s}) = \begin{cases} \lceil (m+sk+1)/(s+1) \rceil, & \text{if } a = 0 \text{ or } a < b; \\ (m+sk+1)/(s+1)+1, & \text{if } 0 < b \leq a. \end{cases}$$

Suppose k is a power of a prime, $k = p^a$. If $p \neq s+1$, by Theorem 14, $\chi(Z, D_{m,k,s}) = \lceil (m+sk+1)/(s+1) \rceil$ for all $m \geq (s+1)k$. If $p = s+1$, that is, $k = (s+1)^a$, then the chromatic number of $G(Z, D_{m,k,s})$ can be completely determined as follows.

Corollary 15 *Suppose $m \geq (s+1)k$, $s+1$ is prime, $k = (s+1)^a$, and $m+sk+1 = (s+1)^b m'$, where m' is not a multiple of $s+1$. Then*

$$\chi(Z, D_{m,k,s}) = \begin{cases} \lceil (m+sk+1)/(s+1) \rceil, & \text{if } b = 0 \text{ or } a < b; \\ (m+sk+1)/(s+1)+1, & \text{if } 0 < b \leq a. \end{cases}$$

Proof. By Theorem 14, we only have to show the case as $b = 0$, which implies $(k, m+sk+1) = 1$. Hence by Corollary 10, the prove is complete. Q.E.D.

Note that when $s+1$ is prime, Theorem 14 determines the value of $\chi(Z, D_{m,k,s})$ unless $a > 0$ and $b = 0$. Thus, for the rest of this section, we shall assume that $a > 0$ and $b = 0$, that is, k is a multiple of $s+1$ but $m+sk+1$ is not. Our next result completely settles the case for $a = 1$.

Theorem 16 *Suppose $s+1$ is prime, let m, s, k, a, b be integers same as defined in Theorem 3. If $a = 1$, then $\chi(Z, D_{m,k,s}) = \lceil (m+sk+1)/(s+1) \rceil$ for all $m \geq (s+1)k$.*

Proof. Let $r = \lceil (m+sk+1)/(s+1) \rceil \bmod (s+1)$. We consider two cases.

Case 1. $r = 0$. There exists an integer $\bar{m} \geq m$ such that $(\bar{m}+sk+1)/(s+1) = \lceil (m+sk+1)/(s+1) \rceil$. The distance graph $G(Z, D_{m,k,s})$ is a subgraph of $G(Z, D_{\bar{m},k,s})$, so $\chi(Z, D_{m,k,s}) \leq \chi(Z, D_{\bar{m},k,s})$. Let $\bar{m}+sk+1 = (s+1)^{\bar{b}} \bar{m}'$, where $\bar{m}'$ is not divisible by $(s+1)$. Since $(\bar{m}+sk+1)/(s+1) \equiv r \equiv 0 \pmod{s+1}$, $\bar{b} \geq 2 > 1 = a$. Thus by Theorems 2 and 3, we have

$$\lceil (m+sk+1)/(s+1) \rceil \leq \chi(Z, D_{m,k,s}) \leq \chi(Z, D_{\bar{m},k,s}) = (\bar{m}+sk+1)/(s+1).$$

Therefore, $\chi(Z, D_{m,k,s}) = \lceil (m+sk+1)/(s+1) \rceil$.

Case 2. $1 \leq r \leq s$. Since $s+1$ is a prime, there exists an integer $1 \leq t \leq s$ such that $tr \equiv 1 \pmod{s+1}$. Define a pre-coloring c of the set Z^* with $s+1$ colors as follows. For each integer $j \in Z^*$, express j uniquely in the form $j = u(s+1)+v$, where $0 \leq v \leq s$. Then let $c(j) = (ut+v) \bmod (s+1)$. We shall show that c satisfies (1) and (2) in Corollary 7 with $n = \lceil (m+sk+1)/(s+1) \rceil$.

Let $j \in Z^*$. Assume, contrary to (1) of Corollary 7, $c(j-hk) = c(j-h'k)$ for some $0 \leq h < h' \leq s$. Let $j-hk = u(s+1)+v$ and $j-h'k = u'(s+1)+v'$, then $ut+v \equiv u't+v' \pmod{s+1}$. Because $a=1$, $(s+1)$ divides k, which implies $j-hk \equiv j-h'k \pmod{s+1}$, so $v=v'$. Hence, $ut-u't \equiv 0 \pmod{s+1}$. This is impossible because $(t, s+1) = 1$ and $0 < u'-u \leq s$.

Now we show that among any $m-k+1$ consecutive integers, there are at most $\lceil (m-k+1)/(s+1) \rceil$ vertices of pre-color 0. Similarly to the proof of Theorem 13, we divide the set Z^* into segments of length $s+1$ by $A_0 = \{0, 1, \cdots, s\}$, $A_1 = \{s+1, s+2, \cdots, 2s+1\}, \cdots, A_i = \{i(s+1), i(s+1)+1, \cdots, (i+1)(s+1)-1\}, \cdots$. Then each of the segments A_i contains exactly one vertex of each pre-color. Indeed, it is straightforward to verify that the pre-colors of the segment A_i are $\{j, j+1, \cdots, s, 0, 1, \cdots, j-1\}$, where $i \equiv v \pmod{s+1}$, $0 \leq v \leq s$, and $j = vt \bmod (s+1)$.

Let $Y = \{y, y+1, \cdots, y+m-k\}$ be a set of $m-k+1$ consecutive non-negative integers. Suppose $y \in A_i$ and $y+m-k \in A_{i'}$. If $|Y \cap A_i| + |Y \cap A_{i'}| \geq s+1$, then Y intersects $\lceil (m-k+1)/(s+1) \rceil$ segments. Hence Y contains at most $\lceil (m-k+1)/(s+1) \rceil$ vertices of pre-color 0.

Assume $|Y \cap A_i| + |Y \cap A_{i'}| < s+1$, then $i'-i = \lceil (m-k+1)/(s+1) \rceil \equiv \lceil (m+sk+1)/(s+1) \rceil \equiv r \pmod{s+1}$. Recall that $tr \equiv 1 \pmod{s+1}$. Hence, if

A_i is pre-colored by colors $\{j, j+1, \cdots, s, 0, 1, \cdots, j-1\}$, then $A_{i'}$ is pre-colored by colors $\{j+1, j+2, \cdots, s, 0, 1, \cdots, j\}$. Since $|Y \cap A_i| + |Y \cap A_{i'}| < s+1$, we conclude that pre-color 0 is used at most once in the set $(Y \cap A_i) \cup (Y \cap A_{i'})$. Therefore, at most $\lceil (m-k+1)/(s+1) \rceil$ vertices of Y have pre-color 0. This completes the proof of Theorem 16. Q.E.D.

In the next result, we write $m-k+1$ in the form $m-k+1 = u(s+1)k + vk + p(s+1) + q$, where u, v, p, q are integers such that $u \geq 0, \quad 0 \leq v \leq s, \quad 0 \leq p < k/(s+1), \quad 0 \leq q \leq s$. It is easy to see that the integers u, v, p, q are uniquely determined by $m-k+1$.

Theorem 17 *Suppose $m \geq (s+1)k$, k is a multiple of the prime $s+1$, but $m+sk+1$ is not. Let u, v, p, q be integers defined as above. If $q \leq v+1$, then $\chi(Z, D_{m,k,s}) = \lceil (m+sk+1)/(s+1) \rceil$.*

Proof. It suffices to show that $G(Z, D_{m,k,s})$ is $\lceil (m+sk+1)/(s+1) \rceil$-colorable. Define a pre-coloring as follows. First, partition the set of Z^* into blocks recursively in such a way that the first k vertices are divided into $k-1$ blocks with $k-2$ single-vertex blocks followed by one block with two vertices. Then repeat the same process to the next k vertices and so on. Next, pre-color the blocks periodically with pre-colors $\{0, 1, 2, \cdots, s\}$, that is, every vertex in the first block is pre-colored by 0 and so on. It is enough to show that the pre-coloring satisfies (1) and (2) of Corollary 7, with $n = \lceil (m+sk+1)/(s+1) \rceil$.

First we prove that for any $j \geq sk$, the $s+1$ vertices $j, \ j-k, \cdots, \ j-sk$ receive distinct pre-colors. Suppose $0 \leq t < t' \leq s$. Let the pre-colors of $j-t'k$ and $j-tk$ be x and y, respectively. Because $s+1$ divides k, and $s+1$ is prime, we have $(s+1, k-1) = 1$. As $(j-tk)-(j-t'k) = (t'-t)k$ and any consecutive k vertices are divided into $k-1$ blocks, so $y \equiv x + (t'-t)(k-1) \pmod{s+1}$. Hence, we conclude that $x \neq y$, since $1 \leq t'-t < s+1$ and $(s+1, k-1) = 1$.

Next we prove that among any $m-k+1$ consecutive vertices, there are at most $\lceil (m-k+1)/(s+1) \rceil$ vertices with pre-color 0. Given a set Y of $m-k+1$ consecutive non-negative integers, we may assume that the first two vertices of Y have pre-color 0. Among the first $u(s+1)k$ vertices of Y, exactly uk of them have pre-color 0, because any consecutive $(s+1)k$ vertices are evenly pre-colored, *i.e.*, there are exactly k vertices of each pre-color.

The assumption that $m+sk+1$ is not a multiple of $s+1$ implies that $m-k+1$ is not a multiple of $s+1$. Because k is a multiple of $s+1$ while $m-k+1$ is not, $p(s+1)+q \geq 1$. If $p(s+1)+q \geq 2$, then among the remaining $vk+p(s+1)+q$ vertices of Y, there are $v+1$ blocks of size 2. If we remove one vertex from each of these blocks of size 2, then the remaining $vk+p(s+1)+q-v-1$ vertices of Y are *almost* evenly pre-colored, that is, the numbers of vertices with same pre-colors differ by at most one. Hence at most $\lceil (vk+p(s+1)+q-v-1)/(s+1) \rceil$ of them have pre-color 0. On the other hand, among the removed vertices, exactly one vertex has precolor 0. Therefore, the total number of vertices of pre-color 0 is at most $uk+1+\lceil (vk+p(s+1)+q-v-1)/(s+1) \rceil = \lceil (m-k+1)/(s+1) \rceil$. Note that the last equality is due to the assumption that $q \leq v+1$.

Finally, we assume $p(s+1)+q = 1$. Then it is straightforward to verify that either $v=0$, or the pre-color of the last vertex is not 0. Consider the remaining $vk+p(s+1)+q = vk+1$ vertices of Y. If $v=0$, then there is one vertex of pre-color 0. If the pre-color of the last vertex is not 0, then among the remaining $vk+1$ vertices of Y, there are v blocks of size 2. If we remove one vertex from each of these blocks of size 2, then the remaining $vk-v$ vertices of Y are almost evenly pre-colored, so at most $\lceil (vk-v)/(s+1) \rceil$ of them have pre-color 0. On the other hand, among the vertices taken away, only one has pre-color 0. Hence, there are at most $1+\lceil (vk-v)/(s+1) \rceil = \lceil (vk+1)/(s+1) \rceil$ (because $v \leq s$) vertices of pre-color 0 in the remaining $vk+1$ vertices of Y. Therefore, we conclude that Y has at most

$uk + \lceil (vk+1)/(s+1) \rceil = \lceil (m-k+1)/(s+1) \rceil$ vertices with pre-color 0. This completes the proof. Q.E.D.

Corollary 18 *Suppose $m \geq (s+1)k$, k is a multiple of the prime $s+1$, but $m+sk+1$ is not. Let u, v, p, q be the same as defined in Theorem 17. If $v \geq s-1$, or $q \leq 1$, then $\chi(Z, D_{m,k,s}) = \lceil (m+sk+1)/(s+1) \rceil$.*

Note that when $s = 1$, then $v \geq s-1$ is always true, hence we have the following corollary which was proved in [2]:

Corollary 19 *Suppose $s = 1$, $m \geq 2k$, $k = 2^a k'$ and $m+k+1 = 2^b m'$, where k' and m' are odd. Then*

$$\chi(Z, D_{m,k,1}) = \begin{cases} \lceil (m+k+1)/2 \rceil, & \text{if } b = 0 \text{ or } a < b; \\ ((m+k+1)/2) + 1, & \text{if } 0 < b \leq a. \end{cases}$$

Proof. The case as $b = 0$ follows from Corollary 18; and the case as $b > 0$ follows from Theorem 14. Q.E.D.

Recall that $k = (s+1)^a k'$ where $a \geq 1$ and k' is not divisible by $s+1$, and $m-k+1$ is not divisible by $s+1$. In order to introduce the next result, we need the following definitions and notations. For any factor x of k', define:

$$\begin{aligned} q(x) &:= \lceil (m-k+1)/((s+1)^a x) \rceil \bmod (s+1); \\ m(t,x) &:= \max\{t(q(x)-1) \bmod (s+1), tq(x) \bmod (s+1)\}, 1 \leq t \leq s; \\ f(x) &:= \min\{m(t,x) : 1 \leq t \leq s\}. \end{aligned}$$

Finally, define $f := \min\{f(x) : x \text{ is a factor of } k'\}$.

Note that for given m, k and s, the integer f in the above is uniquely determined. Similarly as in Theorem 17, we let $q = (m-k+1) \bmod (s+1)$.

Theorem 20 *Given m, k and s where $m \geq (s+1)k$ and $s+1$ is a prime, let f, q be defined as above. If $f + q \leq s+1$, then $\chi(Z, D_{m,k,s}) = \lceil \chi_f(Z, D_{m,k,s}) \rceil = \lceil (m+sk+1)/(s+1) \rceil$.*

Proof. Suppose $f = f(x) = m(t, x)$ for some factor x of k' and some $1 \le t \le s$. Express any integer $j \in Z^*$ in the following form:

$$j = u(s+1)^a x + v(s+1) + w,$$

where $u \ge 0$, $0 \le v < (s+1)^{a-1}x$ and $0 \le w \le s$.

It is easy to see that for each j, the integers u, v, w in the form above are uniquely determined by j. Define a pre-coloring c using the $s+1$ pre-colors $\{0, 1, \cdots, s\}$ by $c(j) = (ut + w) \bmod (s+1)$. In order to prove $G(Z, D_{m,k,s})$ is $\lceil (m+sk+1)/(s+1) \rceil$-colorable, it suffices to show that c satisfies (1) and (2) of Corollary 7, with $n = \lceil (m+sk+1)/(s+1) \rceil$.

First, let j be any non-negative integer, we shall show that $c(j), c(j-k), c(j-2k), \cdots, c(j-sk)$ are all distinct. Let $0 \le p' < p \le s$. If $j - pk = u(s+1)^a x + v(s+1) + w$, then

$$\begin{aligned} j - p'k &= u(s+1)^a x + v(s+1) + w + (p-p')k \\ &= u(s+1)^a x + v(s+1) + w + (p-p')(s+1)^a k' \\ &= u'(s+1)^a x + v(s+1) + w. \end{aligned}$$

Because $(s+1, k') = (p-p', s+1) = 1$, one has $(u'-u, s+1) = 1$. Assume $c(j-pk) = c(j-p'k)$, then $ut + w \equiv u't + w \pmod{s+1}$. Hence $t(u'-u) \equiv 0 \pmod{s+1}$, which is impossible, since $s+1$ is prime and $(t, s+1) = (u'-u, s+1) = 1$. This proves that c satisfies (1) of Corollary 7.

Next, we prove that among any $m-k+1$ consecutive integers, there are at most $\lceil (m-k+1)/(s+1) \rceil$ vertices with pre-color 0. Divide the vertex set Z^* evenly into segments of length $s+1$ by $A_0 = \{0, 1, 2, \cdots, s\}$, $A_1 = \{s+1, s+2, \cdots, 2s+1\}, \cdots, A_i = \{i(s+1), i(s+1)+1, \cdots, (i+1)(s+1)-1\}, \cdots$. Then each of the segments A_i contains exactly one vertex of each pre-color. Indeed, the pre-colors of the segment A_i are $\{j, j+1, \cdots, s, 0, 1, \cdots, j-1\}$, where $j = ut \bmod (s+1)$, and u is the unique integer such that $i = u(s+1)^{a-1}x + v$, $0 \le v < (s+1)^{a-1}x$.

Let Y be a set of $m-k+1$ consecutive integers, $Y = \{y, y+1, \cdots, y+m-k\}$.

Suppose $y \in A_i$ and $y + m - k \in A_{i'}$. If $|Y \cap A_i| + |Y \cap A_{i'}| \geq s+1$, then Y has at most $\lceil (m-k+1)/(s+1) \rceil$ vertices with pre-color 0 (cf. proof of Theorem 16).

Now we assume that $|Y \cap A_i| + |Y \cap A_{i'}| < s+1$, then $|Y \cap A_i| + |Y \cap A_{i'}| = q$. Suppose $i = u(s+1)^{a-1}x + v$ and $i' = u'(s+1)^{a-1}x + v'$, where $0 \leq v, v' < (s+1)^{a-1}x$. Then by the definition of $q(x)$, either $u' - u = q(x)$ or $u' - u = q(x) - 1$. Suppose $\alpha = q(x)t \bmod (s+1)$ and $\beta = (q(x)-1)t \bmod (s+1)$. Then by the choice of x and t, one has $\alpha, \beta \leq f$.

Suppose the pre-colors of A_i are $\{j, j+1, \cdots, s, 0, 1, \cdots, j-1\}$. Then the pre-colors of $A_{i'}$ are either $\{j+\alpha, j+\alpha+1, \cdots, s, 0, 1, \cdots, j+\alpha-1\}$, if $u' - u = q(x)$; or $\{j+\beta, j+\beta+1, \cdots, s, 0, 1, \cdots, j+\beta-1\}$, if $u' - u = q(x) - 1$.

Any other segment different from A_i and $A_{i'}$ is either disjoint from Y or contained in Y. As each segment contains exactly one vertex of each color, to prove that Y has at most $\lceil (m-k+1)/(s+1) \rceil$ vertices with pre-color 0, it suffices to show that the pre-color 0 is used at most once in the union $(Y \cap A_i) \cup (Y \cap A_{i'})$. Assume that 0 is used in both $Y \cap A_i$ and $Y \cap A_{i'}$. Without loss of generality, we may assume that the pre-colors of $A_{i'}$ are $\{j+\alpha, j+\alpha+1, \cdots, s, 0, 1, \cdots, j+\alpha-1\}$. Then one has $|Y \cap A_i| \geq j$ and $|Y \cap A_{i'}| \geq s+1-(j+\alpha-1)$. It follows that $q = |(Y \cap A_i) \cup (Y \cap A_{i'})| \geq s+2-\alpha$, contrary to the assumption that $\alpha + q \leq f + q \leq s+1$. Therefore c satisfies (2) of Corollary 7, with $n = \lceil (m+sk+1)/(s+1) \rceil$. This completes the proof of Theorem 20. Q.E.D.

Corollary 21 *If $m \geq (s+1)k$, $s+1$ is prime, and there is a factor x of k' such that $q(x) \leq 1$, then $\chi(Z, D_{m,k,s}) = \lceil (m+sk+1)/(s+1) \rceil$. In particular, if $\lceil (m-k+1)/k \rceil$ mod $(s+1) \leq 1$, then $\chi(Z, D_{m,k,s}) = \lceil (m+sk+1)/(s+1) \rceil$.*

Proof. According to definition, if $q(x) = 1$, then $m(1, x) = 1$; if $q(x) = 0$, then $m(t, x) = 1$ for some t such that $ts \equiv 1 \pmod{s+1}$. (Such a t exists, because

$(s, s+1) = 1$.) In any of the two cases, $f = 1$, so $f + q \leq s + 1$. Therefore, $\chi(Z, D_{m,k,s}) = \lceil (m + sk + 1)/(s + 1) \rceil$ by Theorem 20. Q.E.D.

Applying Theorem 14 and Corollaries 18 and 21, we are able to completely settle the case $s = 2$.

Corollary 22 *Suppose $s = 2$, $m \geq 3k$, $k = 3^a k'$ and $m + 2k + 1 = 3^b m'$, where k' and m' are not multiples of 3. Then*

$$\chi(Z, D_{m,k,2}) = \begin{cases} \lceil (m + 2k + 1)/3 \rceil, & \text{if } b = 0 \text{ or } a < b; \\ (m + 2k + 1)/3 + 1, & \text{if } 0 < b \leq a. \end{cases}$$

Proof. According to Theorem 14, we only have to show the case as $b = 0$. Suppose $m - k + 1 = u(s + 1)k + vk + p(s + 1) + q$. If $v \neq 0$, then the conclusion follows from Corollary 18. If $v = 0$, then the conclusion follows from Corollary 21. (Because $\lceil (m - k + 1)/k \rceil \bmod (s + 1) \leq 1$.) Q.E.D.

Remarks. New results related to this topic have been obtained since the submission of this paper. In [5], it was proved that $\chi(G(Z, D_{m,k,s})) \leq \lceil (m+sk+1)/(s+1) \rceil + 1$ for all $m \geq (s + 1)k$. Then in [11], the chromatic numbers of all the graphs $G(Z, D_{m,k,s})$ are completely determined. The circular chromatic number of the class of distance graphs $G(Z, D_{m,k,s})$ was studied in [1, 11, 19], and the value of $\chi_c(Z, D_{m,k,s})$ has been completely determined in [19]. (The circular chromatic number $\chi_c(G)$ of a graph G is a refinement of $\chi(G)$, and $\chi(G) = \lceil \chi_c(G) \rceil$ for any graph G. For a survey of research concerning circular chromatic number of graphs, see [20].)

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