Diophantine approximations and its applications to graph colouring problems

Xuding Zhu
Department of Applied Mathematics
National Sun Yat-sen University
Kaohsiung, Taiwan 80424
Email: zhu@ibm18.math.nsysu.edu.tw

Nov. 20, 1998

Abstract

For a real number x, let $||x||$ denote the distance from x to the nearest integer. Suppose $x_1 < x_2 < x_3$ are positive integers with $\gcd(x_1, x_2, x_3) = 1$. This paper proves the following: if $(x_1, x_2, x_3) \neq (1, 2, 3s)$ for an integer s and $x_3 \neq x_1 + x_2$, or $x_3 = x_1 + x_2$ but $x_1 \equiv x_2 \pmod 3$, then there is a real number t such that $||tx_i|| \geq 1/3$ (for $i = 1, 2, 3$). If $(x_1, x_2, x_3) = (1, 2, 3s)$ or $x_3 = x_1 + x_2$ and $x_1 \not\equiv x_2 \pmod 3$, then no such t exists, i.e., for any t, there is an i such that $||tx_i|| < 1/3$. This result is connected to problems of different fields of mathematics. Firstly, it is a strengthening of the $k = 3$ case of Wills' conjecture, which says that for any k positive integers $x_1, x_2, \cdots, x_k$, there is a real number t such that $||tx_i|| \geq \frac{1}{k+1}$. Secondly, it is applied to graph theory in determining the chromatic number of certain distance graphs, which confirms a conjecture proposed independently by Chen, Chang and Huang [J. Graph Theory, 25(1997)287-294] and Voigt [Ars Combinatoria, to appear]. Thirdly, it has an application to the so called *view obstruction* problem in the 3 dimensional Euclidean space. Fourthly, it has an application to the study of flows in graphs and matroids.

1991 Mathematics Subject Classification. Primary 05C15, 11J13, 11J25.

Keywords: diophantine approximation, view-obstruction, flows, distance graphs.

1 Introduction

For any real number x, let $||x||$ denote the distance from x to the nearest integer. For an k-tuple $\vec{x} = (x_1, x_2, \cdots, x_k)$ of positive integers, let

$$\kappa(\vec{x}) = \sup_{t \in R} \min_{i=1}^{k} ||tx_i||.$$

Note that the order of the integers in the k-tuple has no effect on the value of $\kappa(\vec{x})$. So we assume that $x_1 < x_2 < \cdots < x_k$. It is also easy to see that we may assume that $\gcd(x_1, x_2, \cdots, x_k) = 1$. It was conjectured by Wills [46] that for any k-tuple $\vec{x} = (x_1, x_2, \cdots, x_k)$ of positive integers, $\kappa(\vec{x}) \geq \frac{1}{k+1}$. (In the original conjecture, x_i could be any nonzero real number, but Wills showed that it is equivalent to the case that all the x_i's are positive integers.) This purely number theoretic conjecture turns out to be related to many other problems. Cusick [12, 13, 14, 15] studied this conjecture, motivated by a beautiful application in k dimensional geometry-*view obstructions*; Bienia et al [2] studied this conjecture, motivated by another beautiful application in graphs and matroids-*the existence of flows*; this author encountered the problem of evaluating $\kappa(\vec{x})$ when studying the chromatic numbers of distance graphs [16, 50, 51, 52, 53]. It is interesting to see that different people motivated by completely different problems found them interested in a common problem in number theory.

Wills' conjecture has been studied by many authors in the past thirty years [3, 2, 7, 12, 13, 14, 15, 45, 46, 47, 48, 49]. However, it still remains open for $k \geq 5$. For $k = 1$, it is trivial; the case $k = 2$ was proved in [45]; for $k = 3$, it is already difficult and quite a few different proofs have been published [3, 2, 12, 13, 14]; for $k = 4$, the first proof, given by Cusick and Pomerance [15], is a complicated argument involving exponential sums and electronic case checking, but recently, Bienia et al [2] gave an elementary selfcontained proof of this case.

As observed by Wills [45] (and others), the lower bound $\frac{1}{k+1}$ in Wills' conjecture is sharp. For example, if $\vec{x} = (1, 2, \cdots, k)$, then $\kappa(\vec{x}) = \frac{1}{k+1}$. However, it seems that very few k-tuples of integers attain this lower bound. For example, for $k = 2$, $\vec{x} = (1, 2)$ is the only pair for which $\kappa(\vec{x}) = \frac{1}{3}$, and for $k = 3$, $\vec{x} = (1, 2, 3)$ is the only triple for which $\kappa(\vec{x}) = \frac{1}{4}$ [12, 7]. For pairs of integers $\vec{x} = (x_1, x_2)$, the value of $\kappa(\vec{x})$ is determined in [2, 4, 5, 37]. (In [5], the authors were interested in determining the fractional chromatic number and circular chromatic number of some distance graphs. The determination of $\kappa(\vec{x})$ for $\vec{x} = (x_1, x_2)$, which was not mentioned explicitly, is just a step in the proof that leads to the solution of another problem. In [37], the authors were interested in the asymptotic efficiency of T-colorings. Again the determination of $\kappa(\vec{x})$ for $\vec{x} = (x_1, x_2)$ was not mentioned explicitly, and it is just a step in the proof that leads to the solution of their problem. In [4], the authors were interested in determining the density of some special subsets of the set of integers, which they called the D-sets. The problem of evaluating $\kappa(\vec{x})$ was also not mentioned explictly. So these people were motivated by completely different problems, and end up at solving the same problem.) For $k \geq 3$, besides the attempts for proving Wills conjecture, it seems that no efforts have been made to evaluate $\kappa(\vec{x})$. Since Wills conjecture is still open for $k \geq 5$, and is already difficult for $k = 3$ and $k = 4$, it is natural that it is difficult to determine or estimate the value of $\kappa(\vec{x})$ for $\vec{x} = (x_1, x_2, \cdots, x_k)$ ($k \geq 3$). On the other hand, such efforts may help us understand more about the function $\kappa(\vec{x})$, and shed new lights on Wills' conjecture.

In this paper we estimate the value of $\kappa(\vec{x})$ for triples $\vec{x} = (a, b, c)$ of integers. The following theorem is proved:

Theorem 1.1 *Suppose $\vec{x} = (a, b, c)$ is a triple of positive integers with $a < b < c$ and $gcd(a, b, c) = 1$. Then $\frac{1}{4} \leq \kappa(\vec{x}) < \frac{1}{3}$ if either $\vec{x} = (1, 2, 3s)$ for an integer s or $c = a + b$ and $a \not\equiv b \pmod 3$. Otherwise $\frac{1}{3} \leq \kappa(\vec{x}) \leq \frac{1}{2}$. Moreover $\kappa(\vec{x}) = \frac{1}{2}$ if and only if all the three integers are odd.*

Theorem 1.1 has many connections to problems of different fields of mathematics. Firstly, it is a strengthening of the $k = 3$ case of Wills' conjecture, and provides us more information about the function $\kappa(\vec{x})$. Such information might be helpful for approaching Wills's conjecture for $k \geq 5$. Secondly, it has a nice application to the study of the chromatic number of distance graphs, which settles in affirmative a conjecture that was proposed independently by Chen, Chang and Huang [8] and Voigt [43]. This application is indeed the original motivation for this author to study the value of $\kappa(\vec{x})$. Thirdly, it can also be interpreted as a result concerning view-obstruction problems in the 3 dimensional geometry, which was first studied by Cusick [12]. Fourthly, it has an application to the study of flows in graphs and matroids. Moreover, the function $\kappa(\vec{x})$ is also connected to the circular chromatic number and the fractional chromatic number of distance graphs, and through the fractional chromatic of distance graphs, it is connected to the density of D-sets, and to the asymptotic coloring efficiency of the T colorings (or channel assignments), and to the star-extremality of distance graphs and circulant graphs.

In Section 2, we discuss applications of Theorem 1.1. Sections 3-9 are devoted to the proof of Theorem 1.1.

2 Applications of Theorem 1.1

First we apply Theorem 1.1 to settle a conjecture concerning the chromatic number of distance graphs, which was proposed independently by Chen, Chang, Huang [8] and Voigt [43].

Given a subset D of positive integers, we denote by $G(Z, D)$ the graph with vertex set Z in which $i \sim j$ if $|i - j| \in D$. The graph $G(Z, D)$ is called the *distance graph* generated by the *distance set* D. We are interested in determining the chromatic number $\chi(G(Z, D)$ of the graph $G(Z, D)$, i.e., find the minimum integer k, for which there exists a mapping $c : Z \to \{1, 2, \cdots, k\}$ such that $c(i) \neq c(j)$ whenever $|i - j| \in D$. Such a mapping c is called a *k-coloring* of $G(Z, D)$, and $c(x)$ is called the *color* assigned to x.

The problem of determining the chromatic numbers of distance graphs was studied in [5, 6, 8, 16, 17, 18, 19, 20, 21, 22, 29, 30, 42, 43, 44, 50, 51, 52, 53]. Most of the work deal with distance graphs generated by some special distance sets. Here we are interested in the case that $D = \{a, b, c\}$ contains three integers. The class of distance graphs with $D = \{a, b, c\}$ has also been studied by many authors [8, 16, 18, 43, 50, 51]. However, only some very special cases have been settled. The following conjecture concerning the chromatic number of such distance graphs was proposed independently by Chen, Chang, Huang [8] and Voigt [43]:

Conjecture 2.1 *Suppose $D = \{a, b, c\}$, where $a < b < c$ and gcd$(a, b, c) = 1$. Then $\chi(G(Z, D)) = 4$ if $D = (1, 2, 3s)$ for some integer s or $c = a + b$ and $a \not\equiv b \pmod 3$; $\chi(G(Z, D)) = 2$ if a, b, c are odd; and $\chi(G(Z, D)) = 3$ if none of the conditions above is satisfied.*

We now use Theorem 1.1 to prove Conjecture 2.1.

Theorem 2.1 *Conjecture 2.1 is true.*

Proof.

It is easy to verify the following (cf. [8, 43]):

- For $D = \{a, b, c\}$, $2 \leq \chi(G(Z, D)) \leq 4$;
- If $D = (1, 2, 3s)$ for some integer s, or $c = a + b$ and $a \not\equiv b \pmod 3$, then $\chi(G(Z, D)) = 4$;
- $\chi(G(Z, D)) = 2$ if and only if a, b, c are all odd.

Therefore to prove Theorem 2.1, it suffices to show that $G(Z, D)$ is 3-colorable, provided that $D \neq (1, 2, 3s)$, and $c \neq a + b$, or $c = a + b$ but $a \equiv b$ (mod 3). By Theorem 1.1, under these conditions, we have $\kappa(\vec{x}) \geq \frac{1}{3}$, where

$\vec{x} = (a, b, c)$. This means that there is a real number t and there are integers i, j, k such that

$$i + \frac{1}{3} \le ta \le i + \frac{2}{3}, \quad j + \frac{1}{3} \le tb \le j + \frac{2}{3}, \quad k + \frac{1}{3} \le tc \le k + \frac{2}{3}.$$

Thus

$$3t \in [\frac{3i+1}{a}, \frac{3i+2}{a}] \cap [\frac{3j+1}{b}, \frac{3j+2}{b}] \cap [\frac{3k+1}{c}, \frac{3k+2}{c}].$$

Let $r = \frac{1}{3t}$. We partition the real line R into half open intervals of length r, i.e., let

$$I_s = [sr, (s+1)r) \text{ for } \ s = 0, \pm 1, \pm 2, \cdots.$$

Define a mapping $\Delta : Z \to \{1, 2, 3\}$ as follows:

$$\Delta(x) = s \bmod 3, \text{ if } \ x \in I_s.$$

We shall show that Δ is a proper 3-coloring of the distance graph $G(Z, D)$.

Suppose $u \sim v$ in $G(Z, D)$, then $|u - v| \in D$. Without loss of generality, we may assume that $u - v = a$. We shall prove that $\Delta(u) \neq \Delta(v)$.

Assume $u \in I_s$ and $v \in I_{s'}$, and assume to the contrary, that we have $\Delta(u) = \Delta(v)$. Then $s \equiv s' \pmod 3$, i.e., $s - s' = 3h$ for some integer h. Since

$$sr \le u < (s+1)r \ \text{ and } s'r \le v < (s'+1)r,$$

we have

$$(s - s' - 1)r < u - v = a < (s + 1 - s')r.$$

This implies that

$$\frac{3h-1}{a} < \frac{1}{r} < \frac{3h+1}{a},$$

contrary to the assumption that

$$\frac{1}{r} = 3t \in [\frac{3i+1}{a}, \frac{3i+2}{a}]$$

for some integer i. Therefore Δ is a proper 3-coloring of the distance graph $G(Z, D)$, and hence $\chi(G(Z, D)) \le 3$. ∎

Now we discuss an interpretation of Theorem 1.1 in the study of view-obstructions in the 3 dimensional Euclidean space.

Suppose the unit cube C in the k-dimensional Euclidean space E^k has faces which reflect a certain particle, and in the center of C we place a subcube αC, where $0 < \alpha < 1$ and αC is the magnification of C by the factor α. Suppose a particle is ejected from a corner of the cube C, and its movement is not contained in one of the faces of C. How large should α be so that no matter how the particle is ejected, it will hit the subcube ?

This problem is equivalent to the following: Let S^k denote the region $0 < x_i < \infty$ $(i = 1, 2, \cdots, k)$ of the k-dimensional Euclidean space E^k. Partition S^k into k-dimensional cubes of side 1 whose vertices are at the points with integer coordinates. Place a copy of αC into each of these cubes at the center. Let $\Delta(C, \alpha)$ be the union of these copies of αC. To be precise,

$$\Delta(C, \alpha) = \{\alpha C + (m_1 + \frac{1}{2}, m_2 + \frac{1}{2}, \cdots, m_k + \frac{1}{2}) : \ m_i \text{ nonnegative integers}\}.$$

How large should the factor α be so that $\Delta(C, \alpha)$ will "block the sky", i.e., any half line L, given by $x_i = a_i t$, $a_i > 0, t \geq 0$ $(i = 1, 2, \cdots, k)$, will intersect $\Delta(C, \alpha)$?

This minimum α for which every half line L, given by $x_i = a_i t$, $a_i > 0, t \geq 0$ $(i = 1, 2, \cdots, k)$, intersects $\Delta(C, \alpha)$ is a function of k, which is denoted by $\lambda(k)$. It was conjectured by Cusick that $\lambda(k) = \frac{k-1}{k+1}$. Cusick proved that this conjecture is equivalent to Wills' conjecture. So it has been confirmed for $k \leq 4$. If the conjecture is true, then for any $\alpha < \frac{k-1}{k+1}$, there are some half lines L, given by $x_i = a_i t$, $a_i > 0, t \geq 0$ $(i = 1, 2, \cdots, k)$, which do not intersect $\Delta(C, \alpha)$. In other words, these lines give the angles at which one can "see through" the sky.

Now we restrict to the 3 dimensional Euclidean space, i.e., consider the case that $k = 3$. Cusick's conjecture was proved for $k = 3$ [12], i.e., $\lambda(3) = \frac{1}{2}$. Thus if $\alpha < \frac{1}{2}$, then there are some half lines L, given by $x_i = a_i t$, $a_i > 0, t \geq 0$ $(i = 1, 2, 3)$, do not intersect $\Delta(C, \alpha)$. A half line L in R^3, given by $x_i = a_i t$, $a_i > 0, t \geq 0$ $(i = 1, 2, 3)$, is called *critical* if for any $\alpha < \frac{1}{2}$, L does not intersect $\Delta(C, \alpha)$. It was shown in [12] that L is critical if and only if $(a_1, a_2, a_3) = (1, 2, 3)$.

It is obvious that as α becomes smaller and smaller, more and more half lines will not intersect $\Delta(C, \alpha)$. Intuitively, a person standing at the origin will have more angles to "see through the sky". However, our next result shows that provided $\alpha \geq 1/3$, there are not "many" angles one could see through the sky. To be precise, all the half lines that do not intersect $\Delta(C, \alpha)$ either lie on the plane $x_3 = x_1 + x_2$, or has direction $(a, b, c) = (1, 2, 3s)$ for some integer s.

Theorem 2.2 *Let C be the unit cube in R^3, and let $\Delta(C, \frac{1}{3})$ be defined as above. Then a half line L, given by $L = \{(ta, tb, tc) : t > 0, a, b, c \in Z^+, a < b < c, (a, b, c) = 1\}$, does not intersect $\Delta(C, \frac{1}{3})$ if and only if either $(a, b, c) = (1, 2, 3s)$ for some integer s, or $c = a + b$ and $a \not\equiv b \pmod 3$.*

Proof. The line $L = \{(ta, tb, tc) : t > 0, a, b, c \in Z^+, a < b < c, (a, b, c) = 1\}$, does not intersect $\Delta(C, \frac{1}{3})$ if and only if for any $t > 0$, there is an $u \in \{a, b, c\}$ such that

$$2||tu - \frac{1}{2}|| > \frac{1}{3}.$$

Since $||x|| = \frac{1}{2} - ||x - \frac{1}{2}||$, the condition above is equivalent to

$$2(\frac{1}{2} - ||tu||) > \frac{1}{3}$$

for some $u \in \{a, b, c\}$. Therefore the line L does not intersect $\Delta(C, \frac{1}{3})$ if and only if for any $t > 0$, there is an $u \in \{a, b, c\}$ such that $||tu|| < \frac{1}{3}$. Let $\vec{x} = (a, b, c)$. The condition above is equivalent to $\kappa(\vec{x}) < \frac{1}{3}$. By Theorem 1.1, this happens if and only if $(a, b, c) = (1, 2, 3s)$ or $c = a + b$ and $a \not\equiv b \pmod 3$. ∎

Finally, we give an application of Theorem 1.1 to the study of flows in graphs. Given a graph $G = (V, E)$. Let $\vec{G} = (V, \vec{E})$ be an arbitrary orientation of the edges of G. For a vertex $v \in V$, denote by $E^+(v)$ the set of edges which have v as one of their end vertex, and direct from v to the other end vertex, denote by $E^-(v)$ the set of edges which have v as one of their end vertex, and direct to v from the other end vertex. Intuitively, $E^+(v)$ is the set of edges leaving v, and $E^-(v)$ is the set of edges entering v.

A *flow* of $\vec{G}$ is an assignment f of real numbers to the edges $\vec{E}$ of $\vec{G}$ such that for each vertex v, $\Sigma_{e \in E^+(v)} f(e) = \Sigma_{e \in E^-(v)} f(e)$, i.e., the total amount of flow entering v is equal to the total amount of flow leaving v. A *nowhere zero flow* of $\vec{G}$ is a flow f such that $f(e) \neq 0$ for all $e \in \vec{E}$. Given an integer $n \geq 2$, a *nowhere zero k-flow* of $\vec{G}$ is a flow f such that for each $e \in \vec{E}$, $|f(e)| \in \{1, 2, \cdots, k-1\}$. It is known [1] that whether or not $\vec{G}$ has a nowhere zero k-flow does not depend on the orientation of the edges, rather it is a property of the undirected graph G. The theory of nowhere zero k-flow is a major topic in combinatorics. It was proved by Seymour [38] that every bridgeless graph has a nowhere zero 6-flow, and conjectured by Tutte [41] that every bridgeless graph has a nowhere zero 5-flow. By applying Seymour's 6-flow theorem and the $n \leq 4$ cases of Wills' conjecture, Bienia et al [2] proved the following result:

Theorem 2.3 *If G has a nowhere zero flow f such that $|f(e)|$ assumes at most $k-1$ distinct values, then G has a nowhere zero k-flow.*

Given a set S of $k-1$ positive real numbers, we ask the question whether a graph G has a flow f such that $|f(e)| \in S$ for every e. Intuitively, Theorem 2.3 says that among all $(k-1)$ sets, the set $\{1, 2, \cdots, k-1\}$ is the "best", i.e., if the answer for the question above is "yes" for any $(k-1)$ set S, then the answer is "yes" for $\{1, 2, \cdots, k-1\}$. Our next theorem says that for $k = 4$, most other 3 sets are actually merely as good as the set $\{1, 2\}$.

Theorem 2.4 *Assume G has a nowhere zero flow f such that $|f(e)|$ assumes at most 3 distinct values $\{a, b, c\}$, where $a < b < c$. If $\{a, b, c\} \neq \{a, 2a, 3sa\}$ and $c \neq a + b$, then G has a nowhere zero 3-flow. Moreover, if $c = a + b$*

but there is no real number r such that $a/r, b/r, c/r$ are all integers, gcd $(a/r, b/r, c/r) = 1$ and $b/r \not\equiv a/r \pmod 3$, then G also has a nowhere zero 3-flow.

Proof. We shall assume that a, b, c are integers. The case a, b, c are arbitrary real numbers can be reduced to the all-integer-case, by applying Kronecker's approximation theorem. We shall omit this reduction which is similar to the reduction process that can be found in [45] and [51].

Let f be a nowhere zero flow of an orientation $\vec{G}$ of G such that $|f(e)| \in \{a, b, c\}$ for all e. By Theorem 1.1, there is a real number t such that

$$||ta|| \geq \frac{1}{3}, \ ||tb|| \geq \frac{1}{3}, \ ||tc|| \geq \frac{1}{3}.$$

In other words, the fractional part of tf is in the interval $[1/3, 2/3]$. Now tf is also a flow of G. By a theorem of Ford and Fulkerson [23], G has an integer valued flow g such that for each e, $g(e)$ is equal to either $\lceil tf(e) \rceil$ or $\lfloor tf(e) \rfloor$. Now $f' = tf - g$ is also a flow of G, and for every edge e, $|f'(e)| \in [1/3, 2/3]$. By reversing the orientation of those edges e for which $f'(e)$ is negative, and by multiplying $|f'(e)|$ by 3, we obtain a flow f^* of G such that $f^*(e) \in [1, 2]$. Again by the above mentioned theorem of Ford and Fulkerson, G has an integer valued flow with values in $[1, 2]$, i.e., G has a nowhere zero 3-flow. ∎

The same discussion carries out for flows in matroids (cf. [2]).

It was conjectured by Tutte [41] that every bridgeless graph without 3-edge-cut has a nowhere-zero 3-flow. By Theorem 2.4, Tutte's 3-flow conjecture is equivalent to the following conjecture:

Conjecture 2.2 *Every bridgeless graph without 3-edge-cut has a nowhere-zero flow with $|f(e)| \in \{a, b, c\}$, where $a < b < c$ are positive integers such that gcd$(a, b, c) = 1$, $(a, b, c) \neq (1, 2, 3s)$ for an integer s, and $c \neq a + b$, or $c = a + b$ but $a \equiv b \pmod 3$.*

Indeed, if Tutte's 3-flow conjecture is true, then every bridgeless graph without 3-edge-cut has a nowhere-zero 3-flow, and hence has a nowhere zero flow, say, with $|f(e)| \in \{1, 2, 4\}$. Hence Conjecture 2.2 is true. Conversely, if Conjecture 2.2 is true, then it follows from Theorem 2.4 that every bridgeless graph without 3-edge-cut has a nowhere-zero 3-flow, i.e., Tutte's 3-flow conjecture is true.

The function $\kappa(\vec{x})$ is also connected to the circular chromatic number and the fractional chromatic number of distance graphs [5, 6]. Through the fractional chromatic of distance graphs, it is connected to the density of D-sets [4, 39], and to the asymptotic coloring efficency of the T colorings (or channel assignments) [10, 25], and to the star-extremality of distance graphs and circulant graphs [24, 34]. We shall not explore such connections here, and refer interested readers to the cited references for the definitions of these concepts. We remark that the connections between $\kappa(\vec{x})$ and the above mentioned concepts have not been explicitly discussed in the references.

3 Some preliminaries

Sections 3-9 are devoted to the proof of Theorem 1.1. We shall assume that a, b, c are fixed integers, $a < b < c$ and $\gcd(a, b, c) = 1$. Let $\vec{x} = (a, b, c)$. Since Wills' conjecture is true for $k = 3$, we know that $\kappa(\vec{x}) \geq \frac{1}{4}$. (The argument in this paper can be easily extended to a proof of the $k = 3$ case of Wills' conjecture. Indeed, the proof in the remaining part will show that $\kappa(\vec{x}) \geq \frac{1}{3}$ in most cases. Thus to prove Wills' conjecture for the case $k = 3$, it suffices to consider the case that $(a, b, c) = (1, 2, 3s)$ or $c = a + b$. These cases are easy. However, we shall not bother to give another proof of the $k = 3$ case of Wills conjecture, as quite a few nice proofs of this case have been published.) It is also easy to see that $\kappa(\vec{x}) = 1/2$ if and only if a, b, c are all odd. Therefore, to prove Theorem 1.1, it suffices to prove the following:

1. $\kappa(\vec{x}) < 1/3$ if $(a, b, c) = (1, 2, 3s)$ for some integer s, or $c = a + b$ and $a \not\equiv b \pmod 3$;

2. $\kappa(\vec{x}) \geq 1/3$ if (a, b, c) does not satisfy any of the two conditions above.

First we observe that the following two statements are equivalent:

1. $\kappa(\vec{x}) \geq \frac{1}{k}$.

2. There exist three integers i, j, ℓ such that
$$[\frac{ki+1}{a}, \frac{ki+k-1}{a}] \cap [\frac{kj+1}{b}, \frac{kj+k-1}{b}] \cap [\frac{k\ell+1}{c}, \frac{k\ell+k-1}{c}] \neq \emptyset.$$

Indeed, $||ta|| \geq 1/k$ if and only if $kt \in [\frac{ki+1}{a}, \frac{ki+k-1}{a}]$ for some integer i.

For convenience, in the remaining of this paper, we shall let
$$I[x, y; z] = [\frac{x}{z}, \frac{y}{z}],$$
and let
$$F_k(x) = \cup_{i=0}^{\infty} I[ki + 1, ki + k - 1; x].$$
Thus
$$\kappa(\vec{x}) \geq \frac{1}{k} \text{ if and only if } F_k(a) \cap F_k(b) \cap F_k(c) \neq \emptyset.$$

As observed above, our main concern will be whether or not $F_3(a) \cap F_3(b) \cap F_3(c) \neq \emptyset$. This amounts to determine whether or not there exist three integers k, i, j such that $i \equiv j \equiv k \equiv 1 \pmod 3$ and that

$$I[k, k+1; a] \cap I[i, i+1; b] \cap I[j, j+1; c] \neq \emptyset.$$

For this purpose, we first consider those integer pairs (u, v) such that

$$I[u, u+1; b] \cap I[v, v+1; c] \neq \emptyset, \quad \text{and} \quad u \equiv v \pmod 3.$$

We call such a pair of integers (u, v) a *consistent pair*. We shall find all the consistent pairs and for all the consistent pairs (u, v), determine the intersections $I[u, u+1; b] \cap [v, v+1; c]$.

For a real number x, we denote by $\{x\}$ the fractional part of x, i.e., $\{x\} = x - \lfloor x \rfloor$.

In the remaining part of the paper, there are quite a few special numbers determined by a, b, c that we shall use frequently. The numbers m_i, n_i, δ_i defined below are among these frequently used special numbers.

For any integer $i \geq 0$, we let

$$m_i = \lfloor 3ic/(c-b) \rfloor,$$

$$\delta_i = \{3ic/(c-b)\},$$

and let

$$n_i = \lfloor 3ib/(c-b) \rfloor.$$

Then

$$3ic/(c-b) = m_i + \delta_i,$$

and

$$3ib/(c-b) = n_i + \delta_i.$$

Note that $m_i = n_i + 3i$. In particular, $n_i \equiv m_i \pmod 3$. As m_1, n_1, δ_1 will be used more frequently, we let $m = m_1, n = n_1, \delta = \delta_1$. Observe that

$$m_i = \lfloor i(m+\delta) \rfloor, \quad n_i = \lfloor i(n+\delta) \rfloor, \quad \text{and} \quad \delta_i = \{i\delta\}.$$

Lemma 3.1 *For any integer i, we have*

$$I[m_i, m_i+1; c] \subset I[n_i, n_i+1; b].$$

Therefore (n_i, m_i) is a consistent pair and

$$I[n_i, n_i+1; b] \cap I[m_i, m_i+1; c] = I[m_i, m_i+1; c].$$

Proof. It suffices to show that

$$n_i/b \leq m_i/c, \text{ and } (n_i+1)/b \geq (m_i+1)/c.$$

By definition,

$$n_i/b = 3i/(c-b) - \delta_i/b, \text{ and } m_i/c = 3i/(c-b) - \delta_i/c.$$

Since $c > b$, it follows that $n_i/b \leq m_i/c$. Similarly,

$$(n_i+1)/b = 3i/(c-b) + (1-\delta_i)/b$$

and

$$(m_i+1)/c = 3i/(c-b) + (1-\delta_i)/c.$$

Hence $(n_i+1)/b \geq (m_i+1)/c$. ∎

Lemma 3.2 *Suppose $j \geq 0$ is an integer. Then*

$(m_i-j)/c \in I[n_i-j, n_i-j+1; b]$ if and only if $(m_i-j)/c \geq (3i-1)/(c-b)$;

$(m_i+1+j)/c \in I[n_i+j, n_i+j+1; b]$ if and only if $(m_i+1+j)/c \leq (3i+1)/(c-b)$;

$(n_i-j)/b \in I[m_i-j-1, m_i-j; c]$ if and only if $(n_i-j)/b \geq (3i-1)/(c-b)$;

$(n_i+j)/b \in I[m_i+j, m_i+j+1; c]$ if and only if $(n_i+j)/b \leq (3i+1)/(c-b)$.

Proof. By Lemma 3.1, $m_i/c \geq n_i/b$. Hence

$$(m_i-j)/c = m_i/c - j/c \geq n_i/b - j/b = (n_i-j)/b.$$

Thus

$(m_i-j)/c \in I[n_i-j, n_i-j+1; b]$ if and only if $(m_i-j)/c \leq (n_i-j+1)/b$.

Since

$$m_i/c = 3i/(c-b) - \delta_i/c \text{ and } n_i/b = 3i/(c-b) - \delta_i/b,$$

this is equivalent to

$$3i/(c-b) - (\delta_i+j)/c \leq 3i/(c-b) - (\delta_i+j-1)/b.$$

This inequality holds if and only if

$$(\delta_i+j)/c \geq (\delta_i+j-1)/b,$$

which is equivalent to

$$(\delta_i+j)/c \leq 1/(c-b).$$

Therefore

$(m_i-j)/c \in I[n_i-j, n_i-j+1; b]$ if and only if $3i/(c-b)-(\delta_i+j)/c \geq (3i-1)/(c-b)$.

This is equivalent to the condition that $(m_i - j)/c \geq (3i-1)/(c-b)$.

Also by Lemma 3.1, $(m_i+1)/c \leq (n_i+1)/b$, which implies that

$$(m_i + 1 + j)/c \leq (n_i + 1 + j)/b.$$

Therefore

$(m_i+1+j)/c \in I[n_i+j, n_i+j+1; b]$ if and only if $(m_i+1+j)/c \geq (n_i+j)/b$.

Since

$$m_i/c = 3i/(c-b) - \delta_i/c \text{ and } n_i/b = 3i/(c-b) - \delta_i/b,$$

the condition above is equivalent to

$$3i/(c-b) + (j+1-\delta_i)/c \geq 3i/(c-b) + (j-\delta_i)/b.$$

This is equivalent to

$$1/(c-b) \geq (j+1-\delta_i)/c.$$

Hence

$$(m_i + 1 + j)/c \in I[n_i + j, n_i + j + 1; b]$$

if and only if

$$3i/(c-b) + (j+1-\delta_i)/c \leq (3i+1)/(c-b),$$

which means

$$(m_i + 1 + j)/c \leq (3i+1)/(c-b).$$

The other inequalities are proved similarly. ∎

Corollary 3.1 *Suppose u, v, j are non-negative integers.*

- *If $v/c \in I[3j-1, 3j; c-b]$, then $(v-3j, v)$ is a consistent pair, and the intersection $I[v-3j, v-3j+1; b] \cap I[v, v+1; c]$ is equal to either $[v/c, (v-3j+1)/b]$ or $I[v, v+1; c]$;*

- *If $v/c \in I[3j, 3j+1; c-b]$, then $(v-3j-1, v-1)$ is a consistent pair, and the intersection $I[v-3j-1, v-3j; b] \cap I[v-1, v; c]$ is equal to either $[((v-3j-1)/b, v/c]$ or $I[v-1, v; c]$;*

- *If* $u/b \in I[3j-1, 3j; c-b]$, *then* $(u-1, u+3j-1)$ *is a consistent pair, and the intersection* $I[u-1, u; b] \cap I[u+3j-1, u+3j; c]$ *is equal to* $[(u+3j-1)/c, u/b]$*;*

- *If* $u/b \in I[3j, 3j+1; c-b]$, *then* $(u, u+3j)$ *is a consistent pair, and the intersection* $I[u, u+1; b] \cap I[u+3j, u+3j+1; c]$ *is equal to* $[((v-3j)/b, (v+1)/c]$. ∎

Corollary 3.1 determines all the consistent pairs and the corresponding intersections. In other words, we have the following:

Lemma 3.3 *The pair* (u, v) *is a consistent pair if and only if there is an integer* j *such that* $v = u + 3j$ *and one of the following is true:*

1. $v/c \in I[3j-1, 3j; c-b]$.

2. $u/b \in I[3j, 3j+1; c-b]$.

3. $(u+1)/b \in I[3j-1, 3j; c-b]$.

Proof. Corollary 3.1 shows that each of the conditions above is sufficient. So we only need to prove the necessity.

Suppose (u, v) is consistent. Then there is an integer j such that

$$v - u = 3j \text{ and } I[u, u+1; b] \cap I[v, v+1; c] \neq \emptyset.$$

Since $c > b$, the intersection

$$I[u, u+1; b] \cap I[v, v+1; c]$$

is equal to

$$\text{either } I[v, v+1; c], \text{ or } [u/b, (v+1)/c], \text{ or } [v/c, (u+1)/b].$$

If

$$I[u, u+1; b] \cap I[v, v+1; c] = I[v, v+1; c],$$

then

$$u/b \leq v/c \text{ and } (u+1)/b \geq (v+1)/c,$$

i.e.,

$$(v-3j)c \leq vb \text{ and } (v-3j+1)c \geq (v+1)b.$$

This implies that

$$v(c-b) \leq 3jc \text{ and } \ (v+1)(c-b) \geq 3jc.$$

Hence $v/c \in I[3j-1, 3j; c-b]$.

If

$$I[u, u+1; b] \cap I[v, v+1; c] = [u/b, (v+1)/c],$$

then

$$(u+3j)/c = v/c \leq u/b \leq (v+1)/c = (u+3j+1)/c.$$

Therefore

$$3jb \leq u(c-b) \leq 3jb + b.$$

Hence $u/b \in I[3j, 3j+1; c-b]$.

If

$$I[u, u+1; b] \cap I[v.v+1; c] = [v/c, (u+1)/b],$$

then

$$(u+3j)/c = v/c \leq (u+1)/b \leq (v+1)/c = (u+3j+1)/c.$$

Therefore

$$(3j-1)b \leq (u+1)(c-b) \leq 3jb,$$

i.e., $(u+1)/b \in I[3j-1, 3j; c-b]$. ∎

Note that some of the sufficient conditions in Corollary 3.1 for a pair to be consistent do not appear in Lemma 3.3. This is because these conditions are implied by other conditions, so the conditions in Corollary 3.1 have redundency. We keep these conditions, as we shall use them later.

For the remaining of this paper, let d be the largest integer such that $dc \leq (d+1)b$. By our assumption, $c < 2b$. Therefore $d \geq 1$. The integer d is one of the frequently used special numbers.

Lemma 3.4 *For $i = 0, 1, \cdots$, and for $j = 0, 1, \cdots, d$, we have*

$$(m_i - j)/c \in I[n_i - j, n_i - j + 1; b],$$

and

$$(m_i + 1 + j)/c \in I[n_i + j, n_i + j + 1; b].$$

Proof. Since

$$\frac{m_i - j}{c} = \frac{3i}{c-b} - \frac{\delta_i + j}{c} \quad \text{and} \quad \frac{n_i - j}{b} = \frac{3i}{c-b} - \frac{\delta_i + j}{b},$$

it follows that

$$\frac{n_i - j}{b} < \frac{m_i - j}{c}.$$

On the other hand,

$$\frac{n_i - j + 1}{b} = \frac{3i}{c-b} - \frac{\delta_i + j - 1}{b}.$$

Since $j \le d$ and $\delta_i < 1$, we have $\delta_i + j - 1 < d$. It follows from the definition of d that

$$(\delta_i + j - 1)c \le (\delta_i + j)b.$$

Therefore

$$\frac{m_i - j}{c} \le \frac{n_i - j + 1}{b}.$$

This proves that $(m_i - j)/c \in I[n_i - j, n_i - j + 1; b]$. The remaining part can be proved similarly. ∎

Corollary 3.2 *For $i = 0, 1, 2, \cdots$, for $j = 0, 1, \cdots, d$, the pairs of integers $(n_i - j, m_i - j)$ and $(n_i + j, m_i + j)$ are consistent pairs. Moreover, if $j \neq 0$ then*

$$I[n_i - j, n_i - j + 1; b] \cap I[m_i - j, m_i - j + 1; c] = [\frac{m_i - j}{c}, \frac{n_i - j + 1}{b}],$$

and

$$I[n_i + j, n_i + j + 1; b] \cap I[m_i + j, m_i + j + 1; c] = [\frac{n_i + j}{b}, \frac{m_j + j + 1}{c}].$$

∎

4 Some easy cases

First we settle some easy cases of Theorem 1.1.

Lemma 4.1 *If $(a, b, c) = (1, 2, 3s)$ for some integer s, or $c = a + b$ and $a \not\equiv b \pmod 3$, then $\kappa(\vec{x}) < 1/3$.*

Proof. It is straightforward to verify that

$$F_3(1) \cap F_3(2) = \{1, 2, \cdots\}.$$

Since $i \notin I[3j + 1, 3j + 2; 3s]$ for any integers i, j, we conclude that

$$F_3(1) \cap F_3(2) \cap F_3(3s) = \emptyset,$$

which implies that for $\vec{x} = (1, 2, 3s)$, $\kappa(\vec{x}) < 1/3$.

Assume now that $c = a + b$ and $a \not\equiv b \pmod 3$. Assume to the contrary that $\kappa(\vec{x}) \geq 1/3$. Then there are integers k, u, v such that $k \equiv u \equiv v \equiv 1 \pmod 3$ and

$$I[k, k+1; a] \cap I[u, u+1; b] \cap I[v, v+1; c] \neq \emptyset.$$

In particular, (u, v) is a consistent pair. By Lemma 3.3 and Corollary 3.1, either $v/c \in I[3j-1, 3j; c-b]$, and

$$I[u, u+1; b] \cap I[v, v+1; c] = [v/c, (u+1)/b] \text{ or } I[v, v+1; c],$$

or $u/b \in I[3j, 3j+1; c-b]$ and

$$I[u, u+1; b] \cap I[v, v+1; c] = [u/b, (v+1)/c],$$

or $(u+1)/b \in I[3j-1, 3j; c-b]$ and

$$I[u, u+1; b] \cap I[v, v+1; c] = [v/c, (u+1)/b].$$

Note that $a = c - b$. Hence

$$I[u, u+1; b] \cap I[v, v+1; c] \cap I[k, k+1; c-b] \neq \emptyset,$$

where $u \equiv v \equiv k \equiv 1 \pmod 3$. Therefore in the first case, we must have $k = 3j - 2$, and $v/c = (3j-1)/(c-b)$. Suppose $v = 3s + 1$, then $2c - b = 3(jc - s(c-b))$, hence

$$2c - b \equiv 0 \pmod 3,$$

contrary to the assumption that

$$a = c - b \not\equiv b \pmod 3.$$

Similar calculation shows that in the second case, $u/b = (3j+1)/(c-b)$, which again implies that $a \equiv b \pmod 3$, contrary to the assumption. In the third case, we must have

$$(u+1)/b = (3j-1)/(c-b),$$

which implies that $2c \equiv b \pmod 3$, again contrary to the assumption. ∎

To prove Theorem 1.1, it remains to prove that under the condition $(a, b, c) \neq (1, 2, 3k)$ and $c \neq a + b$, or $c = a + b$ but $a \equiv b \pmod 3$, we have $\kappa(\vec{x}) \geq \frac{1}{3}$.

For $x > 0$, we let C_x be the circle obtained from the interval $[0, x]$ by identifying the end points 0 and x, and let $\phi_x : R \to C_x$ be the mapping defined as $\phi(t) = t \bmod x$. The mapping ϕ_x can be viewed as "wrapping" the real line around the circle C_x. Given two points $u, v \in C_x$, we denote by $[u, v]$ (resp. $[u, v), (u, v], (u, v)$) the closed (resp. half closed, open) arc of C_x from u to v along the clockwise direction, i.e., the increasing direction. Thus, $[u, v)$ and $[v, u)$ form a partition of C_x. In case the real number x is clear from the context, we may simply write C for C_x, ϕ for ϕ_x, etc.

Lemma 4.2 *If $c \le 2a$, then $\kappa(\vec{x}) \ge \frac{1}{3}$;*
if $c = a + b$ and $a \equiv b \pmod 3$, then $\kappa(\vec{x}) \ge \frac{1}{3}$;
If $a = 1, b = 2$ and $3 \not| c$, then $\kappa(\vec{x}) \ge \frac{1}{3}$;
if $b = 2a$ and $a \not| c$, then $\kappa(\vec{x}) \ge \frac{1}{3}$.

Proof. If $c \le 2a$, then

$$I[1,2;a] \cap I[1,2;b] \cap I[1,2;c] \ne \emptyset.$$

If $c = a + b$ and $a \equiv b \pmod 3$, then since $\gcd(a,b,c) = 1$, we conclude that none of a, b, c is a multiple of 3, and hence

$$1 \in F_3(a) \cap F_3(b) \cap F_3(c).$$

If $a = 1, b = 2$ and $3 \not| c$, then again

$$1 \in F_3(a) \cap F_3(b) \cap F_3(c).$$

Assume $b = 2a$ and c is not a multiple of a. Then

$$F_3(a) \cap F_3(b) = \{\frac{3i+1}{a}, \frac{3i+2}{a} : i = 0, 1, \cdots\}.$$

Let $C = C_{3/c}$, $\phi = \phi_{3/c}$. and for $i = 0, 1, \cdots$, let

$$f(i) = \phi(\frac{3i+1}{a}).$$

Since $f(i + a) = f(i)$, we know that $f(Z)$ is a finite set.

Let $x_1, x_2, \cdots, x_k$ be the points of $f(Z)$, ordered cyclically in this order on the circle C. Now we show that the arcs $[x_i, x_{i+1}]$ are of the same length. Assume to the contrary that the arc $[x_i, x_{i+1}]$ has length smaller than the length of the arc $[x_j, x_{j+1}]$. Assume that $f(u) = x_i$, $f(v) = x_{i+1}$ and $f(w) = x_j$. It follows from the definition that the arc $[f(w), f(w + v - u)]$ has the same length as the arc $[f(u), f(v)]$. Therefore $f(w + v - u)$ is contained in the arc $[x_j, x_{j+1}]$, contrary to the definition of x_{j+1}. So the arcs $[x_i, x_{i+1}]$ are of the same length, which simply says that the points of $f(Z)$ are evenly distributed on C. (In the later proofs, there are similarly defined mapping f and circle C, and so the points of $f(Z)$ are evenly distributed on C. The proofs will be the same and shall be omitted.)

Since c is not a multiple of a, we conclude that $|f(Z)| \ge 2$. If $|f(Z)| \ge 3$, then since the points of $f(Z)$ are evenly distributed on C, and since the arc $[\frac{1}{c}, \frac{2}{c}]$ has length 1/3 of the total length of C, there is one image point, say $f(i)$, lies on the arc $[\frac{1}{c}, \frac{2}{c}]$. This means that there is an integer j such that

$$\frac{3i+1}{a} \in I[3j+1, 3j+2; c].$$

Therefore
$$\frac{3i+1}{a} \in F_3(a) \cap F_3(b) \cap F_3(c) \neq \emptyset.$$
If $|f(Z)| = 2$, then $f(2) = f(0)$, which implies that $6/a$ is a multiple of $3/c$, i.e., $2c$ is a multiple of a. Because c is not a multiple of a, we conclude that
$$\frac{1}{a} \equiv \frac{1}{2c} \pmod{3/c} \text{ or } \frac{1}{a} \equiv \frac{3}{2c} \pmod{3/c} \text{ or } \frac{1}{a} \equiv \frac{5}{2c} \pmod{3/c}.$$
If $1/a \equiv 3/(2c) \pmod{3/c}$, then
$$f(0) \in [\frac{1}{c}, \frac{2}{c}],$$
and we are done as before. If
$$1/a \equiv 1/(2c) \pmod{3/c}, \text{ or } 1/a \equiv 5/(2c) \pmod{3/c},$$
then
$$\phi(2/a) \in [\frac{1}{c}, \frac{2}{c}],$$
hence
$$\frac{2}{a} \in F_3(a) \cap F_3(b) \cap F_3(c) \neq \emptyset.$$
∎

Lemma 4.3 *If $b \neq 2a$ and $c \geq 2b$, then $\kappa(\vec{x}) \geq \frac{1}{3}$.*

Proof. Assume that $b \neq 2a$ and $c \geq 2b$. Note that if there are integers i, j such that
$$I[3i+1, 3i+2; a] \cap I[3j+1, 3j+2; b] = [x, y]$$
is an interval of length at least $2/c$, then we are done. Indeed, it is easy to see that $F_3(c)$ has nonempty intersection with any interval of length $2/c$. Therefore
$$F_3(a) \cap F_3(b) \cap F_3(c) \supset [x, y] \cap F_3(c) \neq \emptyset.$$

If $b \geq 4a$, then let j be the smallest integer such that $\frac{3j+1}{b} \geq \frac{1}{a}$, it is easy to verify that
$$I[3j+1, 3j+2; b] \subset I[1, 2; a].$$
Hence
$$I[3j+1, 3j+2; b] \cap I[1, 2; a]$$
has length $\frac{1}{b} \geq \frac{2}{c}$, and we are done.

Assume $2a < b < 4a$. If $5a \leq 2b$, then
$$I[4, 5; b] \subset I[1, 2; a],$$

and we are done as above. Assume $5a > 2b$, and let i be the smallest integer such that

$$(6i+5)a \leq (3i+2)b.$$

(such an $i \geq 0$ exists because $b > 2a$.) Then

$$\frac{6i+4}{b} \geq \frac{3i+1}{a},$$

because otherwise we have

$$\frac{6i-1}{b} \leq \frac{3i-1}{a},$$

contrary to the minimality of i. Therefore

$$I[6i+4, 6i+5; b] \subset I[3i+1, 3i+2; a].$$

Hence $I[3j+1, 3j+2; b] \cap I[3i+1, 3i+2; a]$ has length $1/b \geq 2/c$, and we are done.

Assume now that $b < 2a$.

First we consider the case that $c \geq 4b$. If $2b \leq 3a$, then

$$I[1, 2; a] \cap I[1, 2; b] = [\frac{1}{a}, \frac{2}{b}]$$

is an interval of length at least $1/(2b) \geq 2/c$, and hence we are done. Thus we may assume $2b > 3a$. Let $C = C_{3/a}$ and $\phi = \phi_{3/a}$. For $i = 0, 1, \cdots$, let

$$f(i) = \phi(\frac{3i+2}{b}).$$

Then $f(Z)$ is a finite set of points evenly distributed on C. Assume that $|f(Z)| = g$. Then ga is a multiple of b. Because $3a/2 < b < 2a$, we know that $g \geq 5$. If $g \geq 6$, then as the images $f(Z)$ are evenly distributed on the circle C, there is one point, say $f(j)$, contained in the arc $[\frac{3}{2a}, \frac{2}{a}]$. This means that there is an integer i such that

$$\frac{3i+1.5}{a} \leq \frac{3j+2}{b} \leq \frac{3i+2}{a},$$

which implies that

$$I[3j+1, 3j+2; b] \cap I[3i+1, 3i+2; a]$$

is an interval of length at least $1/(2b) \geq 2/c$. Hence we are done. Suppose $|f(Z)| = 5$. Then $5a$ is a multiple of b. Because $3a/2 < b < 2a$, we know that $5a = 3b$. Easy calculation shows that

$$f(3) \in [\frac{3}{2a}, \frac{2}{a}],$$

and we are also done as above.

If $7b/2 \leq c < 4b$, then since $b < 2a$, which implies that $c < 8a$. Hence

$$I[7,8;c] \cap I[1,2;a] \cap I[1,2;b] = I[7,8;c] \cap [\frac{1}{a}, \frac{2}{b}] \neq \emptyset,$$

and we are done.

Assume $3b \leq c < 7b/2$. If $4a \geq 3b$, then

$$I[1,2;a] \cap I[1,2;b] = [\frac{1}{a}, \frac{2}{b}]$$

is an interval of length $2/b - 1/a \geq \frac{2}{3b} > \frac{2}{c}$, and we are done. Assume that $4a < 3b$. Let

$$C = C_{3/a}, \phi = \phi_{3/a}, \quad \text{and } f(i) = \phi(\frac{3i+2}{b}).$$

Since b is not a multiple of a, we know that $|f(Z)| \geq 2$. Since the points of $f(Z)$ are evenly distributed on C, we conclude that there is one point, say $f(j)$, contained in the arc $[\frac{3}{2a}, \frac{2}{a}]$. This implies that there is an integer i such that either

$$I[3j+1, 3j+2; b] \subset I[3i+1, 3i+2; a],$$

or

$$I[3j+1, 3j+2; b] \cap I[3i+1, 3i+2; a]$$

is an interval of length at least $1/(2a)$. In the former case, we are done as $I[3j+1, 3j+2; b]$ has length at least $2/c$; in the latter case, because $4a < 3b$, which implies that $\frac{1}{2a} > \frac{2}{3b} \geq \frac{2}{c}$, we are also done.

Assume $5b/2 \leq c < 3b$. If $5a \geq c$, then easy calculation shows that

$$\frac{5}{c} \in I[4,5;c] \cap I[1,2;b] \cap I[1,2;a],$$

hence we are done. Therefore we may assume $5b/2 < 5a < c < 3b$. Again, let

$$C = C_{3/a}, \phi = \phi_{3/a}, \quad \text{and } f(i) = \phi(\frac{3i+2}{b}).$$

Suppose $f(Z)$ contains g points, then ga is a multiple of b. Since $5b/2 < 5a < 3b$, we know that $g \geq 6$, which implies that there is one point, say $f(j)$, contained in the arc $[\frac{3}{2a}, \frac{2}{a}]$. Similarly to the previous paragraph, we can show that for some integers i, j, we have

$$I[3j+1, 3j+2; b] \cap I[3i+1, 3i+2; a]$$

has length at least $\frac{2}{c}$, and hence we are done.

If $2b \leq c < 5b/2$, then easy calculation shows that

$$\frac{2}{b} \in I[1,2;a] \cap I[4,5;c],$$

and we are done. This completes the proof of Lemma 4.3. ∎

5 Strongly consistent pairs

In the remaining part of the proof of Theorem 1.1, we assume that $2a < c < 2b$, $c \neq a + b$ and $b \neq 2a$. We shall prove that under these conditions, there are three integers u, v, w such that

$$I[u, u+1; a] \cap I[v, v+1; b] \cap I[w, w+1; c] \neq \emptyset$$

and that $u \equiv v \equiv w \equiv 1 \pmod 3$. As shown in Section 3, this is equivalent to the statement that $\kappa(\vec{x}) \geq \frac{1}{3}$.

In Section 3, we have determined all the consistent pairs (u, v) of integers, i.e., those pairs (u, v) such that $I[u, u+1; b] \cap I[v, v+1; c] \neq \emptyset$ and $u \equiv v \pmod 3$. For all the consistent pairs (u, v), we have also determined the intersection $I[u, u+1; b] \cap I[v, v+1; c]$. Now we shall take those intervals $I[3i+1, 3i+2; a]$ into considerations. We say a consistent pair (u, v) is *strongly consistent* if there is an integer i such that

$$I[3i+1, 3i+2; a] \cap I[u, u+1; b] \cap I[v, v+1; c] \neq \emptyset.$$

Using this notion, to prove Theorem 1.1, it suffices to prove Proposition 5.1 below.

Proposition 5.1 *Under the assumption that $c \neq a + b$, $b \neq 2a$, and $2a < c < 2b$, there is a strongly consistent pair (u, v) such that $v \equiv 1 \pmod 3$.*

We shall prove Proposition 5.1 by contradiction, i.e., assuming proposition 5.1 is not true, we shall derive a contradiction. Thus for the remaining part of this paper, we assume the following

General Assumption: *$c \neq a + b$, $b \neq 2a$, $2a < c < 2b$, and there is no strongly consistent pair (u, v) for which $v \equiv 1 \pmod 3$.*

In order to derive a contradiction, we shall try to find as many strongly consistent pairs as possible. Lemma 5.1 below is just a combination of Corollary 3.2 and the definition of strongly consistency.

Lemma 5.1 *Suppose i, j, ℓ are non-negative integers.*

- *If $I[3i+1, 3i+2; a]$ contains the point $3j/(c-b)$, then (n_j, m_j) is strongly consistent.*

- *If $(n_j - \ell)/b \in I[3i+1, 3i+2; a] \cap I[3j-1, 3j; c-b]$, then $(n_j - \ell - 1, m_j - \ell - 1)$ is strongly consistent. In particular, if $\ell \leq d - 1$ and $(n_j - \ell)/b \in I[3i+1, 3i+2; a]$ then $(n_j - \ell - 1, m_j - \ell - 1)$ is strongly consistent.*

- *If* $(n_j + \ell)/b \in I[3i+1, 3i+2; a] \cap I[3j, 3j+1; c-b]$, *then* $(n_j+\ell, m_j+\ell)$ *is strongly consistent. In particular, if* $\ell \le d$ *and* $(n_j+\ell)/b \in I[3i+1, 3i+2; a]$ *then* $(n_j+\ell, m_j+\ell)$ *is strongly consistent.*

- *If* $(m_j - \ell)/c \in I[3i+1, 3i+2; a] \cap I[3j-1, 3j; c-b]$, *then* $(n_j-\ell, m_j-\ell)$ *is strongly consistent. In particular, if* $\ell \le d$ *and* $(m_j-\ell)/c \in I[3i+1, 3i+2; a]$ *then* $(n_j-\ell, m_j-\ell)$ *is strongly consistent.*

- *If* $(m_j + \ell)/c \in I[3i+1, 3i+2; a] \cap I[3j, 3j+1; c-b]$, *then* $(n_j+\ell-1, m_j+\ell-1)$ *is strongly consistent. In particular, if* $\ell \le d+1$ *and* $(m_j+\ell)/c \in I[3i+1, 3i+2; a]$ *then* $(n_j+\ell-1, m_j+\ell-1)$ *is strongly consistent.*

Corollary 5.1 *Suppose* i, j *are non-negative integers and that* $I[3i+1, 3i+2; a]$ *contains the point* $3j/(c-b)$. *If* $(3i+1)(c-b)/a \le 3j - \gamma$ and $m_j \equiv 2 \pmod 3$, *then*

$$\delta_j > \gamma b/(c-b) > \gamma;$$

if $(3i+2)(c-b)/a \ge 3j + \gamma$ and $m_j \equiv 0 \pmod 3$, *then*

$$1 - \delta_j > \gamma b/(c-b) > \gamma.$$

Proof. Suppose i, j are integers such that

$$3j/(c-b) \in I[3i+1, 3i+2; a] \text{ and } (3i+1)(c-b)/a \le 3j - \gamma.$$

First note that $3j/(c-b) \in I[3i+1, 3i+2; a]$ implies that $(3i+1)(c-b)/a \le 3j$. Thus $\gamma \ge 0$.

If $\delta_j \le \gamma b/(c-b)$ then

$$\delta_j/b \le \gamma/(c-b) \le 3j/(c-b) - (3i+1)/a.$$

Since $n_j/b = 3j/(c-b) - \delta_j/b$ and $3j/(c-b) \in I[3i+1, 3i+2; a]$, we conclude that

$$(3i+2)/a \ge 3j/(c-b) \ge n_j/b \ge (3i+1)/a.$$

Hence $n_j/b \in I[3i+1, 3i+2; a]$, and it follows from Lemma 5.1 that (n_j-1, m_j-1) is strongly consistent, which implies that $m_j \not\equiv 2 \pmod 3$. Therefore $m_j \equiv 2 \pmod 3$ implies that $\delta_j > \gamma b/(c-b)$. Since $2b > c$, it follows that $\gamma b/(c-b) > \gamma$.

Assume now $(3i+2)(c-b)/a \ge 3j + \gamma$. Similarly, because $3j/(c-b) \in I[3i+1, 3i+2; a]$, we know that $\gamma \ge 0$. If $1 - \delta_j \le \gamma b/(c-b)$, then

$$(1-\delta_j)/b \le \gamma/(c-b) \le (3i+2)/a - 3j/(c-b).$$

Therefore $(n_j+1)/b = 3j/(c-b) + (1-\delta_j)/b \leq (3i+2)/a$, which implies that (n_j+1, m_j+1) is strongly consistent (by Lemma 5.1). Hence $m_j \not\equiv 0 \pmod 3$. Therefore $m_j \equiv 0 \pmod 3$ and $(3i+2)(c-b)/a \geq 3j+\gamma$ implies that $1-\delta_j > \gamma b/(c-b) > \gamma$. ∎

Lemma 5.2 *Suppose i, j are non-negative integers.*
If $I[3i+1, 3i+2; a] \supset I[3j-1, 3j; c-b]$ then $d=1$ and $m_j \equiv 0 \pmod 3$; if $I[3i+1, 3i+2; a] \supset I[3j, 3j+1; c-b]$ then $d=1$ and $m_j \equiv 2 \pmod 3$.

Proof. If

$$I[3i+1, 3i+2; a] \supset I[3j-1, 3j; c-b],$$

then it follows from Corollary 3.2 that

$$(n_j, m_j), (n_j-1, m_j-1), \cdots, (n_j-d, m_j-d)$$

are strongly consistent. Since $d \geq 1$ and none of the integers $m_j, m_j-1, \cdots, m_j-d$ is equivalent to 1 modulo 3, we must have $d=1$ and $m_j \equiv 0 \pmod 3$.

If

$$I[3i+1, 3i+2; a] \supset I[3j, 3j+1; c-b],$$

then it follows from Corollary 3.2 that

$$(n_j, m_j), (n_j+1, m_j+1), \cdots, (n_j+d, m_j+d)$$

are strongly consistent. Since $d \geq 1$ and none of the integers $m_j, m_j+1, \cdots, m_j+d$ is equivalent to 1 modulo 3, we must have $d=1$ and $m_j \equiv 2 \pmod 3$. ∎

Corollary 5.2 *There are no non-negative integers i, j such that*

$$I[3i+1, 3i+2; a] \supset I[3j-1, 3j+1; c-b].$$

Proof. If

$$I[3i+1, 3i+2; a] \supset I[3j-1, 3j+1; c-b],$$

then

$$I[3i+1, 3i+2; a] \supset I[3j-1, 3j; c-b]$$

which implies that $m_j \equiv 0 \pmod 3$. On the other hand,

$$I[3i+1, 3i+2; a] \supset I[3j, 3j+1; c-b]$$

which implies that $m_j \equiv 2 \pmod 3$, an obvious contradiction. ∎

In the remaining part of the paper, we let

$$C = C_{3/(c-b)},\ \phi = \phi_{3/(c-b)},$$

and for $i = 0, \pm 1, \pm 2, \cdots$, let

$$f(i) = \phi((3i+1)/a).$$

Then $f(Z)$ is a finite set of points that are evenly distributed on the circle C. We assume that $|f(Z)| = q$, and denote these q points, according to their clockwise cyclic ordering, by $x_0, x_1, \cdots, x_{q-1}$. We shall refer the points $x_0, x_1, \cdots, x_{q-1}$ as *image points*.

Since the q points $x_0, x_1, \cdots, x_{q-1}$ are evenly distributed on the circle C, any half open arc $[x, y)$ (or $(x, y]$) of C of length $3/(q(c-b))$ contains exactly one of the image points.

We assume that $f(0) = \phi(1/a) = x_0$, and $f(1) = \phi(4/a) = x_p$. Because $(3i+1)/a - 1/a = i(4/a - 1/a)$, it follows that $f(i) = \phi((3i+1)/a) = x_{ip}$, where the index ip takes the modulo q value of ip. The two integers p, q must be coprime, for otherwise $\phi((3i+1)/a)$ cannot assume the value x_1. Let α, β be the unique integers such that $1 \le \alpha \le q-1, 0 \le \beta \le p-1$ and $\alpha p - \beta q = 1$. Then α is the smallest positive integer such that $\phi((3\alpha+1)/a) = x_1$. Clearly if $p = 1$ then $\alpha = 1$ and $\beta = 0$. We also note that $q = 1$ if and only if $\phi(4/a) = \phi(1/a)$, which is equivalent to the condition that $c - b$ is a multiple of a.

Since $\phi(1/a) = x_0$, $\phi(4/a) = x_p$ and the arc $[x_0, x_p]$ has length $3p/(q(c-b))$, it follows that there is an integer $t \ge 0$ such that

$$\frac{c-b}{a} = t + \frac{p}{q}.$$

The numbers α, β, p, q, t defined above will be used frequently in the remaining of the paper. The circle C, the mapping ϕ and those points $x_0, x_1, \cdots, x_{q-1}$ of C will also be used frequently in the remaining.

6 The case $c - b \ge 2a$

In this section, we assume that $c - b \ge 2a$ and we shall derive a contradiction to our general assumption. First we prove a lemma which will be used in this section as well as in Section 8. For an interval I of the real line R, we shall denote the length of I by length(I).

Lemma 6.1 *If $a < c - b$ then $d = 1$.*

Proof. First we claim that there are integers i, j such that

$$\text{length}(I[3i+1, 3i+2; a] \cap I[3j-1, 3j+1; c-b]) \ge 1/(c-b).$$

If $q = 1$, then $c - b = ta$ for some integer $t \geq 2$. If $t = 2$ then

$$I[1, 2; a] \cap I[2, 4; c - b] = I[2, 4; c - b],$$

which is an interval of length $2/(c - b)$.

If $t = 3$ then

$$I[1, 2; a] \cap I[2, 4; c - b] = I[3, 4; c - b],$$

which is an interval of length $1/(c - b)$.

If $t \geq 4$ then length$(I[1, 2; a]) \geq 4/(c - b)$. Let j be the smallest integer such that $(3j - 1)/(c - b) \geq 1/a$, then it is straightforward to verify that $I[1, 2; a] \supset I[3j - 1, 3j + 1; c - b]$.

If $q = 2$, then $p = 1$ and $(c - b)/a = t + 1/2$ for some integer $t \geq 1$. In particular $(c - b)/a \geq 3/2$. The arc $[1.5/(c - b), 3/(c - b)]$ of C has length $3/(q(c - b))$, and hence contains an image point. Suppose

$$\phi((3i + 1)/a) \in [1.5/(c - b), 3/(c - b)].$$

By the definition of ϕ, it means that for some integer j,

$$(3j - 1.5)/(c - b) \leq (3i + 1)/a \leq 3j/(c - b).$$

Since $(c - b)/a \geq 3/2$, it follows that $1/a \geq 3/(2(c - b))$. Therefore

$$I[3i + 1, 3i + 2; a] \supset I[3j - 1, 3j; c - b].$$

If $q \geq 3$, then the arc $[2/(c - b), 3/(c - b)]$ of C has length at least $3/(q(c - b))$, hence it contains an image point. Suppose

$$\phi((3i + 1)/a) \in [2/(c - b), 3/(c - b)].$$

Then for some integer j,

$$(3j - 1)/(c - b) \leq (3i + 1)/a \leq 3j/(c - b).$$

Therefore either $I[3i + 1, 3i + 2; a] \subset I[3j - 1, 3j + 1; c - b]$, which implies that

$$I[3i + 1, 3i + 2; a] \cap I[3j - 1, 3j + 1; c - b] = I[3i + 1, 3i + 2; a], \text{ or}$$

$$I[3i + 1, 3i + 2; a] \supset I[3j, 3j + 1; c - b].$$

By noting that $a < c - b$, which implies that $1/a > 1/(c - b)$, we have proved the claim. Thus we may assume that i, j are integers such that

$$\text{length}(I[3i + 1, 3i + 2; a] \cap I[3j - 1, 3j + 1; c - b]) \geq 1/(c - b).$$

This implies that $3j/(c-b) \in I[3i+1, 3i+2; a]$. Hence by Lemma 5.1, (n_j, m_j) is strongly consistent.

Assume to the contrary of this Lemma that $d \geq 2$. That means $2c \leq 3b$, and hence $2/b \leq 1/(c-b)$. Since

$$\text{length}(I[3i+1, 3i+2; a] \cap I[3j-1, 3j+1; c-b]) \geq 1/(c-b) \geq 2/b,$$

there are two consecutive integers $u, u+1$ such that

$$u/b,\ (u+1)/b \in I[3i+1, 3i+2; a] \cap I[3j-1, 3j+1; c-b].$$

If both points $u/b,\ (u+1)/b$ are contained in $I[3j-1, 3j; c-b]$, then since $3j/(c-b) \in I[3i+1, 3i+2; a]$, we may assume that $u+1 = n_j$. Therefore both points n_j/b and $(n_j-1)/b$ are contained in the intersection

$$I[3i+1, 3i+2; a] \cap I[3j-1, 3j+1; c-b].$$

By Lemma 5.2, both (n_j-1, m_j-1) and (n_j-2, m_j-2) are strongly consistent. However, we have observed in the second previous paragraph that (n_j, m_j) is also strongly consistent. This is in contrary to the general assumption, as one of the integers m_j, m_j-1, m_j-2 is equivalent to 1 modulo 3.

If $u/b \leq 3j/(c-b)$ and $(u+1)/b > (u+1)/b$, then by applying Lemma 5.2 as above, we conclude that

$$(n_j-1, m_j-1),\ (n_j, m_j),\ (n_j+1, m_j+1)$$

are all strongly consistent, contrary to the general assumption. If $u/b > 3j/(c-b)$, then since $3j/(c-b) \in I[3i+1, 3i+2; a]$, we may assume that $u+1 = n_j+1$. Similarly as above, we can conclude that

$$(n_j, m_j),\ (n_j+1, m_j+1),\ (n_j+2, m_j+2)$$

are all strongly consistent, contrary to the general assumption. ∎

Thus for the remaining part of this section, we may assume that $d = 1$. This implies that $3b < 2c$ and hence $c-b > b/2$ and $c-b > c/3$.

Lemma 6.2 $3a \geq c-b$.

Proof. Assume to the contrary that $3a < c-b$. We shall show that, contrary to Corollary 5.2, there exist integers i, j such that

$$I[3j-1, 3j+1; c-b] \subset I[3i+1, 3i+2; a].$$

If $q \geq 3$, then the arc $[1/(c-b), 2/(c-b)]$ of C contains an image point. Suppose

$$\phi((3i+1)/a) \in [1/(c-b), 2/(c-b)].$$

Then for some integer j,

$$(3j-2)/(c-b) \leq (3i+1)/a \leq (3j-1)/(c-b).$$

Because $1/a > 3/(c-b)$, i.e.,

$$\text{length}(I[3i+1, 3i+2; a]) \geq 3/(c-b),$$

we conclude that

$$I[3j-1, 3j+1; c-b] \subset I[3i+1, 3i+2; a].$$

If $q = 2$, then $p = 1$ and $(c-b)/a = t + 1/2$ for some integer $t \geq 3$. Hence $(c-b)/a \geq 3.5$. The arc $[1/(2(c-b)), 2/(c-b)]$ of C has length $3/(q(c-b))$, hence it contains an image point. Suppose

$$\phi((3i+1)/a) \in [1/(2(c-b)), 2/(c-b)].$$

Then for some integer j,

$$(3j-2.5)/(c-b) \leq (3i+1)/a \leq (3j-1)/(c-b).$$

Because $1/a \geq 3.5/(c-b)$, i.e.,

$$\text{length}(I[3i+1, 3i+2; a]) \geq 3.5/(c-b),$$

we conclude that

$$I[3j-1, 3j+1; c-b] \subset I[3i+1, 3i+2; a].$$

If $q = 1$ then $c - b = ta$ for some integer $t \geq 4$. If $t \geq 5$ then

$$\text{length}(I[1, 2; a]) \geq 5/(c-b).$$

Then it is obvious that for some integer j,

$$I[3j-1, 3j+1; c-b] \subset I[1, 2; a].$$

If $t = 4$ then

$$I[5, 7; c-b] \subset I[1, 2; a].$$

∎

Lemma 6.3 *$3a \neq c - b$.*

Proof. Assume to the contrary that $3a = c - b$. Then

$$I[1, 2; a] \supset I[3, 4; c - b] \cup I[5, 6; c - b].$$

By Lemma 5.2,

$$m_1 = m \equiv 2 \pmod 3, \text{ and } m_2 = \lfloor 2(m + \delta) \rfloor \equiv 0 \pmod 3.$$

This is an obvious contradiction, as $0 \leq 2\delta < 2$. ∎

Lemma 6.4 $c - b \leq 2a$.

Proof. Assume to the contrary that $2a < c - b < 3a$. (Note that we have already shown that $c - b < 3a$.) Since $(c - b)/a = t + p/q$ for some integer t, we have $t = 2$. First we assume that $p \geq 3$. The arc $[2/(c - b) - 3/(q(c - b)), 2/(c - b)]$ of C has length $3/(q(c - b))$. Therefore it contains an image point. Suppose

$$\phi((3i + 1)/a) \in [2/(c - b) - 3/(q(c - b)), 2/(c - b)].$$

Then for some integer j,

$$\frac{3j - 1 - \frac{3}{q}}{c - b} \leq \frac{3i + 1}{a} \leq \frac{3j - 1}{c - b}.$$

Since the interval $I[3i + 1, 3i + 2; a]$ has length

$$\frac{1}{a} = \frac{2 + \frac{p}{q}}{c - b} \geq \frac{2 + \frac{3}{q}}{c - b},$$

we conclude that, in contrary to Corollary 5.2,

$$I[3j - 1, 3j + 1; c - b] \subset I[3i + 1, 3i + 2; a].$$

If $p = 2$ then $q = 2s + 1$ for some integer s. Therefore

$$\frac{c - b}{a} = 2 + \frac{2}{2s + 1}.$$

Then it is straightforward to verify that

$$I[6s + 5, 6s + 7; c - b] \subset I[3s + 1, 3s + 2; a],$$

again in contrary to Corollary 5.2.

Finally we assume that $p = 1$. Then $(c - b)/a = 2 + 1/q$. It is straightforward to verify that

$$I[3, 4; c - b] \subset I[1, 2; a], \text{ and}$$

$$I[6q-1, 6q; c-b] \subset I[3(q-1)+1, 3(q-1)+2; a].$$

By Lemma 5.2,

$$m \equiv 2 \pmod 3, \text{ and } m_{2q} \equiv 0 \pmod 3.$$

As $(c-b)/a = 3-(q-1)/q$, by Corollary 5.1 we have $\delta > (q-1)/q$. Then

$$2q(m+1) > 2q(m+\delta) > 2q(m+1)-2,$$

which implies that

$$m_{2q} = \lfloor 2q(m+\delta) \rfloor \not\equiv 0 \pmod 3$$

(note that $m+1 \equiv 0 \pmod 3$), contrary to the previous conclusion. ∎

Lemma 6.5 $2a \neq c-b$.

Proof. If $2a = c-b$ then $I[1,2;a] = I[2,4;c-b]$, contrary to Corollary 5.2. ∎

Thus we have proved that under the general assumption, we have $c-b < 2a$.

7 A technical lemma

The results in Sections 4 and 6 prove Theorem 1.1 for the cases

$$c-b \geq 2a, \; c \leq 2a, \; b = 2a, \; c = a+b.$$

It remains to show that Theorem 1.1 is also true for those triples a, b, c such that

$$2a < c < 2a+b, \text{ and } c \neq a+b, b \neq 2a.$$

We note that the intersection $F_3(a) \cap F_3(b) \cap F_3(c)$ is becoming smaller as the triple (a,b,c) is closer to the plane $c = a+b$. Indeed, if the triple (a,b,c) is on the plane $c = a+b$ and $a \equiv b \pmod 3$, then

$$F_3(a) \cap F_3(b) \cap F_3(c) = \emptyset,$$

and hence $\kappa(\vec{x}) < \frac{1}{3}$. Therefore it is natural that it becomes more difficult to prove that $F_3(a) \cap F_3(b) \cap F_3(c) \neq \emptyset$ when the triples become closer to the plane $c = a+b$. The proofs for the remaining cases are more complicated and involved. In this section, we prove a technical lemma which will be crucial to the proofs in the next two sections.

Lemma 7.1 *Let* $\lfloor (tq+p)(m+\delta)\rfloor = w$ *and* $\{(tq+p)(m+\delta)\} = \delta'$. *Then* $w \equiv 0 \pmod 3$ *and* $\delta' = 0$, *i.e.,* $(tq+p)(m+\delta)$ *is an integer equivalent to* 0 *modulo* 3. *In particular* $(tq+p)\delta$ *is an integer.*

Proof. Recall that $(c-b)/a = t + \frac{p}{q}$. As we have already proved Theorem 1.1 for $c-b \geq 2a$, it remains to consider the cases that $t = 0, 1$. However, the proof for $t = 1$ is also valid for $t \geq 2$.

First we consider the case $t = 0$. Thus we assume that $(c-b)/a = p/q$, and we shall prove that $p(m+\delta)$ is an integer equivalent to 0 modulo 3. Observe that since $p/q < 1$, we have $1/a < 1/(c-b)$.

Claim 7.1 *There are integers* i, j *such that*

$$I[3i+1, 3i+2; a] \subset I[3j-1, 3j+1; c-b].$$

Indeed, if $q = 2$ then $p = 1$ and

$$I[1, 2; a] \subset [0, 1/(c-b)].$$

If $q \geq 3$ then the arc $[2/(c-b), 3/(c-b)]$ of C contains one of the image points. Suppose $f(i) \in [2/(c-b), 3/(c-b)]$, then it follows from the definition of $f(i)$ that there exists an integer j such that

$$(3j-1)/(c-b) \leq (3i+1)/a \leq 3j/(c-b).$$

Since $1/a < 1/(c-b)$, we conclude that

$$I[3i+1, 3i+2; a] \subset I[3j-1, 3j+1; c-b].$$

This completes the proof of Claim 7.1.

Suppose $I[3i+1, 3i+2; a]$ is contained in $I[3j-1, 3j+1; c-b]$. By symmetry, we may assume that

$$I[3i+1, 3i+2; a] \cap I[3j-1, 3j; c-b] \neq \emptyset.$$

Since $(c-b)/a = p/q$, for any integer k, we have $3kq/a = 3kp/(c-b)$, hence

$$I[3(i+kq)+1, 3(i+kq)+2; a] \subset I[3(j+kp)-1, 3(j+kp)+1; c-b]$$

and that

$$I[3(i+kq)+1, 3(i+kq)+2; a] \cap I[3(j+kp)-1, 3(j+kp); c-b] \neq \emptyset.$$

Let

$$\tau = \max\{0,\ 3jc/(c-b) - (3i+2)c/a\}.$$

Then for all k, we have

$$\max\{0,\ 3(j+kp)/(c-b)-(3(i+kq)+2)/a\}=\tau.$$

Let $\tau^*=0$ when $\tau=0$, and $\tau^*=\tau+1-\{\tau\}$ if $\tau>0$. Alternately, we can define τ^* as $\tau^*=\lceil\tau\rceil$ when $\tau=0$ or τ is not an integer, and $\tau^*=\tau+1$ when τ is a positive integer.

Claim 7.2 *For any integer k, $(n_{j+kp}-\tau^*, m_{j+kp}-\tau^*)$ is strongly consistent.*

Indeed, if $\tau=0$ then

$$3(j+kp)/(c-b)\in I[3(i+kq)+1,3(i+kq)+2;a].$$

By Lemma 5.1, (n_{j+kp}, m_{j+kp}) is strongly consistent. If $\tau>0$ then

$$\tau=3(j+kp)c/(c-b)-(3(i+kq)+2)c/a.$$

Hence $\tau/c=3(j+kp)/(c-b)-(3(i+kq)+2)/a$. Therefore

$$\begin{aligned}\frac{m_{j+kp}-\tau^*}{c} &= \frac{3(j+kp)}{c-b}-\frac{\delta_{j+kp}}{c}-\frac{\tau}{c}-\frac{1-\{\tau\}}{c}\\ &= \frac{3(i+kq)+2}{a}-\frac{\delta_{j+kp}+(1-\{\tau\})}{c}.\end{aligned}$$

As $2/c<1/a$, $0<\delta_{j+kp}+(1-\{\tau\})<2$, it follows that

$$\begin{aligned}\frac{m_{j+kp}-\tau-1+\{\tau\}}{c} &\in I[3(i+kq)+1,3(i+kq)+2;a]\\ &\cap\ I[3(j+kp)-1,3(j+kp);c-b].\end{aligned}$$

By Lemma 5.1, $(n_{j+kp}-\tau^*, m_{j+kp}-\tau^*)$ is strongly consistent. This completes the proof of Claim 7.2.

Therefore for all k,

$$m_{j+kp}-\tau^*\not\equiv 1\pmod 3.$$

Since

$$\begin{aligned}m_{j+kp} &= \lfloor m_j+\delta_j+kp(m+\delta)\rfloor\\ &= m_j+\lfloor k(w+\delta')+\delta_j\rfloor,\end{aligned}$$

we have

$$m_{j+kp}-\tau^*-(m_j-\tau^*)=\lfloor k(w+\delta')+\delta_j\rfloor.$$

Claim 7.3 $w\not\equiv 2\pmod 3$.

Assume to the contrary that $w \equiv 2 \pmod 3$. If $m_j - \tau^* \equiv 0 \pmod 3$, then for all $k \geq 0$,

$$\begin{aligned} m_{j+kp} - \tau^* &\equiv \lfloor k(w + \delta') + \delta_j \rfloor \\ &\equiv 2k + \lfloor k\delta' + \delta_j \rfloor \pmod 3. \end{aligned}$$

Let $k_0 \geq 2$ be the least integer such that $k_0\delta' + \delta_j < k_0 - 1$ (such an integer k exists because $0 \leq \delta_j, \delta' < 1$). Then we have

$$k_0 - 2 \leq k_0\delta' + \delta_j < k_0 - 1.$$

This implies that

$$m_{j+k_0p} - \tau^* \equiv 2k_0 + \lfloor (k_0 - 1)\delta' + \delta_j \rfloor = 3k_0 - 2 \equiv 1 \pmod 3,$$

contrary to the fact that for all k,

$$m_{j+kp} - \tau^* \not\equiv 1 \pmod 3.$$

If $m_j - \tau^* \equiv 2 \pmod 3$, then

$$\begin{aligned} m_{j+kp} - \tau^* &\equiv 2 + \lfloor k(w + \delta') + \delta_j \rfloor \\ &\equiv 2 + 2k + \lfloor k\delta' + \delta_j \rfloor \pmod 3. \end{aligned}$$

Let $k_0 \geq 1$ be the least integer such that $k_0\delta' + \delta_j < k_0$. Then

$$k_0 - 1 \leq k_0\delta' + \delta_j < k_0.$$

Hence

$$m_{j+k_0p} - \tau^* \equiv 1 \pmod 3,$$

again contrary to the fact that for all k,

$$m_{j+kp} - \tau^* \not\equiv 1 \pmod 3.$$

This completes the proof of Claim 7.3.

Claim 7.4 *If $w \equiv 0 \pmod 3$, then $\delta' = 0$.*

If $w \equiv 0 \pmod 3$, then

$$m_{j+kp} - \tau^* \equiv (m_j - \tau^*) + \lfloor k\delta' \rfloor \pmod 3.$$

If $m_j - \tau^* \equiv 0 \pmod 3$, then

$$m_{j+kp} - \tau^* \equiv \lfloor k\delta' \rfloor \pmod 3.$$

Assume to the contrary that $\delta' \neq 0$. Let k_0 be the least integer such that $k_0\delta' \geq 1$. Then

$$\begin{aligned} m_{j+k_0p} - \tau^* &\equiv \lfloor k\delta' \rfloor \\ &\equiv 1 \pmod 3, \end{aligned}$$

contrary to the fact that for all k,

$$m_{j+kp} - \tau^* \not\equiv 1 \pmod 3.$$

If $m_j - \tau^* \equiv 2 \pmod 3$, then

$$m_{j+kp} - \tau^* \equiv 2 + \lfloor k\delta' \rfloor \pmod 3.$$

Assume to the contrary that $\delta' \neq 0$. Let $k_0 \geq 3$ be the least integer such that $k_0\delta' \geq 2$. Then

$$\begin{aligned} m_{j+k_0p} - \tau^* &\equiv 2 + \lfloor k\delta' \rfloor \\ &\equiv 1 \pmod 3, \end{aligned}$$

contrary to the fact that for all k,

$$m_{j+kp} - \tau^* \not\equiv 1 \pmod 3.$$

This completes the proof Claim 7.4

Claim 7.5 $w \not\equiv 1 \pmod 3$.

Assume to the contrary that $w \equiv 1 \pmod 3$. We shall only consider the case when $\tau > 0$, i.e., $(3i+2)/a < 3j/(c-b)$. The case when $\tau = 0$ needs to be discussed separately, but the idea is the same. We shall just point out the difference at the appropriate places of the proof, and omit the details for that case.

By Claim 7.2,

$$m_j - \tau^* \not\equiv 1 \pmod 3.$$

Assume first that

$$m_j - \tau^* \equiv 0 \pmod 3.$$

Then

$$\begin{aligned} m_{j+kp} - \tau^* &\equiv \lfloor k(w+\delta') + \delta_j \rfloor \\ &\equiv k + \lfloor k\delta' + \delta_j \rfloor \pmod 3. \end{aligned}$$

We now prove by induction that for all k,

$$\begin{aligned} m_{j+2kp} - \tau^* &\equiv 0 \pmod 3, \\ m_{j+(2k+1)p} - \tau^* &\equiv 2 \pmod 3, \\ 2k\delta' + \delta_j &< k+1, \\ (2k+1)\delta' + \delta_j &\geq k+1. \end{aligned}$$

When $k = 0$, we have $m_j - \tau^* \equiv 0 \pmod 3$ by assumption. Moreover,

$$m_{j+p} - \tau^* \equiv 1 + \lfloor \delta' + \delta_j \rfloor \pmod 3.$$

Since $m_{j+p} - \tau^* \not\equiv 1 \pmod 3$, we must have

$$m_{j+p} - \tau^* \equiv 2 \pmod 3$$

and hence

$$\delta' + \delta_j \geq 1.$$

Suppose the above statement is true for integers $\leq k$. Then

$$m_{j+2(k+1)p} - \tau^* \equiv 2(k+1) + \lfloor 2(k+1)\delta' + \delta_j \rfloor \pmod 3.$$

By the induction hypothesis, we have

$$2k\delta' + \delta_j < k+1, \text{ and } (2k+1)\delta' + \delta_j \geq k+1.$$

This implies that

$$k+1 \leq 2(k+1)\delta' + \delta_j < k+3.$$

If $2(k+1)\delta' + \delta_j \geq k+2$, then we would have

$$\lfloor 2(k+1)\delta' + \delta_j \rfloor = k+2,$$

and hence

$$m_{j+2(k+1)p} - \tau^* \equiv 1 \pmod 3,$$

contrary to our previous conclusion. Therefore we have

$$2(k+1)\delta' + \delta_j < k+2$$

and

$$m_{j+2(k+1)p} - \tau^* \equiv 0 \pmod 3.$$

Also we have

$$m_{j+(2k+3)p} - \tau^* \equiv 2k+3 + \lfloor (2k+3)\delta' + \delta_j \rfloor \pmod 3.$$

Since $k+1 \leq 2(k+1)\delta' + \delta_j < k+2$, we have

$$k+1 \leq (2k+3)\delta' + \delta_j < k+3.$$

Because

$$m_{j+(2k+3)p} - \tau^* \not\equiv 1 \pmod 3,$$

we conclude that

$$k+2 \leq (2k+3)\delta' + \delta_j < k+3$$

and

$$m_{j+(2k+3)p} - \tau^* \equiv 2 \pmod 3.$$

Now $2k\delta' + \delta_j < k+1$ and $(2k+1)\delta' + \delta_j \geq k+1$ for all k implies that $\delta' = 1/2$ and $\delta_j \geq 1/2$.

Next we show that $\delta_j < \{\tau\}$. Assume to the contrary that $\delta_j \geq \{\tau\}$. Then

$$\delta_j - \{\tau\} \geq 0.$$

Recall that

$$\tau = 3jc/(c-b) - (3i+2)c/a,$$

hence

$$\tau/c = 3j/(c-b) - (3i+2)/a.$$

Therefore

$$\begin{aligned} \frac{m_j - \tau^* + 1}{c} &= \frac{3j}{c-b} - \frac{\delta_j}{c} - \frac{\tau}{c} + \frac{\{\tau\}}{c} \\ &= \frac{3i+2}{a} - \frac{\delta_j - \{\tau\}}{c}. \end{aligned}$$

As $2/c < 1/a$, $0 \leq \delta_j - \{\tau\} < 1$, it follows that

$$\begin{aligned} \frac{m_j - \tau^* + 1}{c} &\in I[3i+1, 3i+2; a] \\ &\cap I[3j-1, 3j; c-b]. \end{aligned}$$

By Lemma 5.1, $(n_j - \tau^* + 1, m_j - \tau^* + 1)$ is strongly consistent. However $m_j - \tau^* + 1 \equiv 1 \pmod 3$, contrary to the general assumption.

Summing up the discussion above, we have proved that

$$\delta' = 1/2, \ \delta_j \geq 1/2, \ \text{and } \delta_j < \{\tau\}.$$

It follows then that

$$m_{j+p} - \tau^* \equiv 1 + \lfloor \delta_j + \delta' \rfloor \equiv 2 \pmod 3,$$

and

$$\delta_{j+p} = \delta_j + \delta' - 1 = \delta_j - 1/2 < 1/2.$$

Since $\{\tau\} > \delta_j \geq 1/2$, we conclude that

$$\delta_{j+p} - \{\tau\} < 0.$$

Then the same calculation as above shows that

$$\begin{aligned} \frac{m_{j+p} - \tau^* - 1}{c} &= \frac{3(j+p)}{c-b} - \frac{\delta_{j+p}}{c} - \frac{\tau}{c} - \frac{2 - \{\tau\}}{c} \\ &= \frac{3(i+q)+2}{a} - \frac{2 - \{\tau\} + \delta_{j+p}}{c}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{m_{j+p} - \tau^* - 1}{c} &\in I[3(i+q)+1, 3(i+q)+2; a] \\ &\cap \ I[3(j+p)-1, 3(j+p); c-b]. \end{aligned}$$

By Lemma 5.1, $(n_{j+p} - \tau^* - 1, m_{j+p} - \tau^* - 1)$ is strongly consistent. However $m_{j+p} - \tau^* - 1 \equiv 1 \pmod 3$, contrary to the general assumption. This completes the proof for the case when $m_j - \tau^* \equiv 0 \pmod 3$.

We note that in the proof of this case, we assumed that $\tau > 0$. In case $\tau = 0$, then instead of showing that $\{\tau\} > \delta_j$, we should prove that

$$(3i+1)/a \le 3j/(c-b) - \delta_j/a,$$

and instead of considering $(m_j - \tau^* + 1)/c$ and $(m_{j+p} - \tau^* - 1)/c$, we should consider $(n_j + 1)/b$ and n_{j+p}/b, respectively. The rest of the argument is the same.

Now we consider the case when $m_j - \tau^* \equiv 2 \pmod 3$. Then

$$\begin{aligned} m_{j+p} - \tau^* &= \lfloor (j+p)(m+\delta) \rfloor - \tau^* \\ &= m_j + w - \tau^* + \lfloor \delta_j + \delta' \rfloor \\ &\equiv \lfloor \delta_j + \delta' \rfloor \pmod 3. \end{aligned}$$

Since $m_{j+p} - \tau^* \not\equiv 1$, we conclude that

$$m_{j+p} - \tau^* \equiv 0 \pmod 3.$$

Thus we may replace j by $j + p$ in the proof for the case when $m_j - \tau^* \equiv 0 \pmod 3$, and obtain a contradiction to the general assumption. This completes the proof of Claim 7.5, as well as the proof of the $t = 0$ case of Lemma 7.1.

The case $t \ge 1$ of Lemma 7.1 is proved similarly. The following two paragraphs point out the difference.

If $t \ge 1$, then $1/a > 1/(c-b)$. We observe first that there are integers i, j such that

$$3j/(c-b) \in I[3i+1, 3i+2; a].$$

Indeed, if $q \ge 3$ then there is an image point contained in the arc $[2/(c-b), 3/(c-b)]$ of C. If

$$f(i) \in [2/(c-b), 3/(c-b)],$$

then the image arc

$$[\phi((3i+1)/a), \phi((3i+2)/a)]$$

contains the points $3/(c-b)$ which implies that for some j, $I[3i+1, 3i+2; a]$ contains the point $3j/(c-b)$. The case $q = 1$ or 2 are easy, and we omit the details.

It is easy to see that the fact that $I[3i+1, 3i+2; a]$ contains the point $3j/(c-b)$ implies that for all $k \geq 0$,

$$3(j+k(tq+p))/(c-b) \in I[3(i+qk)+1, 3(i+qk)+2; a].$$

By Lemma 5.1, $m_{j+k(tq+p)} \not\equiv 1 \pmod 3$ for all integer k. The rest of the proof is the same as the corresponding part of the proof for the case when $t = 0$, and we omit the detail. ∎

The following corollary is an easy consequence of the fact that $\delta_j = \{j\delta\}$ for any integer j.

Corollary 7.1 *For any j, $(tq+p)\delta_j$ is an integer. Therefore $\delta_j = s_j/(tq+p)$ for some integer s_j. In particular,*

$$0 \leq \delta_j \leq (tq+p-1)/(tq+p).$$

∎

8 The case $2a > c-b > a$

In this section, we assume that $2a > c-b > a$, and we shall derive a contradiction to the general assumption. Since $2a > c-b > a$ and $(c-b)/a = t + p/q$ for some integer t, we must have $t = 1$, i.e.,

$$(c-b)/a = 1 + p/q.$$

We also note that by Lemma 6.1, we may assume that $d = 1$, hence $3b/2 < c < 2b$. In particular, $3 < 3b/(c-b) = n + \delta < 6$.

Lemma 8.1 *$q \neq 2$.*

Proof. If $q = 2$, then $p = 1$ and $(c-b)/a = 3/2$. Therefore

$$I[2, 3; c-b] \subset I[1, 2; a], \text{ and}$$

$$[6, 7; c-b] \subset I[4, 5; a].$$

By Lemma 5.2, we have $m \equiv 0 \pmod 3$ and $m_2 \equiv 2 \pmod 3$. However if $m \equiv 0 \pmod 3$ then $m_2 = \lfloor 2(m+\delta) \rfloor \equiv \lfloor 2\delta \rfloor \not\equiv 2 \pmod 3$. ∎

Lemma 8.2 *If $q = 3$ then $p \neq 2$.*

Proof. Assume to the contrary that $q = 3$ and $p = 2$. Then $(c-b)/a = 5/3$.Therefore

$$I[1,2;a] \supset I[2,3;c-b].$$

By Lemma 5.2, $m \equiv n \equiv 0 \pmod 3$. By Lemma 7.1,

$$(q+p)(m+\delta) = 5(m+\delta) \equiv 5\delta \equiv 0 \pmod 3.$$

Therefore $\delta = 0$ or $3/5$. If $\delta = 0$, then $3 < 3b/(c-b) = n < 6$, contrary to the previous conclusion that $n \equiv m \equiv 0 \pmod 3$. If $\delta = 3/5$, then $3b/(c-b) = n + \delta = 3 + 3/5$ (because $n \equiv 0 \pmod 3$ and $b > c - b > b/2$). Therefore

$$\frac{b}{a} = \frac{b}{c-b}\frac{c-b}{a} = 2,$$

contrary to the general assumption. ∎

Lemma 8.3 $q \neq 3$.

Proof. Assume to the contrary that $q = 3$. Then since $p \neq 2$ we must have $p = 1$. Therefore $(c-b)/a = 4/3$, hence $6/(c-b) \in I[4,5;a]$. By Lemma 5.1, we have $m_2 \not\equiv 1 \pmod 3$. Moreover, since $4(c-b)/a = 6 - 2/3$ and $5(c-b)/a = 6 + 2/3$, it follows from Corollary 5.1 that if $m_2 \equiv 2 \pmod 3$, then $\delta_2 > 2/3$; if $m_2 \equiv 0 \pmod 3$, then $\delta_2 < 1/3$.

Next we show that $m_3 \not\equiv 0 \pmod 3$. Since $(c-b)/a = 4/3$, we have $7(c-b)/a = 9 + 1/3$. Recall that $2c < 3b$, which implies $1/c > 1/(3(c-b))$. As

$$m_3/c = 9/(c-b) - \delta_3/c,$$

we have

$$\begin{aligned} (m_3+2)/c &= 9/(c-b) + (2-\delta_3)/c \\ &> 9/(c-b) + 1/(3(c-b)) \\ &= 7/a. \end{aligned}$$

On the other hand, as $c < 2b$ we have

$$(2-\delta_3)/c < 2/c < 1/(c-b),$$

hence $(m_3+2)/c < 10/(c-b) < 8/a$. Therefore

$$(m_3+2)/c \in I[7,8;a] \cap I[9,10;c-b].$$

By Lemma 5.1, (n_3+1, m_3+1) is strongly consistent. Therefore $m_3 \not\equiv 0 \pmod 3$.

Similarly, $m/c = 3/(c-b) - \delta/c$ implies that

$$\begin{aligned}(m-1)/c &= 3/(c-b) - (1+\delta)/c \\ &> 3/(c-b) - 2/c > 2/(c-b) \\ &= 1.5/a,\end{aligned}$$

and

$$\begin{aligned}(m-1)/c &\leq 3/(c-b) - 1/c \\ &< 8/(3(c-b)) = 2/a.\end{aligned}$$

Hence

$$(m-1)/c \in I[1,2;a] \cap I[2,3;c-b].$$

By Lemma 5.1, $(n-1, m-1)$ is strongly consistent, hence $m \not\equiv 2 \pmod 3$.

By Lemma 7.1,

$$4(m+\delta) \equiv 0 \pmod 3.$$

If $m \equiv 0 \pmod 3$, then $4\delta \equiv 0 \pmod 3$. Hence either $\delta = 0$ or $\delta = 3/4$. If $\delta = 3/4$ then $m_2 \equiv 1 \pmod 3$, contrary to our previous conclusion. If $\delta = 0$ then $m_3 \equiv 0 \pmod 3$, again contrary to our previous conclusion.

If $m \equiv 1 \pmod 3$, then $4(m+\delta) \equiv 0 \pmod 3$ implies that $4\delta \equiv 2 \pmod 3$, hence $\delta = 1/2$. But then $m_2 \equiv 2 \pmod 3$ and $\delta_2 = 0$, contrary to the first paragraph of this proof. ∎

For the remaining part of this section, we assume that $q \geq 4$. First we consider the case when $p = 1$.

Lemma 8.4 *If $p = 1$ then $q \not\equiv 1 \pmod 3$.*

Proof. Assume to the contrary that $p = 1$ and $q = 3s+1$ for some integer $s \geq 1$. Then

$$(c-b)/a = 1 + 1/q = (3s+2)/(3s+1).$$

It follows that for $i \geq 0$,

$$\begin{aligned}\frac{3(s+i+1)a}{c-b} &= \frac{3(s+i+1)(3s+1)}{3s+2} \\ &= 3(s+i+1) - \frac{3(s+i+1)}{3s+2} \\ &= 3(s+i) + 2 - \frac{3i+1}{3s+2}.\end{aligned}$$

Therefore for $i = 0, 1, \cdots, s$,

$$3(s+i+1)/(c-b) \in I[3(s+i)+1, 3(s+i)+2; a],$$

and that

$$I[3s+1, 3s+2; a] \supset I[3s+2, 3s+3; c-b], \text{ and}$$

$$I[6s+1, 6s+2; a] \supset I[3(2s+1), 3(2s+1)+1; c-b].$$

By Lemma 5.2, $m_{s+1} \equiv 0 \pmod 3$ and $m_{2s+1} \equiv 2 \pmod 3$, and by Lemma 5.1, $m_{s+i+1} \not\equiv 1 \pmod 3$ for $i = 0, 1, \cdots, s$.

Assume first that $m \equiv 0 \pmod 3$. Then since $m_{s+1} \equiv 0 \pmod 3$, for $i = 0, 1, \cdots, s$, we have

$$\begin{aligned} m_{s+i+1} &\equiv \lfloor m_{s+1} + \delta_{s+1} + im + i\delta \rfloor \\ &\equiv \lfloor \delta_{s+1} + i\delta \rfloor \pmod 3. \end{aligned}$$

Since $m_{2s+1} \equiv 2 \pmod 3$, we have $\delta_{s+1} + s\delta \geq 2$. Let i_0 be the least integer such that $\delta_{s+1} + i_0\delta \geq 1$. Then $1 \leq i_0 < s$, and

$$1 \leq \delta_{s+1} + i_0\delta < 2,$$

hence

$$m_{s+i_0+1} \equiv 1 \pmod 3,$$

contrary to the previous conclusion.

Assume now that $m \equiv 1 \pmod 3$. Then

$$\begin{aligned} m_{s+2} &= \lfloor m_{s+1} + \delta_{s+1} + m + \delta \rfloor \\ &\equiv \lfloor 1 + \delta_{s+1} + \delta \rfloor \pmod 3. \end{aligned}$$

Since $q \geq 4$ which implies that $s \geq 1$, we know that $s+2 \leq 2s+1$ and hence $m_{s+2} \not\equiv 1 \pmod 3$. Therefore

$$\delta_{s+1} + \delta > 1, \text{ and } m_{s+2} \equiv 2 \pmod 3.$$

Moreover, since

$$(3(s+1)+1)(c-b)/a = 3(s+2) - (q-3)/q,$$

it follows from Corollary 5.1 that

$$\delta_{s+2} = \delta_{s+1} + \delta - 1 > (q-3)/q.$$

Therefore $\delta > (q-3)/q$.

If $s \geq 2$ then $s+3 < 2s+1$, which implies that $m_{s+3} \not\equiv 1 \pmod 3$. However,

$$\begin{aligned} m_{s+3} &= \lfloor m_{s+2} + \delta_{s+2} + m + \delta \rfloor \\ &\equiv \lfloor \delta + \delta_{s+2} \rfloor \pmod 3. \end{aligned}$$

Since $q \geq 7$ and $\delta > (q-3)/q$, $\delta_{s+2} > (q-3)/q$, we conclude that $m_{s+3} \equiv 1 \pmod 3$, which is a contradiction. Thus we may assume that $s = 1$, i.e., $q = 4$. By Lemma 7.1, $(p+q)(m+\delta) = 5(m+\delta) \equiv 0 \pmod 3$. Since $m \equiv 1 \pmod 3$ we have $5\delta \equiv 1 \pmod 3$. Hence $\delta = 1/5$ or $4/5$. However since

$$m_{s+1} = m_2 = \lfloor 2(m+\delta) \rfloor \equiv 0 \pmod 3,$$

we conclude that $2\delta \geq 1$, hence $\delta = 4/5$. Then $3b/(c-b) = n + \delta = 4 + 4/5$ implies that $b/(c-b) = 8/5$. Therefore

$$n_3 = \lfloor 9b/(c-b) \rfloor \equiv 2$$

and

$$\delta_3 = \{9b/(c-b)\} = 2/5.$$

However, $7(c-b)/a = 9 - 1/4$. It follows from Corollary 5.1 that $\delta_3 > (1/4)b/(c-b) = 2/5$, a contradiction.

Finally we consider the case when $m \equiv 2 \pmod 3$. Since $3b/(c-b) = m + \delta$ and $3b/2 < c < 2b$, we have $m = 5$, i.e., $3b/(c-b) = 5 + \delta$. Now for $i = 0, 1, \cdots$,

$$\begin{aligned} m_{s+i+1} &= \lfloor m_{s+1} + \delta_{s+1} + i(m+\delta) \rfloor \\ &\equiv 2i + \lfloor i\delta + \delta_{s+1} \rfloor \pmod 3. \end{aligned}$$

If $s\delta + \delta_{s+1} < s - 1$, then let j^* be the smallest integer such that $j^*\delta + \delta_{s+1} < j^* - 1$. Then $2 \leq j^* \leq s$ and by the minimality of j^*, we have

$$j^* - 2 \leq j^*\delta + \delta_{s+1} < j^* - 1.$$

This implies that

$$m_{s+j^*+1} \equiv 1 \pmod 3,$$

contrary to our previous conclusion. Therefore,

$$s\delta + \delta_{s+1} \geq s - 1,$$

which implies that

$$\delta \geq 1 - (1 + \delta_{s+1})/s.$$

As $q = 3s + 1$, straightforward calculation shows that

$$\begin{aligned} (q+1)\delta &= (3s+2)\delta \\ &\geq (q+1) - 3 - 3\delta_{s+1} - (2 + 2\delta_{s+1})/s \\ &> q - 9. \end{aligned}$$

By Lemma 7.1, $(q+1)(m+\delta) \equiv 0 \pmod 3$. Since $(q+1) \equiv 2 \pmod 3$ and $m \equiv 2 \pmod 3$, it follows that $(q+1)\delta \equiv 2 \pmod 3$. Therefore the possible values for $(q+1)\delta$ are $q-2, q-5$ and $q-8$.

If $(q+1)\delta = q-8$ then $q \geq 8$. Hence $s \geq 3$, which implies that $(q+1)\delta \geq q-6$, a contradiction.

If $(q+1)\delta = q-2$, then $\delta = (q-2)/(q+1)$. This implies that

$$\delta_{2s+1} = \{(2s+1)\delta\} = 1/(q+1).$$

However, since $(c-b)/a = (3s+2)/(3s+1)$, straightforward calculation shows that $(6s+1)(c-b)/a = 3(2s+1)-1/q$. Since $m_{2s+1} \equiv 2 \pmod 3$, by Corollary 5.1, we should have $\delta_{2s+1} > 1/q$, which is an obvious contradiction.

If $(q+1)\delta = q-5$ then $q \geq 5$ and $\delta = (q-5)/(q+1)$. This implies that $3b/(c-b) = 5 + (q-5)/(q+1)$, hence $b/(c-b) = 2q/(q+1)$. Therefore $b = 2a$, contrary to our general assumption. ∎

Lemma 8.5 *If $p = 1$ then $q \not\equiv 2 \pmod 3$.*

Proof. Assume to the contrary that $p = 1$ and $q = 3s+2$ for some integer $s \geq 1$. Then

$$(c-b)/a = (p+q)/q = (3s+3)/(3s+2).$$

It follows that for $i \geq 0$,

$$\begin{aligned} \frac{3(s+i+1)a}{c-b} &= \frac{3(s+i+1)(3s+2)}{3s+3} \\ &= 3(s+i+1) - \frac{3(s+i+1)}{3s+3} \\ &= 3(s+i)+2-\frac{3i}{3s+3}. \end{aligned}$$

Therefore for $i = 0, 1, \cdots, s+1$,

$$3(s+i+1)/(c-b) \in I[3(s+i)+1, 3(s+i)+2; a],$$

and that

$$I[3s+1, 3s+2; a] \supset I[3s+2, 3s+3; c-b], \text{ and}$$
$$I[3(2s+1)+1, 3(2s+1)+2; a] \supset I[3(2s+2), 3(2s+2)+1; c-b].$$

By Lemma 5.2, $m_{s+1} \equiv 0 \pmod 3$ and $m_{2s+2} \equiv 2 \pmod 3$, and by Lemma 5.1, $m_{s+i+1} \not\equiv 1 \pmod 3$ for $i = 0, 1, \cdots, s+1$.

Assume first that $m \equiv 0 \pmod 3$. Then for $i = 0, 1, \cdots, s+1$,

$$\begin{aligned} m_{s+i+1} &= \lfloor m_{s+1} + \delta_{s+1} + i(m+\delta) \rfloor \\ &\equiv \lfloor i\delta + \delta_{s+1} \rfloor \pmod 3. \end{aligned}$$

Since $m_{2s+2} \equiv 2 \pmod 3$, we conclude that $(s+1)\delta + \delta_{s+1} \geq 2$. let j^* be the smallest integer such that $j^*\delta + \delta_{s+1} \geq 1$. Then $1 \leq j^* \leq s$ and by the minimality of j^*, we have $1 \leq j^*\delta + \delta_{s+1} < 2$. This implies that

$$m_{s+j^*+1} \equiv 1 \pmod 3,$$

contrary to our previous conclusion.

If $m \equiv 1 \pmod 3$, then

$$\begin{aligned} m_{s+2} &= \lfloor m_{s+1} + \delta_{s+1} + m + \delta \rfloor \\ &\equiv \lfloor 1 + \delta_{s+1} + \delta \rfloor \pmod 3. \end{aligned}$$

Since $m_{s+2} \not\equiv 1 \pmod 3$, we have $\delta_{s+1} + \delta > 1$ and $m_{s+2} \equiv 2 \pmod 3$. Moreover, since

$$(3(s+1)+1)(c-b)/a = 3(s+2) - (q-2)/q,$$

it follows from Corollary 5.1 that $\delta_{s+2} > (q-2)/q$. Now

$$\delta_{s+2} = \{\delta_{s+1} + \delta\} = \delta_{s+1} + \delta - 1.$$

Hence $\delta + \delta_{s+1} > 1 + (q-2)/q$, which implies that $\delta > (q-2)/q$ and $\delta_{s+1} > (q-2)/q$. Therefore

$$m_{s+3} = \lfloor m_{s+2} + \delta_{s+2} + m + \delta \rfloor \equiv \lfloor \delta + \delta_{s+2} \rfloor \equiv 1 \pmod 3.$$

Since $s \geq 1$. Hence $s+3 \leq 2s+2$, which implies that $m_{s+3} \not\equiv 1 \pmod 3$, a contradiction.

Finally we consider the case when $m \equiv 2 \pmod 3$. Then for $i = 0, 1, \cdots, s+1$,

$$\begin{aligned} m_{s+i+1} &= \lfloor m_{s+1} + \delta_{s+1} + i(m+\delta) \rfloor \\ &\equiv 2i + \lfloor i\delta + \delta_{s+1} \rfloor \pmod 3. \end{aligned}$$

If $(s+1)\delta + \delta_{s+1} < s$, then let j^* be the smallest integer such that $j^*\delta + \delta_{s+1} < j^* - 1$. Then $2 \leq j^* \leq s+1$, and

$$j^* - 2 \leq j^*\delta + \delta_{s+1} < j^* - 1.$$

Hence

$$\begin{aligned} m_{s+j^*+1} &\equiv 2i + i - 2 \\ &\equiv 1 \pmod 3, \end{aligned}$$

contrary to our previous conclusion. Therefore

$$(s+1)\delta + \delta_{s+1} \geq s.$$

This implies that $\delta \geq 1 - (1 + \delta_{s+1})/(s+1)$, and hence

$$\begin{aligned} (q+1)\delta &\geq q - 2 - 3\delta_{s+1} \\ &> q - 5. \end{aligned}$$

By Lemma 7.1, $(q+1)(m+\delta) \equiv 0 \pmod 3$. Since $q+1 \equiv 0 \pmod 3$, we conclude that $(q+1)\delta \equiv 0 \pmod 3$. Therefore $(q+1)\delta = q-2$, and hence $\delta = (q-2)/(q+1) = s/(s+1)$. This implies that $\delta_{2s+2} = \{(2s+2)\delta\} = 0$. Since $m_{2s+2} \equiv 2$, and

$$3(2s+2)/(c-b) \in I[3(2s+1)+1, 3(2s+1)+2; a],$$

this is in contrary to Corollary 5.1. ∎

Lemma 8.6 *If* $p = 1$ *then* $q \not\equiv 0 \pmod 3$.

Proof. Assume to the contrary that $p = 1$ and $q = 3s$ for some integer s. Since $q \neq 3$, we know that $s \geq 2$. As $(c-b)/a = (3s+1)/(3s)$, it follows that for $i \geq 0$,

$$\begin{aligned}
\frac{3(s+i+1)a}{c-b} &= \frac{3(s+i+1)(3s)}{3s+1} \\
&= 3(s+i+1) - \frac{3(s+i+1)}{3s+1} \\
&= 3(s+i)+2 - \frac{3i+2}{3s+1}.
\end{aligned}$$

Therefore for $i = 0, 1, \cdots, s-1$,

$$3(s+i+1)/(c-b) \in I[3(s+i)+1, 3(s+i)+2; a].$$

It follows from Lemma 5.1 that $m_{s+i+1} \not\equiv 1 \pmod 3$ for $i = 0, 1, \cdots, s-1$. We consider three cases:

Case 1. $m \equiv 0 \pmod 3$.

Assume first that $m_{s+1} \equiv 0 \pmod 3$. Then

$$\begin{aligned}
m_{s+i+1} &= m_{s+1} + \lfloor i(m+\delta) + \delta_{s+1} \rfloor \\
&\equiv \lfloor i\delta + \delta_{s+1} \rfloor \pmod 3.
\end{aligned}$$

If $(s-1)\delta + \delta_{s+1} \geq 1$, then let j^* be the smallest integer such that $j^*\delta + \delta_{s+1} \geq 1$, then $1 \leq j^* \leq s-1$, and

$$m_{s+j^*+1} \equiv 1 \pmod 3,$$

contrary to our previous conclusion. Therefore we have $(s-1)\delta + \delta_{s+1} < 1$, which implies that $\delta < 1/(s-1)$, and $m_{2s} \equiv 0 \pmod 3$. Since

$$m_{2s} \equiv \lfloor 2s\delta \rfloor,$$

we conclude that $2s\delta < 1$, i.e., $\delta < 1/(2s)$. By Lemma 7.1,

$$(q+p)(m+\delta) \equiv (3s+1)\delta \equiv 0 \pmod 3.$$

Because $\delta < 1/(2s)$, we conclude that $\delta = 0$. Hence $3b/(c-b) = n_1 \equiv 0 \pmod 3$, i.e., $b/(c-b)$ is an integer. This is in contrary to the fact that $b > c-b > b/2$.

Assume now that

$$m_{s+1} \equiv \lfloor (s+1)\delta \rfloor \equiv 2 \pmod 3.$$

Since $(3s+1)(c-b)/a = 3(s+1) - (q-1)/q$, by Corollary 5.1, we have

$$\delta_{s+1} > \frac{q-1}{q}\frac{b}{c-b} > \frac{q-1}{q}.$$

Since $(q+1)\delta_{s+1}$ is an integer (by Lemma 7.1), we must have $\delta_{s+1} = q/(q+1)$. Therefore

$$q/(q+1) = \delta_{s+1} > \frac{q-1}{q}\frac{b}{c-b}.$$

It follows that $b/(c-b) < q^2/(q^2-1)$. Because $n \equiv m \equiv 0 \pmod 3$ and that

$$3b/(c-b) = n + \delta < 3q^2/(q^2-1),$$

we conclude that $n = 3$ and $\delta < 3/(q^2-1)$. As $q = 3s$, this is in contrary to our previous conclusion that $\delta_{s+1} = \{(s+1)\delta\} = q/(q+1)$.

Case 2. $m \equiv 1 \pmod 3$.

If $m \equiv 1 \pmod 3$ then, because $b > c-b > b/2$ and $3b/(c-b) = m+\delta$, we conclude that

$$3b/(c-b) = 4 + \delta \geq 4.$$

If $m_{s+1} \equiv 2 \pmod 3$, then since $(3s+1)(c-b)/a = 3(s+1) - (q-1)/q$, by Corollary 5.1, we have

$$\begin{aligned} \delta_{s+1} &> \frac{q-1}{q}\frac{b}{c-b} \\ &\geq \frac{4(q-1)}{3q}. \end{aligned}$$

This is impossible, because $q \geq 6$ and $\delta_{s+1} < 1$. Therefore $m_{s+1} \equiv 0 \pmod 3$.

Since

$$(3s+2)(c-b)/a = 3(s+1) + 2/q,$$

by Corollary 5.1, we have $(1-\delta_{s+1}) > 2/q$, i.e., $\delta_{s+1} < (q-2)/q$. Since

$$\begin{aligned} m_{s+2} &\equiv \lfloor m_{s+1} + 1 + \delta_{s+1} + \delta \rfloor \\ &\equiv \lfloor 1 + \delta_{s+1} + \delta \rfloor \pmod 3, \end{aligned}$$

and since $s \geq 2$, which implies that $m_{s+2} \not\equiv 1 \pmod 3$, we must have $m_{s+2} \equiv 2 \pmod 3$ and

$$\delta_{s+2} = \delta_{s+1} + \delta - 1.$$

As $(c-b)/a = (3s+1)/(3s)$, it follows that

$$(3s+4)(c-b)/a = 3(s+2) - (q-4)/q.$$

By Corollary 5.1, we have $\delta_{s+2} > (q-4)/q$. Since $\delta_{s+1} < (q-2)/q$, we have $\delta > (q-2)/q$.

If $s \geq 3$, then

$$\begin{aligned} m_{s+3} &\equiv \lfloor m_{s+2} + m + \delta_{s+2} + \delta \rfloor \\ &\equiv \lfloor \delta_{s+2} + \delta \rfloor \\ &\equiv 1 \pmod 3, \end{aligned}$$

because $q \geq 9$, $\delta_{s+2} > (q-4)/q$ and $\delta > (q-2)/q$. However, $s \geq 3$ implies that $m_{s+3} \not\equiv 1 \pmod 3$, a contradiction. Therefore we have $s = 2$ and $q = 6$. But then since $\delta > (q-2)/q = 2/3$, we have

$$m_{s+1} = m_3 = \lfloor 3(m+\delta) \rfloor \equiv \lfloor 3\delta \rfloor \equiv 2 \pmod 3,$$

contrary to our previous conclusion.

Case 3. $m \equiv 2 \pmod 3$.

If $m \equiv 2 \pmod 3$ then $3b/(c-b) = 5 + \delta$. Hence

$$b/(c-b) = (5+\delta)/3 \geq 5/3.$$

If $m_{s+1} \equiv 2 \pmod 3$, then as in Case 1, we have that

$$\delta_{s+1} > \frac{q-1}{q}\frac{b}{c-b}.$$

But this is impossible as $\delta_{s+1} < 1$, $b/(c-b) \geq 5/3$ and $q \geq 6$. Therefore $m_{s+1} \equiv 0 \pmod 3$.

If $m_{2s} \equiv 0 \pmod 3$, then since

$$(6s-1)(c-b)/a = 3(2s) + (1 - \frac{1}{3s}),$$

by Corollary 5.1, we have

$$1 - \delta_{2s} > \frac{q-1}{q}\frac{b}{c-b},$$

which is impossible (as $1 - \delta_{2s} \leq 1$ $q \geq 6$, $b/(c-b) \geq 5/3$). Hence we have $m_{2s} \equiv 2 \pmod 3$.

Since $m_{s+1} \equiv 0 \pmod 3$ and $m_{2s} \equiv 2 \pmod 3$, and since

$$m_{2s} \equiv 2(s-1) + \lfloor \delta_{s+1} + (s-1)\delta \rfloor, \pmod 3$$

we conclude that $(s-1)\delta + \delta_{s+1} < s-1$. If $(s-1)\delta + \delta_{s+1} < s-2$,, then let j^* be the least integer such that $j^*\delta + \delta_{s+1} < j^* - 1$. Then $2 \le j^* \le s-1$. By the minimality of j^*, we have $j^* - 2 \le j^*\delta + \delta_{s+1} < j^* - 1$, which implies that

$$m_{s+j^*+1} \equiv 1 \pmod 3,$$

contrary to our previous conclusion. Therefore we have

$$s - 2 \le (s-1)\delta + \delta_{s+1} < s - 1.$$

This implies that $\delta \ge 1 - (1 + \delta_{s+1})/(s-1)$ and hence

$$\begin{aligned}(q+1)\delta &\ge q + 1 - 3 - 3\delta_{s+1} - (4 + 4\delta_{s+1})/(s-1) \\ &\ge q - 13.\end{aligned}$$

By Lemma 7.1,

$$(q+1)(m+\delta) \equiv 0 \pmod 3.$$

As $q \equiv 0 \pmod 3$ and $m \equiv 2 \pmod 3$, we know that

$$(q+1)\delta \equiv 1 \pmod 3.$$

Therefore the possible values for $(q+1)\delta$ are $q-2, q-5, q-8$ and $q-11$.

If $(q+1)\delta = q - 11$ then $q \ge 11$. Hence $s \ge 4$, which implies that $(q+1)\delta \ge q - 8$, a contradiction.

If $(q+1)\delta = q - 8$ then $q \ge 8$, hence $s \ge 3$. Therefore

$$q - 8 = (q+1)\delta \ge q - 4 - 5\delta_{s+1}.$$

This implies that $\delta_{s+1} \ge 4/5$. Since $\delta_{s+1} = \{(s+1)\delta\}$, $q = 3s$ and $\delta = (q-8)/(q+1)$, it follows that

$$\delta_{s+1} = 1 - 6/(3s+1).$$

Now $4/5 \le \delta_{s+1} = 1 - 6/(3s+1)$ implies that $3s + 1 \ge 30$, hence $s \ge 10$. But then

$$(q+1)\delta \ge q + 1 - 3 - 3\delta_{s+1} - (4 + 4\delta_{s+1})/(s-1) > q - 6,$$

an obvious contradiction.

If $(q+1)\delta = q - 2$, then $\delta = (q-2)/(q+1)$. This implies that

$$\delta_{2s} = \{2s\delta\} = 2/(q+1).$$

However,

$$(3(2s-1)+1)(c-b)/a = 6s - 2/q.$$

Since $m_{2s} \equiv 2 \pmod 3$, this is in contrary to Corollary 5.1

Assume now that $(q+1)\delta = q-5$. Then $q \geq 5$ and $\delta = (q-5)/(q+1)$. Hence $3b/(c-b) = 5+\delta = 6-6/(q+1)$, which implies that $(c-b)/b = (q+1)/(2q)$ and hence $b = 2a$, contrary to our general assumption. ∎

Combining Lemmas 8.4, 8.5, 8.6, we conclude that $p \neq 1$. For the remaining part of this section, we assume that $p \geq 2$.

Since the integer $\alpha+\beta$ will be very frequently used in the remaining part of this section (more than 50 times), we use a single letter h in place of it, i.e., we let $h = \alpha+\beta$. Since $\alpha p = \beta q+1$, we have $\alpha(p+q) = \beta q+\alpha q+1 = hq+1$. This formula will be frequently used in the the following.

Let

$$i_0 = \lceil (q-2p)/3 \rceil, \text{ and } j_0 = \lfloor (2q-p)/3 \rfloor.$$

(Note that i_0 could be negative). Since $p \geq 2$, we know that

$$(c-b)/a = (p+q)/q \geq 1+2/q.$$

For and integer j, we have

$$\begin{aligned} \frac{(3j\alpha+1)(c-b)}{a} &= \frac{(3j\alpha+1)(p+q)}{q} \\ &= 3jh + \frac{p+q+3j}{q} \\ &= 3(jh+1) + \frac{p-2q+3j}{q} \end{aligned}$$

For $i_0 \leq j \leq j_0$, since $(q-2p)/3+2/3 \leq i_0$ and $j_0 \geq (2q-p)/3-2/3$, it follows that

$$\frac{3(jh+1)}{c-b} \in I[3j\alpha+1, 3j\alpha+2; a].$$

Moreover, we have

$$I[3i_0h+2, 3i_0h+3; c-b] \subset I[3i_0\alpha+1, 3i_0\alpha+2; a],$$

and

$$I[3j_0h+3, 3j_0h+4; c-b] \subset I[3j_0\alpha+1, 3j_0\alpha+2; a].$$

By Corollary 5.1, we have

$$\begin{aligned} m_{i_0h+1} &\equiv 0 \pmod 3, \\ m_{j_0h+1} &\equiv 2 \pmod 3, \\ m_{jh+1} &\not\equiv 1 \pmod 3, \text{ for } j = i_0, i_0+1, \cdots, j_0. \end{aligned}$$

By Lemma 7.1, $(q+p)(m+\delta) \equiv 0 \pmod 3$. In particular, $(q+p)\delta$ is an integer.

Suppose $h(m+\delta)=\mu+\epsilon$, where

$$\mu=\lfloor h(m+\delta)\rfloor$$

and

$$\epsilon=\{h(m+\delta)\}=\{h\delta\}.$$

Since $\alpha p=\beta q+1$, we have

$$\begin{aligned}q(\mu+\epsilon)+m+\delta &= qh(m+\delta)+m+\delta\\ &= q(\alpha+\beta)(m+\delta)+m+\delta\\ &= (\alpha q+\beta q+1)(m+\delta)\\ &= \alpha(q+p)(m+\delta)\equiv 0 \pmod 3.\end{aligned}$$

Hence $q\epsilon+\delta$ is an integer. Also observe that

$$(q+p)(\epsilon+\mu)=(q+p)h(m+\delta)\equiv 0 \pmod 3.$$

In particular, we have $(q+p)\epsilon$ is an integer.

For $j\geq 0$,

$$\begin{aligned}m_{(i_0+j)h+1} &= m_{i_0h+1}+\lfloor jh(m+\delta)+\delta_{i_0h+1}\rfloor\\ &= \lfloor(j(\mu+\epsilon)+\delta_{i_0h+1}\rfloor\\ &\equiv \lfloor j\epsilon+\delta_{i_0h+1}\rfloor \pmod 3.\end{aligned}$$

Lemma 8.7 $\mu\not\equiv 0 \pmod 3$.

Proof. Assume to the contrary that $\mu\equiv 0 \pmod 3$. Then for $j\geq 0$,

$$\begin{aligned}m_{(i_0+j)h+1} &= m_{i_0h+1}+\lfloor jh(m+\delta)+\delta_{i_0h+1}\rfloor\\ &= \lfloor(j(\mu+\epsilon)+\delta_{i_0h+1}\rfloor\\ &\equiv \lfloor j\epsilon+\delta_{i_0h+1}\rfloor \pmod 3.\end{aligned}$$

Since

$$m_{j_0h+1}\equiv 2 \pmod 3,$$

we conclude that

$$(j_0-i_0)\epsilon+\delta_{i_0h+1}\geq 2.$$

Let j^* be the least integer such that $j^*\epsilon+\delta_{i_0h+1}\geq 1$, then $1\leq j^*<j_0-i_0$, and by the minimality of j^*, we have

$$m_{(i_0+j^*)h+1}\equiv 1 \pmod 3,$$

contrary to our previous conclusion. ∎

Lemma 8.8 $\mu\not\equiv 1 \pmod 3$.

Proof. Assume to the contrary that $\mu \equiv 1 \pmod 3$. Since $q \geq 4$ we know that $j_0 \geq i_0 + 1$. Hence $m_{(i_0+1)h+1} \not\equiv 1 \pmod 3$. On the other hand,

$$\begin{aligned} m_{(i_0+1)h+1} &= \lfloor m_{i_0h+1} + \delta_{i_0h+1} + \mu + \epsilon \rfloor \\ &\equiv 1 + \lfloor \epsilon + \delta_{i_0h+1} \rfloor \pmod 3. \end{aligned}$$

Therefore

$$\begin{aligned} \epsilon + \delta_{i_0h+1} &\geq 1, \\ m_{(i_0+1)h+1} &\equiv 2 \pmod 3, \\ \delta_{(i_0+1)h+1} &= \epsilon + \delta_{i_0h+1} - 1. \end{aligned}$$

Since $i_0 \leq (q-2p)/3 + 2/3$, $\alpha p = \beta q + 1$, $h = \alpha + \beta$, $(c-b)/a = (p+q)/q$, it follows that

$$3((i_0+1)h+1)/(c-b) \in I[3(i_0+1)\alpha + 1, 3(i_0+1)\alpha + 2; a], \text{ and}$$

$$(3(i_0+1)\alpha + 1)(c-b)/a \leq 3((i_0+1)h+1) - (q+p-5)/q.$$

By Corollary 5.1,

$$\delta_{(i_0+1)h+1} > (q+p-5)/q,$$

which implies that

$$\epsilon > (q+p-5)/q \geq (q-3)/q,$$

and

$$\delta_{i_0h+1} > (q-3)/q.$$

First we consider the case $q \geq 6$. Then $j_0 \geq i_0 + 2$ and hence

$$m_{(i_0+2)h+1} \not\equiv 1 \pmod 3.$$

However

$$m_{(i_0+2)h+1} \equiv 2 + \lfloor 2\epsilon + \delta_{i_0h+1} \rfloor \pmod 3.$$

As

$$2 \leq 2(q-3)/q + (q-3)/q < 2\epsilon + \delta_{i_0h+1} < 3,$$

we conclude that

$$\lfloor 2\epsilon + \delta_{i_0h+1} \rfloor = 2.$$

Hence $m_{(i_0+2)h+1} \equiv 1 \pmod 3$, which is a contradiction.

If $q = 4$ then $p = 3$ (as $p \geq 2$ and $\gcd(p,q) = 1$), and $(c-b)/a = 7/4$. This implies that

$$I[1,2;a] \supset I[2,3;c-b].$$

By Lemma 5.2, $m \equiv 0 \pmod 3$. By Lemma 7.1

$$(q+p)(m+\delta) \equiv 0 \pmod 3,$$

which implies that $7\delta \equiv 0 \pmod 3$. Since $(c-b)/a = 7/4$, which implies that $2(c-b)/a = 3 + 1/2$, by Corollary 5.1 we have $1 - \delta > 1/2$, hence $\delta < 1/2$. Therefore $\delta = 0$ or $\delta = 3/7$. It is straightforward to verify that

$$I[18, 19; c-b] \subset I[10, 11; a], \text{ and } 10(c-b)/a = 18 - 1/2.$$

It follows from Lemma 5.2 that

$$m_6 \equiv 2 \pmod 3$$

and follows from Corollary 5.1 that

$$\delta_6 = \{6\delta\} > 1/2.$$

This implies that $\delta \neq 0$, hence $\delta = 3/7$. Note that $1 < b/(c-b) < 2$ and $3b/(c-b) = m + \delta$. Since $m \equiv 0 \pmod 3$, we conclude that

$$3b/(c-b) = 3 + \delta = 24/7,$$

which implies that $b = 2a$, contrary to the general assumption.

Assume that $q = 5$. Then $p = 2, 3$, or 4. If $q = 5$ and $p = 2$, then $(c-b)/a = 7/5$. Hence

$$I[4, 5; a] \supset I[6, 7; c-b]$$

and $4(c-b)/a = 6 - 2/5$. By Lemma 5.2, $m_2 = \lfloor 2(m+\delta) \rfloor \equiv 2 \pmod 3$ and by Corollary 5.1 $\delta_2 > 2/5$. This implies that $m \not\equiv 0 \pmod 3$ (for otherwise we would have $m_2 \equiv \lfloor 2\delta \rfloor \pmod 3$).

It is also straightforward to verify that $I[10, 11; a] \supset I[14, 15; c-b]$, which implies that $m_5 \equiv 0 \pmod 3$ (by Lemma 5.2).

If $m \equiv 1 \pmod 3$ then by 7.1, $7(1+\delta) \equiv 0 \pmod 3$, hence

$$7\delta \equiv 2 \pmod 3.$$

Therefore $\delta = 2/7$ or $\delta = 5/7$. If $\delta = 5/7$, then $m_5 \equiv \lfloor 5(1 + \frac{5}{7}) \rfloor \equiv 2 \pmod 3$, contrary to our previous conclusion. If $\delta = 2/7$ then $3b/(c-b) = 4 + \delta = 30/7$, which implies that $b = 2a$, contrary to our general assumption.

If $m \equiv 2 \pmod 3$, then

$$7(2 + \delta) \equiv 0 \pmod 3.$$

Hence $\delta = 1/7$ or $4/7$. In any case, $\delta_2 = \{2\delta\} < 2/5$, contrary to our previous conclusion.

If $q = 5$ and $p = 3$, then $(c-b)/a = 8/5$. Hence

$$I[2, 3; c] \subset I[1, 2; a],$$

and $2(c-b)/a = 3+1/5$. By Lemma 5.2, $m \equiv 0 \pmod 3$ and by Corollary 5.1, $\delta < 4/5$. By Lemma 7.1, $8\delta \equiv 0 \pmod 3$. Therefore $\delta = 0$ or $3/8$ or $6/8$. However, $\delta = 0$ would imply that $3b/(c-b) = 3$, and hence $c = 2b$, contrary to our assumption; and $\delta = 6/8$ would imply that $3b/(c-b) = 3+6/8 = 30/8$, which implies that $b = 2a$, again contrary to our assumption. Therefore $\delta = 3/8$.

Now $m \equiv 0 \pmod 3$ and $\delta = 3/8$ implies that $m_4 \equiv \lfloor 4\delta \rfloor \equiv 1 \pmod 3$. However, straightforward calculation shows that $12/(c-b) \in I[7,8;a]$. By Lemma 5.1, (n_4, m_4) is strongly consistent, contrary to the general assumption.

If $q = 5$ and $p = 4$, then $(c-b)/a = 9/5$. Then

$$I[2,3;c-b] \subset I[1,2;a],$$

which implie that $m \equiv 0 \pmod 3$ (by Lemma 5.1). Moreover, $2(c-b)/a = 3+3/5$, which implies that $1-\delta > 3/5$ (by Corollary 5.1), hence $\delta < 2/5$. By Lemma 7.1, $9(m+\delta) \equiv 0 \pmod 3$, which implies that $\delta = 0$ or $3/9$ or $6/9$. Since $\delta < 2/5$, so $\delta \neq 6/9$. Also $\delta = 0$ would imply that $3b/(c-b) = 3$ which implies that $c = 2b$, contrary to our assumption. Therefore $\delta = 3/9 = 1/3$. But this implies that $3b/(c-b) = 3+1/3$, and hence $b = 2a$, again contrary to our general assumption. ∎

Lemma 8.9 $\mu \not\equiv 2 \pmod 3$.

Proof. Assume to the contrary that $\mu \equiv 2 \pmod 3$. Then for $j = 0, 1, 2, \cdots, j_0 - i_0$,

$$\begin{aligned} m_{(i_0+j)h+1} &\equiv m_{i_0h+1} + \lfloor \delta_{i_0h+1} + j(\mu+\epsilon) \rfloor \\ &\equiv 2j + \lfloor j\epsilon + \delta_{i_0h+1} \rfloor \pmod 3. \end{aligned}$$

If there is an integer $1 \leq j \leq j_0 - i_0$ such that

$$j\epsilon + \delta_{i_0h+1} < j - 1,$$

then let j^* be the least such integer, we would have

$$j^* - 2 \leq j^*\epsilon + \delta_{i_0h+1} < j^* - 1.$$

This implies that

$$m_{(i_0+j^*)h+1} \equiv 1 \pmod 3,$$

contrary to our previous conclusion. Therefore for $1 \leq j \leq j_0 - i_0$,

$$j\epsilon + \delta_{i_0h+1} \geq j - 1.$$

In particular,

$$(j_0 - i_0)\epsilon \geq (j_0 - i_0) - 1 - \delta_{i_0h+1}.$$

Therefore we have

$$q + p > (q+p)\epsilon \geq q + p - (1 + \delta_{i_0 h+1})(q+p)/(j_0 - i_0).$$

Recall that

$$(q+p)(\epsilon + \mu) \equiv (q+p)(\epsilon + 2) \equiv 0 \pmod 3.$$

It follows that $(q+p)\epsilon = q + p - 3i$ for some integer $i \geq 1$.

We consider three cases:

Case 1: $q + p \equiv 0 \pmod 3$.

If $q + p \equiv 0 \pmod 3$, then $i_0 = (q-2p)/3$, $j_0 = (2q-p)/3$ and $(j_0 - i_0) = (q+p)/3$ and hence

$$(q+p)\epsilon \geq q + p - 3 - 3\delta_{i_0 h+1} > q + p - 6.$$

Therefore $(q+p)\epsilon = q + p - 3$ and $\epsilon = 1 - 3/(q+p)$. Since $q\epsilon + \delta$ is an integer, we know that $\delta = \{3q/(q+p)\}$. Because

$$\delta_{j_0 h+1} = \{j_0 \epsilon + \delta\}, \text{ and } j_0 = (2q-p)/3,$$

straightforward calculation shows that $\delta_{j_0 h+1} = 0$. As $m_{j_0 h+1} \equiv 2 \pmod 3$ and $3m_{j_0 h+1}/(c-b) \in I[3j_0\alpha + 1, 3j_0\alpha + 2; a]$, this is in contrary to Corollary 5.1.

Case 2: $q + p \equiv 1 \pmod 3$.

If $q + p \equiv 1 \pmod 3$, then $i_0 = (q - 2p + 2)/3$, $j_0 = (2q - p - 2)/3$ and $j_0 - i_0 = (q + p - 4)/3$. Hence

$$(q+p)\epsilon \geq q + p - 3 - 3\delta_{i_0 h+1} - \frac{12 + 12\delta_{i_0 h+1}}{q + p - 4}.$$

Since $p \geq 2$, $q \geq 4$ and $\gcd(q,p) = 1$, we know that $q + p \geq 7$, hence

$$(12 + 12\delta_{i_0 h+1})/(q + p - 4) < 8.$$

Therefore the possible values for $(q+p)\epsilon$ are $q + p - 3i$ for $i = 1, 2, 3, 4$.

If $(q+p)\epsilon = q + p - 12$ then $q + p \geq 12$ and hence

$$\begin{aligned} q + p - 3 - 3\delta_{i_0 h+1} - \frac{12 + 12\delta_{i_0 h+1}}{q + p - 4} &> q + p - 6 - \frac{24}{8} \\ &= q + p - 9, \end{aligned}$$

which is a contradiction.

If $(q+p)\epsilon = q + p - 9$ then $q + p \geq 9$. Since $q + p \equiv 1 \pmod 3$, we have $q + p \geq 10$. This implies that

$$q + p - 9 \geq q + p - 3 - 3\delta_{i_0 h+1} - (12 + 12\delta_{i_0 h+1})/6.$$

Therefore $\delta_{i_0h+1} \geq 4/5$. Since

$$i_0 = (q-2p+2)/3, \ \delta_{i_0h+1} = \{\delta + i_0\epsilon\}, \ \delta = \{9q/(q+p)\}$$

(which follows from the fact that $q\epsilon+\delta$ is an integer and $\epsilon = 1-9/(p+q)$), we conclude that

$$\begin{aligned}\delta_{i_0h+1} &= \{9q/(q+p) - 3(q-2p+2)/(q+p)\} \\ &= 1-6/(q+p).\end{aligned}$$

Therefore $6/(q+p) < 1/5$, and hence $q+p > 30$. But then

$$\begin{aligned}(q+p)\epsilon &\geq q+p-3-3\delta_{i_0h+1} - \frac{12+12\delta_{i_0h+1}}{q+p-4} \\ &> q+p-6-\frac{24}{26},\end{aligned}$$

which is a contradiction.

If $(q+p)\epsilon = q+p-3$, then $\epsilon = 1-3/(p+q)$. Since $q\epsilon+\delta$ is an integer, we have $\delta = \{3q/(q+p)\}$. Since $j_0 = (2q-p-2)/3$ and $\delta_{j_0h+1} = \{(j_0\epsilon+\delta\}$, easy calculation shows that

$$\delta_{j_0h+1} = 2/(q+p).$$

However, since $(c-b)/a = (q+p)/q, \ j_0 = (2q-p-2)/3$ and $\alpha p = \beta q+1$, it is straightforward to verify that

$$(3j_0\alpha+1)(c-b)/a = (3(j_0h+1)-2/q.$$

Since $m_{j_0h+1} \equiv 2 \pmod 3$, this is in contrary to Corollary 5.1.

Now we consider the case when $(q+p)\epsilon = q+p-6$. If $q > 5p$ then since $q\epsilon+\delta$ is an integer, we conclude that

$$\delta = (q-5p)/(q+p).$$

If $m \equiv 2 \pmod 3$ then

$$3b/(c-b) = 5+\delta = 6q/(q+p),$$

which implies that $b = 2a$, contrary to the general assumption. If $m \equiv 0 \pmod 3$ then, contrary to our previous conclusion, we have

$$\begin{aligned}m_{i_0h+1} &= \lfloor i_0(\mu+\epsilon)+m+\delta \rfloor \\ &\equiv \lfloor \frac{q-5p}{q+p} - \frac{q-2p+2}{3}\frac{6}{q+p} \rfloor \\ &\equiv 1 \pmod 3.\end{aligned}$$

If $m \equiv 1 \pmod 3$ then, contrary to our previous conclusion, we have

$$\begin{aligned} m_{i_0h+1} &= \lfloor i_0(\mu+\epsilon)+\delta\rfloor \\ &\equiv 1+\lfloor i_0(2+\epsilon)+\delta\rfloor \\ &\equiv 1+\lfloor \frac{q-5p}{q+p}-\frac{q-2p+2}{3}\frac{6}{q+p}\rfloor \\ &\equiv 2 \pmod 3. \end{aligned}$$

Thus we may assume that $q < 5p$. If $5p > q > 2p$ then $\delta = (2q-4p)/(q+p)$.

If $m \equiv 2 \pmod 3$, then, contrary to our previous conclusion, we have

$$\begin{aligned} m_{i_0h+1} &= \lfloor i_0(\mu+\epsilon)+m+\delta\rfloor \\ &\equiv 2+\lfloor i_0(2+\epsilon)+\delta\rfloor \\ &\equiv 2+\lfloor \frac{2q-4p}{q+p}-\frac{q-2p+2}{3}\frac{6}{q+p}\rfloor \\ &\equiv 1 \pmod 3. \end{aligned}$$

If $m \equiv 1 \pmod 3$, then

$$3b/(c-b) = 4+\delta$$

and hence

$$b/(c-b) = 2q/(q+p),$$

which implies that $b = 2a$, contrary to our general assumption.

If $m \equiv 0 \pmod 3$ then, contrary to our previous conclusion, we have

$$\begin{aligned} m_{i_0h+1} &= \lfloor i_0(\mu+\epsilon)+m+\delta\rfloor \\ &\equiv \lfloor i_0(2+\epsilon)+\delta\rfloor \\ &\equiv \lfloor \frac{2q-4p}{q+p}-\frac{q-2p+2}{3}\frac{6}{q+p}\rfloor \\ &\equiv 2 \pmod 3. \end{aligned}$$

If $q < 2p$, then $\delta = 3(q-p)/(q+p)$, and

$$I[1,2;a] \supset I[2,3;c-b].$$

By Lemma 5.2, $m \equiv 0 \pmod 3$. Hence

$$3b/(c-b) = 3+\delta = 6q/(q+p).$$

This implies that $b = 2a$, contrary to the general assumption. This completes the proof of Case 2.

Case 3: $q+p \equiv 2 \pmod 3$.

If $q+p\equiv 2 \pmod 3$ then

$$i_0=(q-2p+1)/3,\ j_0=(2q-p-1)/3$$

and

$$j_0-i_0=(q+p-2)/3.$$

Hence

$$q+p>(q+p)\epsilon>q+p-3-3\delta_{i_0h+1}-\frac{6+6\delta_{i_0h+1}}{q+p-2}.$$

Since $q+p\geq 7$, we have

$$(q+p)\epsilon\geq q+p-8.$$

Therefore the possible values for $(q+p)\epsilon$ are $q+p-3$ and $q+p-6$.

If $(q+p)\epsilon=q+p-3$, then $\epsilon=1-3/(q+p)$ and hence $\delta=\{(q-2p)/(q+p)\}$ (recall that $q\epsilon+\delta$ is an integer). Because

$$\delta_{j_0h+1}=\{j_0\epsilon+\delta\}$$

and $j_0=(2q-p-1)/3$, straightforward calculation shows that

$$\delta_{j_0h+1}=1/(q+p).$$

However, it is easy to verify that

$$(3j_0\alpha+1)(c-b)/a=(3(j_0h+1)-1/q.$$

Since $m_{j_0h+1}\equiv 2 \pmod 3$, this is in contrary to Corollary 5.1.

It remains to consider the case when $(q+p)\epsilon=q+p-6$. The proof is similar to the corresponding part of the proof for Case 2. If $q>5p$ then $\delta=(q-5p)/(q+p)$ (by using the fact that $q\epsilon+\delta$ is an integer). If $m\equiv 2 \pmod 3$ then

$$3b/(c-b)=5+\delta=6q/(q+p),$$

which implies that $b=2a$, contrary to the general assumption. If $m\equiv 0 \pmod 3$ then, contrary to our previous conclusion, we have

$$\begin{aligned} m_{i_0h+1} &= \lfloor i_0(\mu+\epsilon)+m+\delta\rfloor \\ &\equiv \lfloor\frac{q-5p}{q+p}-\frac{q-2p+1}{3}\frac{6}{q+p}\rfloor \\ &\equiv 1 \pmod 3. \end{aligned}$$

If $m\equiv 1 \pmod 3$ then, contrary to our previous conclusion, we have

$$\begin{aligned} m_{i_0h+1} &= \lfloor i_0(\mu+\epsilon)+\delta\rfloor \\ &\equiv 1+\lfloor i_0(2+\epsilon)+\delta\rfloor \\ &\equiv 1+\lfloor\frac{q-5p}{q+p}-\frac{q-2p+1}{3}\frac{6}{q+p}\rfloor \\ &\equiv 2 \pmod 3. \end{aligned}$$

Thus we may assume that $q < 5p$. If $5p > q > 2p$ then

$$\delta = (2q - 4p)/(q + p).$$

If $m \equiv 2 \pmod 3$, then, contrary to our previous conclusion, we have

$$\begin{aligned} m_{i_0 h+1} &= \lfloor i_0(\mu + \epsilon) + m + \delta \rfloor \\ &\equiv 2 + \lfloor i_0(2 + \epsilon) + \delta \rfloor \\ &\equiv 2 + \lfloor \frac{2q - 4p}{q + p} - \frac{q - 2p + 1}{3} \frac{6}{q + p} \rfloor \\ &\equiv 1 \pmod 3. \end{aligned}$$

If $m \equiv 1 \pmod 3$, then $3b/(c - b) = 4 + \delta$ and hence

$$b/(c - b) = 2q/(q + p),$$

which implies that $b = 2a$, contrary to our general assumption. If $m \equiv 0 \pmod 3$ then, contrary to our previous conclusion, we have

$$\begin{aligned} m_{i_0 h+1} &= \lfloor i_0(\mu + \epsilon) + m + \delta \rfloor \\ &\equiv \lfloor i_0(2 + \epsilon) + \delta \rfloor \\ &\equiv \lfloor \frac{2q - 4p}{q + p} - \frac{q - 2p + 1}{3} \frac{6}{q + p} \rfloor \\ &\equiv 2 \pmod 3. \end{aligned}$$

If $q < 2p$, then $\delta = 3(q - p)/(q + p)$, and

$$I[1, 2; a] \supset I[2, 3; c - b].$$

By Lemma 5.2, $m \equiv 0 \pmod 3$. Hence

$$3b/(c - b) = 3 + \delta = 6q/(q + p).$$

This implies that $b = 2a$, contrary to the general assumption. ∎

Combining Lemmas 8.7, 8.8, 8.9, we obtain the final contradiction. Therefore under the general assumption, we cannot have $2a > c - b > a$.

9 The case $a > c - b$

In this section we assume that $a > c - b$, and derive a contradiction to our general assumption. Since $a > c - b$ and $(c - b)/a = t + p/q$, we have $t = 0$ and $(c - b)/a = p/q$.

Lemma 9.1 *If there are integers* i, j *such that*

$$I[3i+1, 3i+2; a] \subset I[3j-1, 3j; c-b],$$

or

$$I[3i+1, 3i+2; a] \subset I[3j, 3j+1; c-b],$$

then $c < 3a$. *In general, we have* $c < 4a$.

Proof. Suppose to the contrary that there are integers i, j such that

$$I[3i+1, 3i+2; a] \subset I[3j-1, 3j; c-b], \text{ and } c \geq 3a.$$

Since $I[3i+1, 3i+2; a]$ has length $1/a \geq 3/c$, it follows that there is an integer u such that

$$u/c, (u+1)/c, (u+2)/c \in I[3i+1, 3i+2; a] \cap I[3j-1, 3j; c-b].$$

By Lemma 5.1,

$$(u-3j, u), (u+1-3j, u+1), (u+2-3j, u+2)$$

are all strongly consistent pairs. One of the three integers $u, u+1, u+2$ is equivalent to 1 modulo 3, which is in contrary to the general assumption.

The case when $I[3i+1, 3i+2; a] \subset I[3j, 3j+1; c-b]$ is proved similarly.

Next we show that in general we have $c < 4a$.

If $q = 2$, then $1/a = 1/2(c-b)$. Hence $I[4, 5; a] \subset I[2, 3; c-b]$, which implies that $c < 3a$.

If $q \geq 3$, then since the arc $[2/(c-b), 3/(c-b)]$ of C has length at least $3/(q(c-b))$, it contains an image point. Suppose

$$f(i) = \phi((3i+1)/a) \in [2/(c-b), 3/(c-b)].$$

This means that for some integer j,

$$(3j-1)/(c-b) \leq (3i+1)/a \leq 3j/(c-b).$$

Since $1/a < 1/(c-b)$, it follows that

$$I[3i+1, 3i+2; a] \subset I[3j-1, 3j+1; c-b].$$

Assume to the contrary of the Lemma that $c \geq 4a$, i.e., $1/a \geq 4/c$. It follows that there is an integer u such that for $\ell = 0, 1, 2, 3$,

$$(u+\ell)/c \in I[3i+1, 3i+2; a].$$

Now it is easy to see that either there is an $\ell \in \{0,1,2,3\}$ such that $u+\ell \equiv 1 \pmod 3$ and

$$(u+\ell)/c \in I[3i+1, 3i+2; a] \cap I[3j-1, 3j; c-b]$$

or there is an $\ell \in \{0,1,2,3\}$ such that $u+\ell \equiv 2 \pmod 3$ and

$$(u+\ell)/c \in I[3i+1, 3i+2; a] \cap I[3j, 3j+1; c-b].$$

Applying Lemma 5.1, we obtain a contrary to the general assumption. ∎

Let $\beta(m+\delta) = \mu + \epsilon$. Then

$$p(\mu+\epsilon) = p\beta(m+\delta) \equiv 0 \pmod 3$$

(cf. Lemma 7.1). In particular, $p\epsilon$ is an integer.

Also observe that

$$\begin{aligned} q(\mu+\epsilon)+m+\delta &= q\beta(m+\delta)+m+\delta \\ &= (q\beta+1)(m+\delta) \\ &= \alpha p(m+\delta) \\ &\equiv 0 \pmod 3. \end{aligned}$$

In particular, $q\epsilon + \delta$ is an integer.

Recall that $m_j + \delta_j = 3jc/(c-b) = j(m+\delta)$ and $n_j + \delta_j = 3jb/(c-b) = j(n+\delta)$. In the following, we shall frequently use the formula that

$$m_{j\beta+1} + \delta_{j\beta+1} = j(\mu+\epsilon) + m + \delta.$$

Lemma 9.2 $p \neq 1$.

Proof. If $p = 1$ then $q \geq 2 = 2p$, i.e., $(c-b)/a = p/q \leq 1/2$. Hence $2/a \leq 1/(c-b)$, which implies that

$$I[1,2;a] \subset [0, 1/(c-b)].$$

By Lemma 9.1, $c < 3a$. By Lemma 7.1, we have

$$p(m+\delta) = m+\delta \equiv 0 \pmod 3.$$

Therefore $\delta = 0$ and $m \equiv 0 \pmod 3$. Recall that $m+\delta = 3c/(c-b) = 3b/(c-b)+3 = n+\delta+3$, it follows that $b/(c-b) = n/3 = s$ and $c/(c-b) = m/3 = s+1$ for some integer s. Since $c > 2a$ and $a/(c-b) = q$, it follows that $s \geq 2q$. Therefore $b \geq 2a$. It follows that $4/b \leq 2/a$. Since $4/b > 4/c > 1/a$, we conclude that

$$4/b \in I[1,2;a] \subset [0, 1/(c-b)].$$

Hence, by Lemma 5.1, $(4,4)$ is strongly consistent, contrary to the general assumption. ∎

Lemma 9.3 *$p \neq 2$.*

Proof. Assume to the contrary that $p = 2$. Since $\gcd(p, q) = 1$, we know that $q = 2u + 1$ for some integer u. It follows that $(3u + 3/2)/a = 3/(c - b)$, i.e., the point $3/(c-b)$ is the middle point of the interval $I[3u+1, 3u+2; a]$. By lemma 5.1, (n, m) is strongly consistent, and hence $m \not\equiv 1 \pmod 3$. Since $1/a > 1/b$, at least one of the points $n/b, (n+1)/b$ is contained in $I[3u + 1, 3u + 2; a]$.

If $(n + 1)/b \in I[3u + 1, 3u + 2; a]$, then by Lemma 5.1, $(n + 1, m + 1)$ is strongly consistent, and hence $m \equiv 2 \pmod 3$. By Lemma 7.1, $2(m+\delta) \equiv 0 \pmod 3$. But this is impossible because $0 \leq 2\delta < 2$.

If $n/b \in I[3u+1, 3u+2; a]$ then by Lemma 5.1, $(n-1, m-1)$ is strongly consistent, and hence $m \equiv 0 \pmod 3$. Thus $m = 3s$ for some integer s. By Lemma 7.1, $2(m + \delta) \equiv 0 \pmod 3$. Therefore $2\delta \equiv 0 \pmod 3$ hence $\delta = 0$. Then it follows from the definition of m, n and δ that

$$b/(c - b) = s, \quad c/(c - b) = s + 1.$$

Since $1/c < 1/(2a)$, we conclude that

$$q/c < q/(2a) = 1/(c - b) = s/b = (s + 1)/c.$$

Therefore $q \leq s$, and hence $1/b \leq 1/(2a)$. This implies (by the definition of n) that

$$(n + 1)/b \leq 3/(c - b) + 1/b \leq (3u + 2)/a,$$

hence $(n + 1)/b \in I[3u + 1, 3u + 2; a]$, and by Lemma 5.1, $(n + 1, m + 1)$ is strongly consistent, contrary to the general assumption (as $m + 1 \equiv 1 \pmod 3$). ∎

For the remaining, we assume that $p \geq 3$. Let $i_0 = \lceil q - 2p/3 \rceil$ and $j_0 = \lfloor q - p/3 \rfloor$. In other words,
if $p \equiv 0 \pmod 3$, *then* $i_0 = q - 2p/3$ *and* $j_0 = q - p/3$*;*
if $p \equiv 1 \pmod 3$, *then* $i_0 = q - 2p/3 + 2/3$ *and* $j_0 = q - p/3 - 2/3$*;*
if $p \equiv 2 \pmod 3$, *then* $i_0 = q - 2p/3 + 1/3$ *and* $j_0 = q - p/3 - 1/3$. Since $p \geq 3$, we know that $i_0 \leq j_0$, and that $i_0 = j_0$ if and only if $p = 4$.

Since $\alpha p = \beta q + 1$, and $(c - b)/a = p/q$, we have

$$\begin{aligned} \frac{3j\alpha(c-b)}{a} &= \frac{3j\alpha p}{q} \\ &= \frac{3j(\beta q + 1)}{q} \\ &= 3j\beta + \frac{3j}{q}, \end{aligned}$$

or equivalently,

$$\frac{3j\alpha}{a} = \frac{3j\beta}{c-b} + \frac{3j}{q(c-b)}.$$

This formula will be used frequently in the remaining.

Lemma 9.4 *For* $i_0 \leq j \leq j_0$, $m_{j\beta+1} \not\equiv 1 \pmod 3$.

Proof. Note that

$$\frac{(3j\alpha+1)(c-b)}{a} = 3j\beta + \frac{3j+p}{q},$$

and

$$\frac{(3j\alpha+2)(c-b)}{a} = 3j\beta + \frac{3j+2p}{q}.$$

Since

$$3q - 2p \leq 3i_0 \leq 3j \leq 3j_0 \leq 3q - p,$$

we conclude that

$$\frac{(3j\alpha+1)(c-b)}{a} \leq 3(j\beta+1),$$

and

$$\frac{(3j\alpha+2)(c-b)}{a} \geq 3(j\beta+1).$$

Hence

$$3(j\beta+1)/(c-b) \in I[3j\alpha+1, 3j\alpha+2; a].$$

By Lemma 5.1, $(n_{j\beta+1}, m_{j\beta+1})$ is a strongly consistent pair for $j = i_0, i_0 + 1, \cdots, j_0$. Hence $m_{j\beta+1} \not\equiv 1 \pmod 3$ for $j = i_0, i_0+1, \cdots, j_0$. ∎

Lemma 9.5 *If* $p \equiv 1 \pmod 3$, *then* $(n_{(i_0-1)\beta+1} - 1, m_{(i_0-1)\beta+1} - 1)$ *is strongly consistent.*

Proof. Assume $p \equiv 1 \pmod 3$. Then $3i_0 = 3q - 2p + 2$, and hence

$$\begin{aligned}
\frac{(3(i_0-1)\alpha+1)(c-b)}{a} &= 3(i_0-1)\beta + \frac{3(i_0-1)}{q} + \frac{p}{q} \\
&= 3(i_0-1)\beta + 3 - \frac{p+1}{q} \\
&\geq 3((i_0-1)\beta+1) - 1.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\frac{(3(i_0-1)\alpha+2)(c-b)}{a} \\
&\qquad = 3((i_0-1)\beta+1) - \frac{1}{q} \\
&\qquad < 3((i_0-1)\beta+1).
\end{aligned}$$

This implies that

$$\begin{aligned}
& I[3(i_0-1)\alpha+1, 3(i_0-1)\alpha+2; a] \\
\subset\ & I[3((i_0-1)\beta+1)-1, 3((i_0-1)\beta+1); c-b].
\end{aligned}$$

Since $c > 2a$, we have

$$\begin{aligned}
\frac{m_{(i_0-1)\beta+1}-1}{c} &= \frac{3((i_0-1)\beta+1)}{c-b} - \frac{1+\delta_{(i_0-1)\beta+1}}{c} \\
&> \frac{3((i_0-1)\beta+1)}{c-b} - \frac{2}{c} \\
&> \frac{3((i_0-1)\beta+1)}{c-b} - \frac{1}{a} \\
&> \frac{3(i_0-1)\alpha+2}{a} - \frac{1}{a} \\
&= \frac{3(i_0-1)\alpha+1}{a}.
\end{aligned}$$

Since $(c-b)/a = p/q$, $p \geq 4$ and $c < 4a$ (by Lemma 9.1), we have

$$1/c > 1/(4a) = p/(4q(c-b)) \geq 1/(q(c-b)).$$

This implies that

$$\begin{aligned}
\frac{m_{(i_0-1)\beta+1}-1}{c} &\leq \frac{3((i_0-1)\beta+1)}{c-b} - \frac{1}{c} \\
&< \frac{3((i_0-1)\beta+1)}{c-b} - \frac{1}{q(c-b)} \\
&= \frac{3(i_0-1)\alpha+2}{a}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{m_{(i_0-1)\beta+1}-1}{c} &\in I[3(i_0-1)\alpha+1, 3(i_0-1)\alpha+2; a] \\
&\subset I[3((i_0-1)\beta+1)-1, 3((i_0-1)\beta+1); c-b].
\end{aligned}$$

By Lemma 5.1, $(n_{(i_0-1)\beta+1}-1, m_{(i_0-1)\beta+1}-1)$ is strongly consistent. ∎

Lemma 9.6 $m_{i_0\beta+1} \equiv 0 \pmod 3$.

Proof. We consider three cases.

Case 1 $p \equiv 0 \pmod 3$.

In this case, $3i_0 = 3q - 2p$. It follows that

$$\begin{aligned} \frac{(3i_0\alpha + 2)(c-b)}{a} &= \frac{(3i_0\alpha+2)p}{q} \\ &= \frac{3i_0\beta q + 3i_0 + 2p}{q} \\ &= 3(i_0\beta + 1). \end{aligned}$$

Therefore

$$\frac{(3i_0\alpha+2)}{a} = \frac{3(i_0\beta+1)}{c-b}.$$

Since $1/a < 1/(c-b)$, it follows that

$$I[3i_0\alpha+1, 3i_0\alpha+2; a] \subset I[3(i_0\beta+1)-1, 3(i_0\beta+1); c-b].$$

Recall that $2/c < 1/a$. Therefore

$$\begin{aligned} \frac{m_{i_0\beta+1}-1}{c} &= \frac{3(i_0\beta+1)}{c-b} - \frac{1+\delta_{i_0\beta+1}}{c} \\ &\geq \frac{3(i_0\beta+1)}{c-b} - \frac{2}{c} \\ &\geq \frac{3(i_0\beta+1)}{c-b} - \frac{1}{a} \\ &= \frac{(3i_0\alpha+1)}{a} \end{aligned}$$

Since

$$\frac{m_{i_0\beta+1}-1}{c} \leq \frac{3(i_0\beta+1)}{c-b} = \frac{(3i_0\alpha+2)}{a},$$

we conclude that

$$\frac{m_{i_0\beta+1}-1}{c} \in I[3i_0\alpha+1, 3i_0\alpha+2; a] \cap I[3i_0\beta+2, 3i_0\beta+3; c-b].$$

By Lemma 5.1, $(n_{i_0\beta+1}-1, m_{i_0\beta+1}-1)$ is strongly consistent. Therefore

$$m_{i_0\beta+1} - 1 \not\equiv 1 \pmod 3.$$

By Lemma 9.4, $m_{i_0\beta+1} \not\equiv 1 \pmod 3$. Therefore

$$m_{i_0\beta+1} \equiv 0 \pmod 3.$$

Case 2 $p \equiv 1 \pmod 3$.

In this case, $3i_0 = 3q - 2p + 2$. It follows that

$$\begin{aligned} \frac{(3i_0\alpha + 1)(c-b)}{a} &= \frac{(3i_0\alpha+1)p}{q} \\ &= \frac{3i_0\beta q + 3i_0 + p}{q} \\ &= 3(i_0\beta + 1) - (p-2)/q. \end{aligned}$$

Assume to the contrary that $m_{i_0\beta+1} \not\equiv 0 \pmod 3$. Then by Lemma 9.4, $m_{i_0\beta+1} \equiv 2 \pmod 3$. By Corollary 5.1,

$$\delta_{i_0\beta+1} > \frac{(p-2)b}{q(c-b)}.$$

As $1/a = p/(q(c-b))$, we conclude that

$$\delta_{i_0\beta+1} > \frac{p-2}{q}\frac{b}{c-b} > \frac{p-2}{q}\frac{a}{c-b} = (p-2)/p.$$

By Corollary 7.1, $\delta_{i_0\beta+1} = s/p$ for some integer s, hence we must have

$$\delta_{i_0\beta+1} = (p-1)/p.$$

Now $(p-1)/p = \delta_{i_0\beta+1} > (p-2)b/(pa)$ implies that

$$(p-1)a > (p-2)b.$$

As $p \geq 4$, we have $3a > 2b$, hence

$$4/c < 2/a = 2p/(q(c-b)) < 3/b.$$

This implies that

$$q < 2p.$$

Indeed, if $q \geq 2p$, then

$$4/c < 2p/(q(c-b)) \leq 1/(c-b),$$

which implies that $3/b < 4/c$, contrary to the above.

By Lemma 9.5, $(n_{(i_0-1)\beta+1} - 1, m_{(i_0-1)\beta+1} - 1)$ is strongly consistent. Now

$$m_{(i_0-1)\beta+1} = m_{i_0\beta+1} - \mu + \lfloor \delta_{i_0\beta+1} - \epsilon \rfloor.$$

Since $\epsilon \leq (p-1)/p$ and $\delta_{i_0\beta+1} = (p-1)/p$, we conclude that

$$m_{(i_0-1)\beta+1} \equiv m_{i_0\beta+1} - \mu \pmod 3.$$

Since $m_{(i_0-1)\beta+1} \not\equiv 2 \pmod 3$ and $m_{i_0\beta+1} \equiv 2 \pmod 3$, we conclude that $\mu \not\equiv 0 \pmod 3$.

Thus either $\mu \equiv 1 \pmod 3$ or $\mu \equiv 2 \pmod 3$. Assume first that $\mu \equiv 1 \pmod 3$. Since $\delta_{i_0\beta+1} = (p-1)/p$, we conclude that $\delta \neq 0$, and hence $\epsilon \neq 0$ (as $q\epsilon + \delta$ is an integer). Because $p\epsilon$ is an integer, $\epsilon \geq 1/p$, and hence

$$\lfloor \epsilon + \delta_{i_0\beta+1} \rfloor = 1.$$

This implies that

$$\begin{aligned} m_{(i_0+1)\beta+1} &= m_{i_0\beta+1} + u + \lfloor \delta_{i_0\beta+1} + \epsilon \rfloor \\ &\equiv 1 \pmod 3. \end{aligned}$$

By applying Lemma 9.4, we conclude that $j_0 = i_0$, i.e., $p = 4$. Since $p < q < 2p$, and $\gcd(p,q) = 1$, the possible values for q are 5 and 7.

If $q = 7$ then $\alpha = 2$, $\beta = 1$ and $i_0 = q - 2p/3 + 2/3 = 5$. Hence $\mu + \epsilon = m + \delta$, i.e., $\mu = m, \epsilon = \delta$. Moreover

$$\begin{aligned} m_{i_0\beta+1} + \delta_{i_0\beta+1} &= i_0(\mu + \epsilon) + m + \delta \\ &= 6(m + \delta). \end{aligned}$$

Since $m_{i_0\beta+1} \equiv 2 \pmod 3$ and $\delta_{i_0\beta+1} = (p-1)/p = 3/4$, we conclude that $6\delta = 2 + 3/4 = 11/4$. However, by Lemma 7.1, 4δ is an integer, which is an obvious contradiction.

If $q = 5$ then $\alpha = 4$, $\beta = 3$ and $i_0 = q - 2p/3 + 2/3 = 3$. Since

$$\beta(m + \delta) = 3(m + \delta) = \mu + \epsilon$$

and $\mu \equiv 1 \pmod 3$, we conclude that $3\delta = 1 + \epsilon$. Therefore $\delta \neq 0$, which implies that $\epsilon \neq 0$ (because $q\epsilon + \delta$ is an integer.) As 4ϵ is an integer, we know that $\epsilon \geq 1/4$. Because 4δ is an integer, we know that $\delta = s/4$ for some integer s. Now $3\delta = 1 + \epsilon$ implies that $\delta = \epsilon = 1/2$. But $i_0 = 3$ implies that $\delta_{i_0\beta+1} = \{3\epsilon + \delta\} = 0$, contrary to the previous conclusion that $\delta_{i_0\beta+1} = (p-1)/p = 3/4$.

Assume next that $\mu \equiv 2 \pmod 3$. If $p = 4$, then $q = 5$ or 7. If $q = 7$, we obtain the same contradiction as above. If $q = 5$ then $\alpha = 4$, $\beta = 3$ and $i_0 = 3$. Since

$$\beta(m + \delta) = 3(m + \delta) = \mu + \epsilon$$

and $\mu \equiv 2 \pmod 3$, we conclude that $3\delta = 2 + \epsilon$. Since $\epsilon \geq 1/4$ and 4δ is an integer, it follows that $\delta = 3/4$ and $\epsilon = 1/4$. But $i_0 = 3$ implies that $\delta_{i_0\beta+1} = \{3\epsilon + \delta\} = 1/2$, contrary to the previous conclusion that $\delta_{i_0\beta+1} = (p-1)/p = 3/4$.

Thus we may assume that $p \geq 7$, and hence $j_0 - i_0 \geq 1$. For $i_0 \leq j \leq j_0$, we have

$$\begin{aligned} m_{j\beta+1} &= m_{i_0\beta+1} + \lfloor (j - i_0)(\mu + \epsilon) + \delta_{i_0\beta+1} \rfloor \\ &\equiv 2 + 2(j - i_0) + \lfloor (j - i_0)\epsilon + \delta_{i_0\beta+1} \rfloor \pmod 3 \end{aligned}$$

If

$$(j_0 - i_0)\epsilon + \delta_{i_0\beta+1} < j_0 - i_0,$$

then let j^* be the least integer such that

$$(j^* - i_0)\epsilon + \delta_{i_0\beta+1} < j^* - i_0.$$

Then $i_0 + 1 \leq j^* \leq j_0$, and by the minimality of j^*, we have $\lfloor (j^* - i_0)\epsilon + \delta_{i_0\beta+1} \rfloor = j^* - i_0 - 1$. It follows that $m_{j^*\beta+1} \equiv 1 \pmod 3$, contrary to Lemma 9.4. Therefore we have

$$(j_0 - i_0)\epsilon + \delta_{i_0\beta+1} \geq j_0 - i_0,$$

and hence $m_{j_0\beta+1} \equiv 2 \pmod 3$. It follows from above that

$$\begin{aligned} \epsilon &\geq 1 - \delta_{i_0\beta+1}/(j_0 - i_0) \\ &= 1 - (p-1)/(p(j_0 - i_0)). \end{aligned}$$

Since $j_0 - i_0 = (p-4)/3$, it follows that

$$p\epsilon \geq p - 3(p-1)/(p-4).$$

Since $p \geq 7$, we conclude that $p\epsilon \geq p - 6$. Because $p(\mu + \epsilon) \equiv 0 \pmod 3$, we conclude that $p\epsilon = p-3$ or $p-6$. If $p\epsilon = p-6$, then $3(p-1)/(p-4) \geq 6$, and it follows that $p = 7$ and $\epsilon = 1/7$. Hence $j_0 - i_0 = 1$ and

$$\delta_{j_0\beta+1} = \{\delta_{i_0\beta+1} + \epsilon\} = \{6/7 + 1/7\} = 0.$$

Since $m_{j_0\beta+1} \equiv 2 \pmod 3$, and (by the proof of Lemma 9.4)

$$3(j_0\beta + 1)/(c-b) \in I[3j_0\alpha + 1, 3j_0\alpha + 2; a],$$

this is in contrary to Corollary 5.1.

If $p\epsilon = p - 3$, then $\delta = \{3q/p\}$ (because $q\epsilon + \delta$ is an integer) and hence

$$\begin{aligned} \delta_{i_0\beta+1} &= \{i_0\epsilon + \delta\} \\ &= \{\frac{3q-2p+2}{3}(1 - \frac{3}{p}) + \frac{3q}{p}\} \\ &= \frac{p-2}{p}, \end{aligned}$$

contrary to the previous conclusion that $\delta_{i_0\beta+1} = (p-1)/p$.

Case 3 $p \equiv 2 \pmod 3$.

Assume to the contrary that $m_{i_0\beta+1} \equiv 2 \pmod 3$. As $p \equiv 2 \pmod 3$, we have $3i_0 = 3q - 2p + 1$. Therefore

$$\frac{(3i_0\alpha + 1)(c-b)}{a} = \frac{(3i_0\alpha + 1)p}{q}$$

$$
\begin{aligned}
&= \frac{3i_0\beta q + 3i_0 + p}{q} \\
&= \frac{3i_0\beta q + 3q - p + 1}{q} \\
&= 3(i_0\beta + 1) - \frac{p-1}{q}.
\end{aligned}
$$

Because (by the proof of Lemma 9.4)

$$3(i_0\beta + 1)/(c - b) \in I[3i_0\alpha + 1, 3i_0\alpha + 2; a],$$

by Corollary 5.1,

$$\delta_{i_0\beta+1}(c - b)/b > (p - 1)/q.$$

As $1/a = p/(q(c - b))$, we conclude that

$$
\begin{aligned}
\delta_{i_0\beta+1} &> \frac{p-1}{q}\frac{b}{c-b} \\
&> \frac{p-1}{q}\frac{q}{p} \\
&= \frac{p-1}{p}.
\end{aligned}
$$

However, by Corollary 7.1, $\delta_{i_0\beta+1} = s/p$ for some integer s, which is a contradiction. ∎

Lemma 9.7 *If* $p \equiv 0 \pmod 3$, *then* $m_{j_0\beta+1} \equiv 2 \pmod 3$.
If $p \equiv 2 \pmod 3$ *and* $m_{j_0\beta+1} \equiv 0 \pmod 3$, *then* $\delta_{j_0\beta+1} = 0$.
If $p \equiv 1 \pmod 3$ *and* $m_{j_0\beta+1} \equiv 0 \pmod 3$, *then either* $\delta_{j_0\beta+1} = 0$, *or*

$$\delta_{j_0\beta+1} = 1/p,\ 2p > q \text{ and } (p - 1)a > (p - 2)b.$$

Proof. If $p \equiv 0 \pmod 3$, then $3j_0 = 3q - p$, and it follows that

$$(3j_0\alpha + 2)(c - b)/a = 3(j_0\beta + 1) + p/q.$$

If $m_{j_0\beta+1} \equiv 0 \pmod 3$, then by Corollary 5.1,

$$1 - \delta_{j_0\beta+1} > \frac{pb}{q(c-b)}.$$

As $b/(c - b) > a/(c - b) = q/p$, we conclude that

$$1 - \delta_{j_0\beta+1} > \frac{p}{q}\frac{b}{c-b} > \frac{p}{q}\frac{q}{p} = 1,$$

which is an obvious contradiction.

If $p \equiv 2 \pmod 3$, then $3j_0 = 3q - p - 1$, and it follows that

$$\begin{aligned}\frac{(3j_0\alpha+2)(c-b)}{a} &= \frac{(3j_0\alpha+2)p}{q} \\ &= \frac{3j_0\beta q + 3j_0 + 2p}{q} \\ &= 3(j_0\beta+1) + (p-1)/q.\end{aligned}$$

If $m_{j_0\beta+1} \equiv 0 \pmod 3$, then by Corollary 5.1,

$$1 - \delta_{j_0\beta+1} > \frac{(p-1)b}{q(c-b)}.$$

As $1/a = p/(q(c-b))$, we conclude that

$$\begin{aligned}1 - \delta_{j_0\beta+1} &> \frac{p-1}{q}\frac{b}{c-b} = \frac{p-1}{q}\frac{b}{a}\frac{a}{c-b} \\ &> \frac{p-1}{q}\frac{a}{c-b} = \frac{p-1}{p}.\end{aligned}$$

As $p\delta_j$ is an integer for all j, we conclude that

$$\delta_{j_0\beta+1} = 0.$$

If $p \equiv 1 \pmod 3$, then $3j_0 = 3q - p - 2$. Therefore

$$(3j_0\alpha+2)(c-b)/a = 3(j_0\beta+1) + (p-2)/q.$$

If $m_{j_0\beta+1} \equiv 0 \pmod 3$, then by Corollary 5.1,

$$1 - \delta_{j_0\beta+1} > \frac{(p-2)b}{q(c-b)}.$$

As $1/a = p/(q(c-b))$, we conclude that

$$1 - \delta_{j_0\beta+1} > \frac{p-2}{q}\frac{b}{c-b} > \frac{p-2}{q}\frac{a}{c-b} = (p-2)/p.$$

Hence $\delta_{j_0\beta+1} < 2/p$. By Corollary 7.1, $\delta_{j_0\beta+1} = s/p$ for some integer s, hence either

$$\delta_{j_0\beta+1} = 0,$$

or

$$\delta_{j_0\beta+1} = 1/p.$$

If $\delta_{j_0\beta+1} = 1/p$, then $(p-1)/p = 1 - \delta_{i_0\beta+1} > (p-2)b/(pa)$ implies that $(p-1)a > (p-2)b$. This implies that $q < 2p$. Indeed, if $q \geq 2p$ then since $2/c < 1/a$, we conclude that

$$4/c < 2/a = 2p/(q(c-b)) \leq 1/(c-b).$$

It is easy to see that $4/c < 1/(c-b)$ implies that $3/b < 4/c$. Hence $3/b < 2/a$, which implies that $2b > 3a$, contrary to the previous conclusion that $(p-1)a > (p-2)b$ (as $p \geq 4$). This completes the proof of Lemma 9.7. ∎

Lemma 9.8 $p \neq 4$.

Proof. Suppose $p = 4$, then $i_0 = j_0 = q - 2$. By Lemma 9.6, $m_{i_0\beta+1} = m_{j_0\beta+1} \equiv 0 \pmod 3$. By Lemma 9.7, either $\delta_{j_0\beta+1} = 0$ or

$$\delta_{j_0\beta+1} = 1/p,\ 2p > q \text{ and } (p-1)a > (p-2)b.$$

Assume first that $\delta_{i_0\beta+1} = 1/p = 1/4$. Then $p < q < 2p$. Therefore either $q = 5$ or $q = 7$.

If $q = 5$ then $\alpha = 4$, $\beta = 3$ and $i_0 = 3$. Hence $i_0\beta + 1 = 10$, and $1/4 = \delta_{i_0\beta+1} = \{10\delta\}$. This is impossible, because by Lemma 7.1, $\delta = s/4$ for some integer s.

If $q = 7$ then $\alpha = 2$, $\beta = 1$ and $i_0 = 5$. Hence $i_0\beta + 1 = 6$, and we obtain the same contradiction.

It remains to consider the case that $\delta_{i_0\beta+1} = 0$. Because $\gcd(p, q) = 1$, we know that either $q = 4s + 1$ or $q = 4s + 3$ for some integer $s \geq 1$.

Note that

$$\frac{(3i_0\alpha + 1.5)(c - b)}{a} = 3(i_0\beta + 1).$$

As $(c - b)/a = 4/q$, we have

$$\frac{3i_0\alpha + 1}{a} = \frac{3(i_0\beta + 1)}{c - b} - \frac{2}{q(c - b)},$$

and

$$\frac{3i_0\alpha + 2}{a} = \frac{3(i_0\beta + 1)}{c - b} + \frac{2}{q(c - b)}.$$

First we show that $2a > b$. Indeed, if to the contrary that $2a \leq b$, then because $\delta_{i_0\beta+1} = 0$, we have

$$\begin{aligned}
\frac{n_{i_0\beta+1} + 1}{b} &= \frac{3(i_0\beta + 1)}{c - b} + \frac{1}{b} \\
&\leq \frac{3(i_0\beta + 1)}{c - b} + \frac{1}{2a} \\
&= \frac{3(i_0\beta + 1)}{c - b} + \frac{2}{q(c - b)} \\
&= \frac{3i_0\alpha + 2}{a}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{n_{i_0\beta+1} + 1}{b} &\in I[3(i_0\beta + 1), 3(i_0\beta + 1) + 1; c - b] \\
&\cap\ I[3i_0\alpha + 1, 3i_0\alpha + 2; a].
\end{aligned}$$

Hence by Lemma 5.1, $(n_{i_0\beta+1}+1, m_{i_0\beta+1}+1)$ is strongly consistent. As $m_{i_0\beta+1}+1 \equiv 1 \pmod 3$, this is in contrary to the general assumption.

If $q = 4s+1$, then $\alpha = 3s+1$, $\beta = 3$ and $i_0 = 4s-1$. If $q = 4s+3$, then $\alpha = s+1$, $\beta = 1$ and $i_0 = 4s+1$. In any case,

$$i_0\beta + 1 \equiv 2 \pmod 4.$$

Since $\delta_{i_0\beta+1} = \{(i_0\beta+1)\delta\} = 0$, it follows that $\delta \neq 3/4, 1/4$. Since 4δ is an integer, it follows that the possible values for δ are 0 and 1/2.

First we consider the case that $\delta = 0$. Then $\delta_j = 0$ for all j, and $\epsilon = 0$. It is straightforward to verify that

$$\begin{aligned} \frac{3(i_0+1)\alpha+1}{a} &= \frac{3((i_0+1)\beta+1)}{c-b} + \frac{1}{q(c-b)}, \\ \frac{3(i_0+1)\alpha+2}{a} &= \frac{3((i_0+1)\beta+1)}{c-b} + \frac{5}{q(c-b)}. \end{aligned}$$

As $q \geq 5$, it follows that

$$\begin{aligned} & I[3(i_0+1)\alpha+1, 3(i_0+1)\alpha+2; a] \\ \subset\ & I[3((i_0+1)\beta+1), 3((i_0+1)\beta+1)+1; c-b]. \end{aligned}$$

By lemma 9.1, $2a < c < 3a$. Recall that $m_{(i_0+1)\beta+1} + \delta_{(i_0+1)\beta+1} = m_{i_0\beta+1} + \delta_{i_0\beta+1} + \mu + \epsilon$. Therefore

$$m_{(i_0+1)\beta+1} \equiv \mu \pmod 3,$$

and

$$\delta_{(i_0+1)\beta+1} = 0.$$

By using the facts that $\delta_{(i_0+1)\beta+1} = 0$, $2a < c < 3a$ and $a < b < 2a$, it is easy to verify that

$$\frac{m_{(i_0+1)\beta+1}+1}{c} \in I[3(i_0+1)\alpha+1, 3(i_0+1)\alpha+2; a],$$

and

$$\frac{n_{(i_0+1)\beta+1}+1}{b} \in I[3(i_0+1)\alpha+1, 3(i_0+1)\alpha+2; a].$$

By Lemma 5.1, $(n_{(i_0+1)\beta+1}, m_{(i_0+1)\beta+1})$ is strongly consistent, and $(n_{(i_0+1)\beta+1}+1, m_{(i_0+1)\beta+1}+1)$ is strongly consistent. As $n_{(i_0+1)\beta+1} \equiv \mu \pmod 3$, by the general assumption, we have $\mu \equiv 2 \pmod 3$.

If $2/b \leq 5/(q(c-b))$, then it is easy to verify that

$$\frac{n_{(i_0+1)\beta+1}+2}{b} \in I[3(i_0+1)\alpha+1, 3(i_0+1)\alpha+2; a],$$

which implies that $(n_{(i_0+1)\beta+1}+2, m_{(i_0+1)\beta+1}+2)$ is strongly consistent, contrary to the general assumption. Thus we may assume that $2/b > 5/(q(c-b))$, which implies that $8a > 5b$.

If $q \geq 11$, then since

$$\frac{1}{c-b} = \frac{q}{4a} \geq \frac{11}{4a} > \frac{11}{2c},$$

we conclude that $11b > 9c > 18a$, contrary to the previous conclusion that $8a > 5b$. Therefore we have $5 \leq q \leq 10$. As $\gcd(q,4) = 1$, the possible values for q are $5, 7$ and 9.

If $q = 5$ or $q = 9$, then $\beta = 3$, and $\mu = \lfloor 3(m+\delta) \rfloor \equiv 2 \pmod 3$, which is an obvious contradiction (as $\delta = 0$). If $q = 7$, then $\beta = 1$, hence $\mu = m \equiv n \equiv 2 \pmod 3$. As $a < b < 14(c-b)/5$, we have $21/4 < n + \delta = 3b/(c-b) < 42/5$. As $n \equiv 2 \pmod 3$ and $\delta = 0$, so $n = 8$, i.e., $3b/(c-b) = 8$. It follows that

$$\begin{aligned}
\frac{m_{(i_0+2)\beta+1}-2}{b} &= \frac{3((i_0+2)\beta+1)}{c-b} - \frac{2}{b} \\
&= \frac{3((i_0+2)\beta+1)}{c-b} - \frac{3}{4(c-b)}.
\end{aligned}$$

It is straightforward to verify that

$$\begin{aligned}
\frac{3(i_0+2)\alpha-2}{a} &= \frac{3(i_0+2)\beta+1)}{c-b} - \frac{8}{7(c-b)}, \\
\frac{3(i_0+2)\alpha-1}{a} &= \frac{3((i_0+2)\beta+1)}{c-b} - \frac{4}{7(c-b)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{m_{(i_0+2)\beta+1}-2}{b} &\in I[3(i_0+2)\alpha-2, 3(i_0+2)\alpha-1; a] \\
&\cap I[3((i_0+2)\beta+1)-1, 3((i_0+2)\beta+1); c-b].
\end{aligned}$$

By Lemma 5.1, $(n_{(i_0+2)\beta+1}-3, m_{(i_0+2)\beta+1}-3)$ is strongly consistent. But $m_{(i_0+2)\beta+1} \equiv 2\mu \equiv 1 \pmod 3$, which is in contrary to the general assumption.

Now we consider the case that $\delta = 1/2$.

If $q = 4s+1$, then $\alpha = 3s+1$, $\beta = 3$ and $i_0 = 4s-1$, and

$$\begin{aligned}
i_0\beta + 1 &\equiv 1 \pmod 3, \\
\frac{1}{2}(i_0\beta+1) &\equiv 2 \pmod 3.
\end{aligned}$$

Because

$$\begin{aligned} m_{i_0\beta+1} &= \lfloor (i_0\beta+1)(m+1/2) \rfloor \\ &= m(i_0\beta+1) + \frac{1}{2}(i_0\beta+1) \\ &\equiv m+2 \\ &\equiv 0 \pmod 3, \end{aligned}$$

we conclude that $n \equiv m \equiv 1 \pmod 3$.

As $2a > b$, we have $3b/(c-b) = n + 1/2 < 6a/(c-b) = 6s + 1 + 1/2$, it follows that $n \le 6s$. As $n \equiv 1 \pmod 3$, we conclude that $n \le 6s - 2$. So

$$\frac{3b}{c-b} \le 6s - 2 + \frac{1}{2},$$

which implies that

$$\frac{2b}{c-b} \le 4s - 1,$$

and hence

$$\frac{c-b}{c} \ge \frac{2}{4s+1}.$$

As $a/(c-b) = (4s+1)/4$, we conclude that

$$\begin{aligned} \frac{a}{c} &= \frac{a}{c-b}\frac{c-b}{c} \\ &\ge \frac{4s+1}{4}\frac{2}{4s+1} \\ &= \frac{1}{2}. \end{aligned}$$

This is in contrary to the general assumption.

Assume now that $q = 4s + 3$. Then $\alpha = s + 1$, $\beta = 1$ and $i_0 = 4s + 1$. Since $\beta = 1$ and $\delta = 1/2$, we have $\mu = m \equiv n \pmod 3$ and $\epsilon = \delta = 1/2$.

Since $2a > b$, we have

$$\frac{3b}{c-b} = n + 1/2 < \frac{6a}{c-b} = 6s + 4 + \frac{1}{2}.$$

Hence $n \le 6s + 3$. If $n \le 6s + 1$, then

$$3b/(c-b) = n + 1/2 \le 6s + 3/2,$$

and

$$2b/(c-b) = n/3 + 1/6 \le 4s + 1.$$

Therefore $(4s+3)b \leq (4s+1)c$ and $c-b \geq 2c/(4s+3)$. It follows that

$$\frac{a}{c} \geq \frac{a}{c-b}\frac{2}{4s+3} = 1/2.$$

This is in contrary to the general assumption.

Thus we may assume that $6s+2 \leq n \leq 6s+3$. If $n = 6s+2$, then

$$\frac{b}{c-b} = 2s + \frac{5}{6} > \frac{3q}{10}$$

and it follows that

$$\frac{2-1/2}{b} < \frac{5}{q(c-b)}.$$

Therefore

$$\begin{aligned}
\frac{n_{(i_0+1)\beta+1}+2}{b} &= \frac{3((i_0+1)\beta+1)}{c-b} + \frac{2-1/2}{b} \\
&< \frac{3((i_0+1)\beta+1)}{c-b} + \frac{5}{q(c-b)} \\
&= \frac{3(i_0+1)\alpha+2}{a} \\
&\leq \frac{3((i_0+1)\beta+1)+1}{c-b}.
\end{aligned}$$

Because

$$\begin{aligned}
\frac{n_{(i_0+1)\beta+1}+2}{b} &= \frac{3((i_0+1)\beta+1)}{c-b} + \frac{2-1/2}{b} \\
&> \frac{3((i_0+1)\beta+1)}{c-b} + \frac{1}{b} \\
&> \frac{3((i_0+1)\beta+1)}{c-b} + \frac{1}{2a} \\
&= \frac{3((i_0+1)\beta+1)}{c-b} + \frac{2}{q(c-b)} \\
&> \frac{3(i_0+1)\alpha+1}{a},
\end{aligned}$$

we conclude that

$$\begin{aligned}
\frac{n_{(i_0+1)\beta+1}+2}{b} &\in I[3(i_0+1)\alpha+1, 3(i_0+1)\alpha+2; a] \\
&\subset I[3((i_0+1)\beta+1), 3((i_0+1)\beta+1)+1; c-b].
\end{aligned}$$

By Lemma 5.1, $(n_{(i_0+1)\beta+1}+2, m_{(i_0+1)\beta+1}+2)$ is strongly consistent.

However, $\mu \equiv n \equiv 2 \pmod 3$. Hence,

$$\begin{aligned}
m_{(i_0+1)\beta+1} &= \lfloor m_{i_0\beta+1} + \delta_{i_0\beta+1} + \mu + \epsilon \rfloor \\
&\equiv 2 \pmod 3,
\end{aligned}$$

which implies that $m_{(i_0+1)\beta+1} + 2 \equiv 1 \pmod 3$, contrary to the general assumption.

If $n = 6s + 3$, then $\mu \equiv n \equiv 0 \pmod 3$. Because

$$\begin{aligned} \frac{n_{(i_0+1)\beta+1}+1}{b} &= \frac{3((i_0+1)\beta+1)}{c-b} + \frac{1-1/2}{b} \\ &< \frac{3((i_0+1)\beta+1)}{c-b} + \frac{5}{q(c-b)} \\ &= \frac{3(i_0+1)\alpha+2}{a} \\ &\leq \frac{3((i_0+1)\beta+1)+1}{c-b}, \end{aligned}$$

and

$$\begin{aligned} \frac{n_{(i_0+1)\beta+1}+1}{b} &= \frac{3((i_0+1)\beta+1)}{c-b} + \frac{1-1/2}{b} \\ &> \frac{3((i_0+1)\beta+1)}{c-b} + \frac{1}{4a} \\ &= \frac{3((i_0+1)\beta+1)}{c-b} + \frac{1}{q(c-b)} \\ &> \frac{3(i_0+1)\alpha+1}{a}, \end{aligned}$$

we conclude that

$$\begin{aligned} \frac{n_{(i_0+1)\beta+1}+1}{b} &\in I[3(i_0+1)\alpha+1, 3(i_0+1)\alpha+2; a] \\ &\subset I[3((i_0+1)\beta+1), 3((i_0+1)\beta+1)+1; c-b]. \end{aligned}$$

By Lemma 5.1, $(n_{(i_0+1)\beta+1}+1, m_{(i_0+1)\beta+1}+1)$ is strongly consistent.

However, $\mu \equiv n \equiv 0 \pmod 3$. Hence, $m_{(i_0+1)\beta+1} + 1 \equiv \mu + 1 \equiv 1 \pmod 3$, which is in contrary to the general assumption. ∎

Lemma 9.9 $\mu \not\equiv 0 \pmod 3$.

Proof. Assume to the contrary that $\mu \equiv 0 \pmod 3$. For any $j \geq 0$,

$$\begin{aligned} m_{(i_0+j)\beta+1} &= m_{i_0\beta+1} + j\mu + \lfloor j\epsilon + \delta_{i_0\beta+1} \rfloor \\ &\equiv \lfloor j\epsilon + \delta_{i_0\beta+1} \rfloor. \end{aligned}$$

If $(j_0 - i_0)\epsilon + \delta_{i_0\beta+1} \geq 1$, then let j^* be the least integer such that

$$j^*\epsilon + \delta_{i_0\beta+1} \geq 1.$$

Then $1 \leq j^* \leq j_0 - i_0$. By the minimality of j^*, it follows that

$$m_{(i_0+j^*)\beta+1} \equiv 1 \pmod 3,$$

contrary to Lemma 9.4. Therefore we have $(j_0 - i_0)\epsilon + \delta_{i_0\beta+1} < 1$, and hence

$$m_{j_0\beta+1} \equiv 0 \pmod 3,$$

and

$$\delta_{j_0\beta+1} = (j_0 - i_0)\epsilon + \delta_{i_0\beta+1}.$$

By Lemma 9.7, $p \not\equiv 0 \pmod 3$.

If $p \equiv 2 \pmod 3$, then by Lemma 9.7, $\delta_{j_0\beta+1} = 0$. Since $j_0 - i_0 \neq 0$, this implies that $\epsilon = 0$. As $q\epsilon + \delta$ is an integer, it follows that $\delta = 0$ and hence $\delta_j = 0$ for all j.

Now $\mu \equiv 0 \pmod 3$ and $\epsilon = 0$ implies that for all j,

$$m_{j\beta+1} \equiv 0 \pmod 3.$$

In particular,

$$m_{(j_0+1)\beta+1} \equiv 0 \pmod 3.$$

Since $c < 4a$, and $p \geq 5$, it follows that

$$1/c > 1/(4a) = p/(4q(c-b)) > 1/(q(c-b)).$$

As $\delta_{(j_0+1)\beta+1} = 0$, we have

$$\begin{aligned}
\frac{m_{(j_0+1)\beta+1} + 2}{c} &= \frac{3((j_0+1)\beta+1)}{c-b} + \frac{2}{c} \\
&> \frac{3((j_0+1)\beta+1)}{c-b} + \frac{2}{q(c-b)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{(3(j_0+1)\alpha+1)(c-b)}{a} &= \frac{(3(j_0+1)\alpha+1)p}{q} \\
&= \frac{(3(j_0+1)\beta q + 3(j_0+1) + p}{q} \\
&= 3((j_0+1)\beta+1) + \frac{2}{q}.
\end{aligned}$$

As $c > 2a$, we have

$$\begin{aligned}
\frac{3(j_0+1)\alpha+2}{a} &> \frac{3((j_0+1)\beta+1)}{c-b} + \frac{1}{a} \\
&> \frac{3((j_0+1)\beta+1)}{c-b} + \frac{2}{c} \\
&= \frac{m_{(j_0+1)\beta+1} + 2}{c}.
\end{aligned}$$

Therefore

$$\begin{aligned}\frac{m_{(j_0+1)\beta+1}+2}{c} &\in I[3(j_0+1)\alpha+1, 3(j_0+1)\alpha+2; a] \\ &\cap I[3((j_0+1)\beta+1), 3((j_0+1)\beta+1)+1; c-b].\end{aligned}$$

By Lemma 5.1, $(n_{(j_0+1)\beta+1}+1, m_{(j_0+1)\beta+1}+1)$ is strongly consistent. However, $m_{(j_0+1)\beta+1}+1 \equiv 1 \pmod 3$, contrary to our general assumption.

If $p \equiv 1 \pmod 3$, then by Lemma 9.7, either $\delta_{j_0\beta+1} = 0$, or

$$\delta_{j_0\beta+1} = 1/p,\ 2p > q \text{ and } (p-1)a > (p-2)b.$$

Since

$$\begin{aligned} m_{(j_0+1)\beta+1} &\equiv \lfloor \epsilon + \delta_{j_0\beta+1} \rfloor \pmod 3, \\ \delta_{(j_0+1)\beta+1} &= \{\epsilon + \delta_{j_0\beta+1}\} \\ \epsilon &\leq (p-1)/p, \end{aligned}$$

we conclude that either

$$m_{(j_0+1)\beta+1} \equiv 0 \pmod 3,$$

or

$$m_{(j_0+1)\beta+1} \equiv 1 \pmod 3, \text{ and } \delta_{(j_0+1)\beta+1} = 0.$$

Since

$$\begin{aligned} 1/c > 1/(4a) &= p/(4q(c-b)) \geq 1/(q(c-b)), \\ j_0 &= q - p/3 - 2/3, \\ (c-b)/a &= p/q, \\ \alpha p &= \beta q + 1, \end{aligned}$$

we have

$$\begin{aligned} \frac{(3(j_0+1)\alpha+1)(c-b)}{a} &= \frac{(3(j_0+1)\alpha+1)p}{q} \\ &= \frac{(3(j_0+1)\beta q + 3(j_0+1)+p}{q} \\ &= 3((j_0+1)\beta+1) + \frac{1}{q}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{m_{(j_0+1)\beta+1}+2}{c} &= \frac{3((j_0+1)\beta+1)}{c-b} + \frac{2-\delta_{(j_0+1)\beta+1}}{c} \\ &> \frac{3((j_0+1)\beta+1)}{c-b} + \frac{1}{c} \\ &> \frac{3((j_0+1)\beta+1)}{c-b} + \frac{1}{q(c-b)} \\ &= \frac{3(j_0+1)\alpha+1}{a}. \end{aligned}$$

On the other hand, as $2/c < 1/a < 1/(c-b)$, we have

$$\begin{aligned} \frac{m_{(j_0+1)\beta+1}+2}{c} &\leq \frac{3((j_0+1)\beta+1)}{c-b} + 2/c \\ &< \min\{\frac{3(j_0+1)\alpha+2}{a}, \frac{3((j_0+1)\beta+1)+1}{c-b}\}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{m_{(j_0+1)\beta+1}+2}{c} &\in I[3(j_0+1)\alpha+1, 3(j_0+1)\alpha+2; a] \\ &\cap \; I[3((j_0+1)\beta+1), 3((j_0+1)\beta+1)+1; c-b]. \end{aligned}$$

By Lemma 5.1, $(n_{(j_0+1)\beta+1}+1, m_{(j_0+1)\beta+1}+1)$ is strongly consistent. Hence $m_{(j_0+1)\beta+1} \not\equiv 0 \pmod 3$,

Therefore we have $m_{(j_0+1)\beta+1} \equiv \lfloor \epsilon + \delta_{j_0\beta+1} \rfloor \equiv 1 \pmod 3$ and $\delta_{(j_0+1)\beta+1} = 0$.

However, by using the fact that $\delta_{(j_0+1)\beta+1} = 0$, the calculation above shows that

$$\begin{aligned} \frac{m_{(j_0+1)\beta+1}+1}{c} &\in I[3(j_0+1)\alpha+1, 3(j_0+1)\alpha+2; a] \\ &\cap \; I[3((j_0+1)\beta+1), 3((j_0+1)\beta+1)+1; c-b]. \end{aligned}$$

By Lemma 5.1, $(n_{(j_0+1)\beta+1}, m_{(j_0+1)\beta+1})$ is strongly consistent, again contrary to the general assumption. ∎

Lemma 9.10 $\mu \not\equiv 1 \pmod 3$.

Proof. Assume to the contrary that $\mu \equiv 1 \pmod 3$. We consider three cases.

Case 1 $p \equiv 0$. Then $j_0 = q - p/3 > q - 2p/3 = i_0$, hence

$$m_{(i_0+1)\beta+1} \not\equiv 1 \pmod 3.$$

As

$$m_{(i_0+1)\beta+1} \equiv 1 + \lfloor \epsilon + \delta_{i_0\beta+1} \rfloor \pmod 3,$$

we conclude that

$$\begin{aligned} m_{(i_0+1)\beta+1} &\equiv 2 \pmod 3, \\ \epsilon + \delta_{i_0\beta+1} &\geq 1 \\ \delta_{(i_0+1)\beta+1} &= \epsilon + \delta_{i_0\beta+1} - 1. \end{aligned}$$

Since $i_0 = q - 2p/3$, $(c-b)/a = p/q$ and $\alpha p = \beta q + 1$, it is straightforward to verify that

$$(3(i_0+1)\alpha+1)(c-b)/a = 3((i_0+1)\beta+1) - (p-3)/q.$$

By Corollary 5.1, we have

$$\delta_{(i_0+1)\beta+1} > \frac{p-3}{q}\frac{b}{c-b} > \frac{p-3}{q}\frac{a}{c-b} = \frac{p-3}{p}.$$

As $\delta_{(i_0+1)\beta+1} = \epsilon + \delta_{i_0\beta+1} - 1$, we conclude that $\epsilon > (p-3)/p$.

If $p \geq 6$ then $j_0 - i_0 = p/3 \geq 2$. Hence $m_{(i_0+2)\beta+1} \not\equiv 1 \pmod 3$. However

$$\begin{aligned} m_{(i_0+2)\beta+1} &\equiv m_{(i_0+1)\beta+1} + \mu + \lfloor \delta_{(i_0+1)\beta+1} + \epsilon \rfloor \\ &\equiv \lfloor \delta_{(i_0+1)\beta+1} + \epsilon \rfloor \pmod 3. \end{aligned}$$

Since $\delta_{(i_0+1)\beta+1} > (p-3)/p \geq 1/2$ and $\epsilon > (p-3)/p \geq 1/2$, we have $m_{(i_0+2)\beta+1} \equiv 1 \pmod 3$, which is in contrary to the general assumption.

If $p = 3$ then by Lemma 7.1, we have $3(m+\delta) \equiv 0 \pmod 3$, which implies that $3\delta \equiv 0 \pmod 3$, hence $\delta = 0$. Then $\delta_{i_0\beta+1} = \epsilon = 0$, contrary to our previous conclusion that $\epsilon + \delta_{i_0\beta+1} \geq 1$.

Case 2 $p \equiv 1 \pmod 3$. By Lemma 9.5, $(n_{(i_0-1)\beta+1} - 1, m_{(i_0-1)\beta+1} - 1)$ is strongly consistent. Hence

$$m_{(i_0-1)\beta+1} \not\equiv 2 \pmod 3.$$

Note that

$$\begin{aligned} m_{(i_0-1)\beta+1} &= m_{i_0\beta+1} - \mu + \lfloor \delta_{i_0\beta+1} - \epsilon \rfloor \\ &\equiv 2 + \lfloor \delta_{i_0\beta+1} - \epsilon \rfloor \pmod 3. \end{aligned}$$

Thus we conclude that

$$m_{(i_0-1)\beta+1} \equiv 1 \pmod 3, \ \epsilon > \delta_{i_0\beta+1} \text{ and } \delta_{(i_0-1)\beta+1} = 1 + \delta_{i_0\beta+1} - \epsilon.$$

Since $i_0 = q - 2p/3 + 2/3$, it is straightforward to verify that

$$(3(i_0-1)\alpha + 2)(c-b)/a = 3((i_0-1)\beta + 1) - 1/q.$$

If $\delta_{(i_0-1)\beta+1}/c \geq 1/(q(c-b))$ then we would have

$$\begin{aligned} \frac{m_{(i_0-1)\beta+1}}{c} &= \frac{3((i_0-1)\beta+1)}{c-b} - \frac{\delta_{(i_0-1)\beta+1}}{c} \\ &\in I[3(i_0-1)\alpha+1, 3(i_0-1)\alpha+2; a] \\ &\cap I[3((i_0-1)\beta+1)-1, 3((i_0-1)\beta+1); c-b]. \end{aligned}$$

which implies that $(n_{(i_0-1)\beta+1}, m_{(i_0-1)\beta+1})$ is strongly consistent, contrary to the general assumption. Therefore we may assume that

$$\delta_{(i_0-1)\beta+1} = 1 + \delta_{i_0\beta+1} - \epsilon < c/(q(c-b)).$$

Since $3(i_0 - 1) = 3q - 2p - 1$, we have

$$\begin{aligned}\frac{(3(i_0-1)\alpha+1)(c-b)}{a} &= 3((i_0-1)\beta+1) - \frac{p+1}{q} \\ &\geq 3((i_0-1)\beta+1) - 1\end{aligned}$$

and

$$\begin{aligned}\frac{(3(i_0-1)\alpha+2)(c-b)}{a} &= 3((i_0-1)\beta+1) - \frac{1}{q} \\ &< 3((i_0-1)\beta+1).\end{aligned}$$

Therefore

$$I[3(i_0-1)\alpha+1, 3(i_0-1)\alpha+2; a] \subset I[3((i_0-1)\beta+1)-1, 3((i_0-1)\beta+1); c-b].$$

By Lemma 9.1, $c < 3a$. Hence $c/(q(c-b)) < 3a/(q(c-b)) = 3/p$. Therefore

$$\epsilon - \delta_{i_0\beta+1} > 1 - 3/p.$$

Since $p\delta_{i_0\beta+1}$, $p\epsilon$ are integers, we conclude that $\epsilon - \delta_{i_0\beta+1} \geq 1-2/p$. Therefore $\delta_{i_0\beta+1} = 0$ or $1/p$, and $\epsilon = 1 - 1/p$ or $1 - 2/p$.

If $\delta_{i_0\beta+1} = 1/p$, then $\epsilon = 1 - 1/p$, which implies that $m_{(i_0+1)\beta+1} \equiv 2 \pmod 3$ and $\delta_{(i_0+1)\beta+1} = 0$. By Lemma 9.8, $p \neq 4$, hence $i_0 + 1 \leq j_0$. By the proof of Lemma 9.4,

$$(3(i_0+1)\beta+1)/(c-b) \in I[3(i_0+1)\alpha+1, 3(i_0+1)\alpha+2; a].$$

This is in contrary to Corollary 5.1.

If $\delta_{i_0\beta+1} = 0$, then $m_{(i_0+1)\beta+1} \equiv 1(\mathrm{mod} 3)$. This is in contrary to Lemma 9.4 (because $p \neq 4$, by Lemma 9.8).

Case 3 $p \equiv 2 \pmod 3$. Then $p \geq 5$, and $j_0 - i_0 \geq 1$. By Lemma 9.4,

$$m_{(i_0+1)\beta+1} \not\equiv 1 \pmod 3.$$

Since

$$m_{(i_0+1)\beta+1} \equiv \lfloor 1 + \delta_{i_0\beta+1} + \epsilon \rfloor \pmod 3,$$

it follows that

$$\delta_{i_0\beta+1} + \epsilon \geq 1,\ m_{(i_0+1)\beta+1} \equiv 2 \pmod 3,\ \text{and}\ \delta_{(i_0+1)\beta+1} = \delta_{i_0\beta+1} + \epsilon - 1.$$

Straightforward calculation shows that

$$(3(i_0+1)\alpha+1)(c-b)/a = 3((i_0+1)\beta+1) - (p-4)/q.$$

By applying Corollary 5.1, we conclude that

$$\delta_{(i_0+1)\beta+1}(c-b)/b > (p-4)/q.$$

Therefore

$$\begin{aligned}\delta_{(i_0+1)\beta+1} &> \frac{p-4}{q}\frac{b}{c-b} > \frac{p-4}{q}\frac{a}{c-b} \\ &= \frac{p-4}{p}.\end{aligned}$$

This implies that $\epsilon > (p-4)/p$.

Assume that $p \neq 5$. Then $p \geq 8$ and $j_0 \geq i_0+2$. Therefore $m_{(i_0+2)\beta+1} \not\equiv 1 \pmod 3$. On the other hand,

$$\begin{aligned}m_{(i_0+2)\beta+1} &\equiv m_{(i_0+1)\beta+1} + \mu + \lfloor \epsilon + \delta_{(i_0+1)\beta+1} \rfloor \\ &\equiv \lfloor \epsilon + \delta_{(i_0+1)\beta+1} \rfloor \pmod 3.\end{aligned}$$

As $m_{(i_0+1)\beta+1} \equiv 2 \pmod 3$, $\mu \equiv 1 \pmod 3$, $\epsilon > (p-4)/p \geq 1/2$ and $\delta_{(i_0+1)\beta+1} > (p-4)/p \geq 1/2$, we conclude that $m_{(i_0+2)\beta+1} \equiv 1 \pmod 3$, a contradiction.

Now we consider the case that $p = 5$. This turns out to be a very subtle case.

Since $i_0 = q - 2p/3 + 1/3$, $(c-b)/a = p/q$ and $\alpha p = \beta q + 1$, it is straightforward to verify that

$$(3i_0\alpha + 2)(c-b)/a = 3(i_0\beta + 1) + 1/q.$$

Since (by Lemma 9.6) $m_{i_0\beta+1} \equiv 0 \pmod 3$, it follows from Corollary 5.1 that

$$1 - \delta_{i_0\beta+1} > \frac{1}{q}\frac{b}{c-b}.$$

Since $b/(c-b) > a/(c-b) = q/p$, we have

$$1 - \delta_{i_0\beta+1} > 1/p.$$

By Lemma 7.1, $p\delta_j$ is an integer for all j. Hence

$$1 - \delta_{i_0\beta+1} \geq 2/p = 2/5.$$

Therefore $\delta_{i_0\beta+1} \leq 3/5$.

On the other hand,

$$m_{(i_0+1)\beta+1} = m_{i_0\beta+1} + \mu + \lfloor \delta_{i_0\beta+1} + \epsilon \rfloor.$$

Since $j_0 - i_0 \geq 1$, by Lemma 9.4, $m_{(i_0+1)\beta+1} \not\equiv 1 \pmod 3$. It follows that

$$m_{(i_0+1)\beta+1} \equiv 2 \pmod 3,$$

and

$$\delta_{(i_0+1)\beta+1} = \delta_{i_0\beta+1} + \epsilon - 1.$$

It is straightforward to verify that

$$(3(i_0+1)\alpha+1)(c-b)/a = 3((i_0+1)\beta+1) - 1/q.$$

By Corollary 5.1,

$$\delta_{(i_0+1)\beta+1} > \frac{1}{q}\frac{b}{c-b} > \frac{1}{p}.$$

Therefore

$$\delta_{(i_0+1)\beta+1} > 1/5.$$

Hence

$$\delta_{(i_0+1)\beta+1} = \delta_{i_0\beta+1} + \epsilon - 1 \geq 2/5.$$

This implies that

$$\delta_{i_0\beta+1} = 3/5,\ \delta_{(i_0+1)\beta+1} = 2/5,\ \text{and } \epsilon = 4/5.$$

Since

$$\frac{2}{5} = 1 - \delta_{i_0\beta+1} > \frac{1}{q}\frac{b}{c-b} = b/5a,$$

we have $2a > b$.

Since $q\epsilon + \delta$ is an integer and $\epsilon = 4/5$, we conclude that $\delta = \{q/5\}$. Suppose $q \equiv i(\text{mod}5)$, where $1 \leq i \leq 4$ (since $\gcd(p,q) = 1$ we know that $i \neq 0$). Suppose $q = 5k+i$. Then $\delta = \{q/5\} = i/5$. By Lemma 7.1, $p(m+\delta) \equiv 0 \pmod 3$. Therefore $5m+i \equiv 0 \pmod 3$, which implies that $m \equiv i \pmod 3$.

Suppose $m = 3s+i$. Then

$$3c/(c-b) = 3s+i+\delta = 3s+i+i/5.$$

This implies that

$$c/(c-b) = s+2i/5,\ \text{and } b/(c-b) = s-1+2i/5.$$

Recall that

$$a/(c-b) = q/p = (5k+i)/5 = k+i/5.$$

If $s \leq 2k$ then $c \leq 2a$, contrary to our assumption. If $s \geq 2k+1$ then $b \geq 2a$, contrary to our previous conclusion. ∎

Lemma 9.11 $\mu \not\equiv 2$.

Proof. Assume to the contrary that $\mu \equiv 2$. Then for $j \geq i_0$,

$$\begin{aligned} m_{j\beta+1} &= m_{i_0\beta+1} + (j-i_0)\mu + \lfloor \delta_{i_0\beta+1} + (j-i_0)\epsilon \rfloor \\ &\equiv 2(j-i_0) + \lfloor \delta_{i_0\beta+1} + (j-i_0)\epsilon \rfloor \pmod 3. \end{aligned}$$

By Lemma 9.8, $p \neq 4$, hence $j_0 > i_0$.

If

$$(j_0 - i_0)\epsilon + \delta_{i_0\beta+1} < j_0 - i_0 - 1,$$

then let j^* be the least integer such that

$$(j^* - i_0)\epsilon + \delta_{i_0\beta+1} < j^* - i_0 - 1,$$

we would have $i_0 + 2 \leq j^* \leq j_0$, and

$$j^* - i_0 - 2 \leq (j^* - i_0)\epsilon + \delta_{i_0\beta+1} < j^* - i_0 - 1.$$

This implies that

$$m_{j^*\beta+1} \equiv 1 \pmod 3,$$

contrary to Lemma 9.4. Therefore we have

$$(j_0 - i_0)\epsilon + \delta_{i_0\beta+1} \geq j_0 - i_0 - 1.$$

So

$$\epsilon \geq 1 - (1 + \delta_{i_0\beta+1})/(j_0 - i_0) > 1 - 2/(j_0 - i_0).$$

If $p \equiv 0 \pmod 3$, then $j_0 - i_0 = p/3$, hence $p\epsilon > p - 6$.

If $p \equiv 1 \pmod 3$, then since $p \neq 4$, we have $p \geq 7$. As $j_0 - i_0 = (p-4)/3$, we conclude that

$$\epsilon \geq 1 - 3/(p-4) - 3\delta_{i_0\beta+1}/(p-4).$$

Therefore

$$p\epsilon \geq p - 3p/(p-4) - 3\delta_{i_0\beta+1}p/(p-4).$$

Since $p \geq 7$, we have $p/(p-4) \leq 7/3$. Hence $p\epsilon > p - 14$.

If $p \equiv 2 \pmod 3$, then $j_0 - i_0 = (p-2)/3$. Since $p \geq 5$ and $p/(p-2) < 5/3$, it follows that

$$p\epsilon \geq p - 3p/(p-2) - 3\delta_{i_0\beta+1}p/(p-2) > p - 10.$$

By Lemma 7.1, $p(\mu + \epsilon) \equiv 0 \pmod 3$. Therefore we conclude that:

if $p \equiv 0 \pmod 3$, *then* $p\epsilon = p - 3$;
if $p \equiv 1 \pmod 3$, *then the possible values of* $p\epsilon$ *are* $p-3, p-6, p-9, p-12$;
if $p \equiv 2 \pmod 3$, *then the possible values of* $p\epsilon$ *are* $p-3, p-6, p-9$.

If $p\epsilon = p - 12$, then $p \geq 12$, which implies that

$$p\epsilon \geq p - 3p/(p-4) - 3\delta_{i_0\beta+1}p/(p-4) > p - 12,$$

which is an obvious contradiction.

If $p\epsilon = p-9$, then $p \not\equiv 0 \pmod 3$). Therefore, $p \geq 10$. When $p \equiv 2 \pmod 3$, then $p/(p-2) < 5/4$, and hence

$$p\epsilon \geq p - 3p/(p-2) - 3\delta_{i_0\beta+1}p/(p-2) > p-9,$$

which is an obvious contradiction.

When $p \equiv 1 \pmod 3$, then $i_0 = q - 2p/3 + 2/3$. As $\epsilon = 1 - 9/p$ and $q\epsilon + \delta$ is an integer, it follows that $\delta = \{9q/p\}$. Therefore

$$\begin{aligned} \delta_{i_0\beta+1} &= \{(i_0\beta+1)\delta\} \\ &= \{\delta + i_0\epsilon\} \\ &= \{9q/p - (q - 2p/3 + 2/3)9/p\} \\ &= 1 - 6/p. \end{aligned}$$

Since $p \geq 10$, which implies that $p/(p-4) \leq 5/3$, and that

$$p\epsilon \geq p - 3p/(p-4) - 3\delta_{i_0\beta+1}p/(p-4),$$

we conclude that

$$\delta_{i_0\beta+1} \geq 4/5.$$

Therefore $6/p \leq 1/5$, hence $p \geq 30$. But this implies that $p/(p-4) \leq 30/26$, hence

$$p\epsilon > p-9,$$

which is an obvious contradiction.

Assume now that $p\epsilon = p-6$. Then $p \not\equiv 0 \pmod 3$, $\epsilon = 1 - 6/p$, and $\delta = \{6q/p\}$ (because $q\epsilon + \delta$ is an integer.)

First we show that

$$2q/p - 1 < b/(c-b) < 2q/p.$$

The first inequality follows easily from the fact that $2/c < 1/a = p/(q(c-b))$. To show that the second inequality holds, we assume to the contrary that $b/(c-b) \geq 2q/p$. Then $2/b \leq p/(q(c-b))$.

If $p \equiv 1 \pmod 3$, then $i_0 = q - p/3 + 2/3$. Since $\delta = \{6q/p\}$, straightforward calculation shows that

$$\delta_{i_0\beta+1} = \{i_0\epsilon + \delta\} = 1 - 4/p.$$

Therefore

$$\begin{aligned} \frac{n_{i_0\beta+1}+1}{b} &= \frac{3(i_0\beta+1)}{c-b} + \frac{1-\delta_{i_0\beta+1}}{b} \\ &= \frac{3(i_0\beta+1)}{c-b} + \frac{4}{pb} \\ &\leq \frac{3(i_0\beta+1)+2/q}{c-b} \\ &= \frac{3i_0\alpha+2}{a}. \end{aligned}$$

As shown in Lemma 9.4,

$$\frac{3i_0\alpha+1}{a} < \frac{3(i_0\beta+1)}{c-b},$$

Therefore

$$\begin{aligned} \frac{n_{i_0\beta+1}+1}{b} &\in I[3i_0\alpha+1, 3i_0\alpha+2; a] \\ &\cap I[3(i_0\beta+1), 3(i_0\beta+1)+1; c-b]. \end{aligned}$$

By Lemma 5.1, we know that $(n_{i_0\beta+1}+1, m_{i_0\beta+1}+1)$ is strongly consistent. By Lemma 9.6, $m_{i_0\beta+1} \equiv 0 \pmod 3$, this is in contrary to the general assumption.

If $p \equiv 2 \pmod 3$, then $i_0 = q - p/3 + 1/3$. However, the same argument as above still works, and derives a contradiction.

Therefore we have proved that

$$2q/p - 1 < b/(c-b) < 2q/p.$$

This implies that

$$6q/p - 3 < 3b/(c-b) < 6q/p.$$

Recall that by definition $\delta = \{3c/(c-b)\} = \{3b/(c-b)\}$. Now $\delta = \{6q/p\}$ implies that

$$\{6q/p\} = \{3b/(c-b)\}.$$

It follows that

$$n + \delta = 3b/(c-b) = 6q/p - 2 \text{ or } 6q/p - 1,$$

and

$$m + \delta = 3b/(c-b) = 6q/p + 1 \text{ or } 6q/p + 2.$$

If $p \equiv 1 \pmod 3$, then $i_0 = q - (2p-2)/3$, and hence

$$\begin{aligned} m_{i_0\beta+1} &= \lfloor i_0(\mu+\epsilon) + m + \delta \rfloor \\ &\equiv \lfloor (q - \frac{2p-2}{3})(3 - \frac{6}{p}) + m + \delta \rfloor \\ &\equiv \lfloor m + \delta - \frac{6q}{p} + \frac{4p-4}{p} \rfloor \pmod 3. \end{aligned}$$

If $m + \delta = 6q/p + 1$ then $m_{i_0\beta+1} \equiv 1 \pmod 3$. If $m + \delta = 6q/p + 2$ then $m_{i_0\beta+1} \equiv 2 \pmod 3$. In any case, this is in contrary to Lemma 9.6.

If $p \equiv 2 \pmod 3$, then $i_0 = q - (2p-1)/3$, and hence

$$\begin{aligned} m_{i_0\beta+1} &= \lfloor i_0(\mu+\epsilon) + m + \delta \rfloor \\ &\equiv \lfloor (q - \frac{2p-1}{3})(3 - \frac{6}{p}) + m + \delta \rfloor \\ &\equiv \lfloor m + \delta - \frac{6q}{p} + \frac{4p-2}{p} \rfloor \pmod 3. \end{aligned}$$

If $m+\delta = 6q/p+1$ then $m_{i_0\beta+1} \equiv 1 \pmod 3$. If $m+\delta = 6q/p+2$ then $m_{i_0\beta+1} \equiv 2 \pmod 3$. In any case, this is in contrary to Lemma 9.6.

Finally we consider the case that $p\epsilon = p-3$. If $p \equiv 0 \pmod 3$, then by Lemma 9.7,

$$m_{j_0\beta+1} \equiv 2 \pmod 3.$$

By the proof of Lemma 9.4, we know that

$$3(j_0\beta+1)/(c-b) \in I[3i_0\alpha+1, 3i_0\alpha+2; a].$$

Therefore, by Corollary 5.1,

$$\delta_{j_0\beta+1} > 0.$$

However, since $\epsilon = 1-3/p$, $\delta = \{3q/p\}$, it follows that

$$\delta_{j_0\beta+1} = \{j_0\epsilon+\delta\} = \{(q-p/3)(1-3/p)+3q/p\} = 0,$$

which is a contradiction.

If $p \equiv 1 \pmod 3$, then $j_0 = q-p/3-2/3$, $i_0 = q-2p/3+2/3$, $\epsilon = 1-3/p$ and $\delta = \{3q/p\}$ (because $q\epsilon+\delta$ is an integer). Therefore

$$\delta_{j_0\beta+1} = \{j_0\epsilon+\delta\} = \{\frac{3q}{p} - \frac{3q-p-2}{p}\} = \frac{2}{p},$$

and

$$\delta_{i_0\beta+1} = \{i_0\epsilon+\delta\} = \{\frac{3q}{p} - \frac{3q-2p+2}{p}\} = \frac{p-2}{p}.$$

This implies that

$$\begin{aligned} m_{j_0\beta+1} &= m_{i_0\beta+1} + (j_0-i_0)\mu + \lfloor (j_0-i_0)\epsilon + \delta_{i_0\beta+1} \rfloor \\ &\equiv \lfloor \delta_{i_0\beta+1} - \frac{3(j_0-i_0)}{p} \rfloor \\ &\equiv \lfloor \frac{p-2}{p} - \frac{p-4}{p} \rfloor \\ &\equiv 0 \pmod 3. \end{aligned}$$

This is in contrary to Lemma 9.7.

If $p \equiv 2 \pmod 3$, then $j_0 = q-p/3-1/3$ and $i_0 = q-2p/3+1/3$. Since $\epsilon = 1-3/p$, and $\delta = \{3q/p\}$, it follows that

$$\delta_{i_0\beta+1} = \{\delta+i_0\epsilon\} = \{\frac{3q}{p} - \frac{3q-2p+1}{p}\} = \frac{p-1}{p},$$

and

$$\delta_{j_0\beta+1} = \{\delta+j_0\epsilon\} = \{\frac{3q}{p} - \frac{3q-p-1}{p}\} = \frac{1}{p}.$$

It follows that

$$\begin{aligned} m_{j_0\beta+1} &= m_{i_0\beta+1} + (j_0 - i_0)\mu + \lfloor (j_0 - i_0)\epsilon + \delta_{i_0\beta+1} \rfloor \\ &\equiv \lfloor \delta_{i_0\beta+1} - \frac{3(j_0 - i_0)}{p} \rfloor \\ &\equiv \lfloor \frac{p-1}{p} - \frac{p-2}{p} \rfloor \\ &\equiv 0 \pmod 3). \end{aligned}$$

This is again in contrary to Lemma 9.7. ∎

References

[1] J. A. Bondy and U.S.R. Murty, **Graph Theory with Application**,Macmillan, London, (1976).

[2] W. Bienia, L. Goddyn, P. Gvozdjak, A. Sebő and M. Tarsi, *Flows, view obstructions, and the lonely runner*, J. Combinatorial Theory (B), 72(1998), 1-9.

[3] U. Betke and J.M.Wills, *Untere schranken für zwei diophantische funktionen*, Montasch. Math. 76(1972), 214-217.

[4] D. G. Cantor and B. Gordon, *Sequences of integers with missing differences*, J. Combinatorial Theory, Ser. (A) 14(1973), 281-287.

[5] , G. Chang, L. Huang and X. Zhu, *The circular chromatic number and fractional chromatic number of distance graphs*, European J. of Combinatorics, to appear.

[6] G. Chang, D. Liu and X. Zhu, *T-colorings and distance graphs*, J. Combinatorial Theory (B), to appear.

[7] Y. G. Chen, *On a conjecture about diophantine approximations, I*, Acta Math. Sinica 33(1990), 712-717. [Chinese]

[8] J. Chen, G. Chang and K. Huang, *Integral distance graphs*, J. Graph Theory, to appear.

[9] K. Collins, *Circulants and sequences*, preprint, 1996.

[10] M. B. Cozzens and F. S. Roberts, *T-colorings of graphs and the channel assignment problems*, Congr. Numer. 35(1982), 191-208.

[11] , H. T. Croft, K. J. Falconer and R. K. Guy, *Unsolved problems in geometry*, Springer-Verlag, New York (1991).

[12] T. W. Cusick, *View-obstruction problems*, Aequationes Math. 9(1973), 165-170.

[13] T. W. Cusick, *View-obstruction problems in n-dimensional geometry*, J. Combinatorial Theory (A), 16(1974), 1-11.

[14] T. W. Cusick, *View-obstruction problems, II*, Proc. Amer. Math. Soc. 84(1982), 25-28.

[15] T. W. Cusick and C. Pomerance, *View-obstruction problems, III*, J. Number Theory 19(1984), 131-139.

[16] W. Deuber and X. Zhu, *The chromatic number of distance graphs*, Disc. Math. 165/166(1997), 195-204.

[17] W. Deuber and X. Zhu, *The chromatic number of distance graphs with distance sets missing multiples*, manuscript, 1997.

[18] R. B. Eggleton, P. Erdös and D. K. Skilton, *Colouring the real line*, J. Combinatorial Theory, Ser. (B) 39(1985) 86-100.

[19] R. B. Eggleton, P. Erdös and D. K. Skilton, *Research problem 77*, Discrete Math. 58(1986) 323.

[20] R. B. Eggleton, P. Erdös and D. K. Skilton, *Update information on research problem 77*, Discrete Math. 69(1988) 105.

[21] R. B. Eggleton, P. Erdös and D. K. Skilton, *Colouring prime distance graphs*, Graphs and Combinatorics, 32(1990) 17-32.

[22] R. B. Eggleton, *New results on 3-chromatic prime distance graphs*, Ars Combinatoria 26B (1988), 153-180.

[23] L. R. Ford and D. R. Fulkerson, *Network flow and systems of representatives*, Canad. J. Math., 10(1958), 78-84.

[24] G. Gao and X. Zhu, *Star-extremal graphs and the lexicographic product*, Discrete Math., 152(1996), 147-156.

[25] J. Griggs and D. Liu, *The channel assignment problem for mutually adjacent sites*, J. Combinatorial Theory, Ser. (A) 68(1994), 169-183.

[26] H. Hadwiger, H. Debrunner and V. Klee, *Combinatorial geometry in the plane*, Holt Rinehart and Winston, New York (1964).

[27] W. K. Hale, *Frequency assignment: theory and applications*, Proc. IEEE 68(1980), 1497-1514.

[28] N. M. Haralambis, *Sets of integers with missing differences*, J. Combinatorial Theory, Ser. (A) 23(1977), 22-33.

[29] L. Huang and G. Chang, *Circular chromatic number of distance graphs with missing multiples in the distance set*, manuscript, 1998.

[30] A. Kemnitz and H. Kolberg, *Coloring of integer distance graphs*, Tech. report, Institute Für Mathematik, Technische Universität Braunschweig, 1996.

[31] D. Liu, *T-colourings of graphs*, Discrete Math. 101(1992), 203-212.

[32] D. Liu, *Graph homomorphisms and the channel assignment problem*, Ph. D. dissertation, University of South Carolina, 1991.

[33] D. Liu, *T-graphs and the channel assignment problem*, Discrete Math. 1996.

[34] K.W.Lih, D. D.-F. Liu and X. Zhu, *Star-extremal circulant graphs*, manuscript, 1998.

[35] D. Liu and X. Zhu, *Distance graphs with distance sets missing multiples*, J. Graph Theory, to appear.

[36] L. Moser and W. Moser, *Solution to problem 10*, Canad. Math. Bull. 4(1961), 187-189.

[37] J. H. Rabinowitz and V. K. Proulx, *An asymptotic approach to the channel assignment problem*, SIAM J. Algebraic Discrete Methods 6(1985), 507-518.

[38] P. Seymour, *Nowhere-zero 6-flows*, J. Combin. Theory Ser. B 30(1981), 130-135.

[39] C. L. Stewart and R. Tijdeman, *On infinite-difference sets*, Canad. J. Math. 31(1979), 879-910.

[40] B. A. tesman, *T-colorings, list T-colorings and set T-colorings of graphs*, Ph. D. dissertation, Rutgers University, 1989.

[41] W.T.Tutte, *A contribution to the theory of chromatic polynomials*, Canad. J. Math. 6(1954), 80-91.

[42] M. Voigt and H. Walther, *Chromatic number of prime distance graphs*, Discrete Math., 51(1994) 197-209.

[43] M. Voigt, *Colouring of distance graphs*, Ars Combinatoria, to appear.

[44] M. Voigt, *On the chromatic number of distance graphs*, J. Inform. Process. Cybernet. EIK 28(1992), 21-28.

[45] J. M. Wills, *Zwei Sätze über inhomogene diophantische appromixation von irrationlzahlen*, Monatsch. Math. 71(1967), 263-269.

[46] J. M. Wills, *Zur simultanen homogenen diophantischen approximation, I*, Monatsch. Math. 72(1968), 254-263.

[47] J. M. Wills, *Zur simultanen homogenen diophantischen approximation, II*, Monatsch. Math. 72(1968), 368-381.

[48] J. M. Wills, *Zur simultanen homogenen diophantischen approximation, III*, Monatsch. Math. 74(1970), 166-171.

[49] J. M. Wills, *Zur simultanen homogenen diophantischen approximation*, Zahlentheorie (Tagung, Math. Forschungsinst, Oberwolfach, 1970), Ber. Math. Forschungsinst., Oberswolfach, (No. 5) pp. 223-227, Bibliographisches Inst., Mannhaim, 1971.

[50] X. Zhu, *Colouring the distance graphs*, manuscript, 1995.

[51] X. Zhu, *Non-integral distance graphs on the real line*, manuscript, 1996.

[52] X. Zhu, *Pattern periodic coloring of distance graphs*, J. Combintorial Theory (B) 72(1998), 195-206.

[53] X. Zhu, *The circular chromatic number of a class of distance graphs*, manuscript, 1998.