Circular perfect graphs

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Dec. 2006

Abstract

For $1 \leq d \leq k$, let $K_{k/d}$ be the graph with vertices $0, 1, \cdots, k-1$, in which $i \sim j$ if $d \leq |i-j| \leq k-d$. The circular chromatic number $\chi_c(G)$ of a graph G is the minimum of those k/d for which G admits a homomorphism to $K_{k/d}$. The circular clique number $\omega_c(G)$ of G is the maximum of those k/d for which $K_{k/d}$ admits a homomorphism to G. A graph G is circular perfect if for every induced subgraph H of G we have $\chi_c(H) = \omega_c(H)$. This paper surveys results on circular perfect graphs.

1 Introduction

Suppose $G = (V, E)$ and $G' = (V', E')$ are graphs. A *homomorphism* from G to G' is a mapping $f : V \to V'$ such that $f(x)f(y) \in E'$ whenever $xy \in E$. Homomorphism of graphs is a generalization of graph colourings. An n-colouring of a graph G is equivalent to a homomorphism of G to K_n. We write $G \preceq G'$ if there exists a homomorphism from G to G'. Two graphs G and G' are *homomorphically equivalent*, written as $G \approx G'$, if $G \preceq G'$ and $G' \preceq G$. The relation $\approx$ is an equivalence relation. Let $\mathcal{F}$ be the set of finite graphs. Then $\preceq$ defines a partial order on $\mathcal{F}/\approx$.

Let $\mathcal{Z}_{\mathcal{G}}$ be the set of complete graphs, i.e., $\mathcal{Z}_{\mathcal{G}} = \{K_1, K_2, \cdots, \}$. Then $\mathcal{Z}_{\mathcal{G}}$ forms an infinite increasing chain in the partial order $(\mathcal{F}/\approx, \preceq)$. Every graph $G \in \mathcal{F}$ admits a homomorphism to some members of the set $\mathcal{Z}_{\mathcal{G}}$, and contains some member of $\mathcal{Z}_{\mathcal{G}}$ as its subgraphs. The *chromatic number* $\chi(G)$ is the minimum n such that $G \preceq K_n$. The *clique number* $\omega(G)$ is the maximum n such that $K_n \preceq G$. We may view the set $\mathcal{Z}_{\mathcal{G}}$ as a representation of natural numbers, with K_n be a representation of the integer n. Then $\chi(G)$ is the least element of the set $\mathcal{Z}_{\mathcal{G}}$ which is "above" G in the order $\preceq$, and $\omega(G)$ is the maximum element of $\mathcal{Z}_{\mathcal{G}}$ which is "below" G in the order $\preceq$. In this sense, we may view the set $\mathcal{Z}_{\mathcal{G}}$ as a scale that measures a dimension of graphs.

Just as the set of natural numbers is extended to the set of rational numbers, we can "extend" the set $\mathcal{Z}_{\mathcal{G}}$ into a larger set. For a fraction k/d with $k \geq 2d$, let $K_{k/d}$ be the graph with vertex set $\{0, 1, \cdots, k-1\}$ and edge set $\{ij : d \leq |i-j| \leq k-d\}$. We shall denote by $\mathcal{Q}_{\mathcal{G}}$ the set $\{K_{k/d} : k \geq 2d\} \cup \{K_1\}$. Note that $K_{k/1} = K_k$. So $\mathcal{Q}_{\mathcal{G}}$ is an extension of $\mathcal{Z}_{\mathcal{G}}$. Moreover, the set $\mathcal{Q}_{\mathcal{G}}$ is also linearly ordered. It was shown in [1, 10] that if $k'/d' \geq 2$ and $k/d \geq 2$, then $k'/d' \leq k/d$ if and only if $K_{k'/d'} \preceq K_{k/d}$. In particular, if $k/d = k'/d'$, then $K_{k/d} \approx K_{k'/d'}$. We call the graphs $K_{k/d}$ *circular cliques* or *circular complete graphs.*

The set $\mathcal{Q}_{\mathcal{G}}$ together with the order $\preceq$ may be viewed as a representation of those rational numbers $r \geq 2$ or $r = 1$ [18]. The *circular chromatic number* $\chi_c(G)$ of a graph is defined as

$$\chi_c(G) = \inf\{k/d : G \preceq K_{k/d}\},$$

and the *circular clique number* $\omega_c(G)$ of G is defined as

$$\omega_c(G) = \sup\{k/d : K_{k/d} \preceq G\}.$$

For any finite graph G, the infimum in this definition of $\chi_c(G)$ and the supremum in the definition of $\omega_c(G)$ are always attained [10, 16]. Hence, $\chi_c(G)$ is the smallest member of $\mathcal{Q}_{\mathcal{G}}$ which is above G, and $\omega_c(G)$ is the largest member of $\mathcal{Q}_{\mathcal{G}}$ which is below G in the order $\preceq$.

We view both $\mathcal{Z}_{\mathcal{G}}$ and $\mathcal{Q}_{\mathcal{G}}$ as scales that measures a dimension of graphs. As $\mathcal{Z}_{\mathcal{G}}$ is a subset of $\mathcal{Q}_{\mathcal{G}}$, the set $\mathcal{Q}_{\mathcal{G}}$ is a finer scale. The circular chromatic number $\chi_c(G)$ of G is a refinement of its chromatic number $\chi(G)$, and the circular clique number $\omega_c(G)$ of G is a refinement of its clique number $\omega(G)$. Indeed, it follows easily from the definitions (and the fact the infimum and the supremum are attained in the definitions) that for any finite graph G,

$$\omega(G) \leq \omega_c(G) \leq \chi_c(G) \leq \chi(G), \ \omega(G) = \lfloor \omega_c(G) \rfloor, \text{ and } \chi(G) = \lceil \chi_c(G) \rceil.$$

A graph G is *perfect* if for every induced subgraph H of G we have $\chi(H) = \omega(H)$. Perfect graphs have been studied extensively in the literature since it was introduced by Berge in 1961, culminating in the proof by Chudnovsky, Robertson, Seymour, and Thomas [2] of the strong perfect graph theorem. The following definition is a natural generalization of this concept to circular colouring.

Definition 1.1 *A graph G is called* circular perfect *if for every induced subgraph H of G we have $\chi_c(H) = \omega_c(H)$.*

The circular chromatic number of a graph was first introduced by Vince [10] under the name "star chromatic number". An equivalent definition of the parameter, using circular colouring, was given in [12]. Circular chromatic number of graphs has attracted considerable attention. Many interesting results are obtained and techniques are developed. Surveys on this topics are given in [13, 19]. The circular clique number of a graph and the concept of circular perfect graphs were introduced in [16]. In this article, we survey results obtained concerning circular perfect graphs. Similar to the strong perfect graph theorem, one might expect a characterization of circular perfect graphs parallel to the characterization of perfect graphs. However, it is unlikely that there is a simple characterization of circular perfect graphs, as we shall explain. If restricted to special classes of graphs, the characterization might be possible. We present some necessary and some sufficient conditions for a graph to be circular perfect, discuss circular perfectness of graphs for special classes of graphs, and present examples of circular perfect graphs with given properties.

2 Basic facts

Since $\omega(G) \leq \omega_c(G) \leq \chi_c(G) \leq \chi(G)$ for any graph G, if G is perfect, then G is circular perfect. So the class of circular perfect graphs contains the class of perfect graphs as a subclass. To study circular perfect graphs, we concentrate on circular perfect graphs that are not perfect.

By definition, if G contains $K_{k/d}$ as a subgraph, then $\omega_c(G) \geq k/d$. The converse is also true, although it is not obvious from the definition.

Lemma 2.1 *If G is a graph with $\omega_c(G) = k/d$ and $gcd(k, d) = 1$, then $K_{k/d}$ is an induced subgraph of G.*

It follows from Lemma 2.1 that determining the circular clique number of a graph can be accomplished by testing all its induced subgraphs. This is certainly not an efficient way, but most likely there is no efficient way as the problem of determining the circular clique number of a graph is NP-hard. Indeed, once the circular clique number of a graph is determined then its clique number is determined. As the latter problem is NP-hard, so is the former one.

Observe that the induced subgraphs of circular complete graphs are usually not circular complete graphs. Thus it is not obvious that circular complete graphs are circular perfect. Nevertheless, we have the following result:

Lemma 2.2 *If G is an induced subgraph of a circular complete graph $K_{k/d}$, then $\chi_c(G) = \omega_c(G)$.*

Corollary 2.1 *Suppose $k \geq 2d$. The circular complete graph $K_{k/d}$ is circular perfect.*

In particular, for $k \geq 2$, the circular complete graphs $K_{(2k+1)/k}$ and $K_{(2k+1)/2}$, which are odd holes and odd anitholes, are circular perfect. As odd holes and odd antiholes are not perfect, the class of perfect graphs is a proper subclass of the class of circular perfect graphs. Although a circular perfect graph need not be perfect, the neighbourhood of any vertex in such a graph must be a perfect graph. For a vertex x of a graph G, let $N_G[x]$ (or $N[x]$) denote the close neighbourhood of x.

Theorem 2.1 *If G is circular perfect then for every vertex x of G, $N_G[x]$ induces a perfect graph.*

3 Some sufficient conditions for a graph to be circular perfect

The necessary condition in Theorem 2.1 for a graph to be circular perfect is not sufficient. By adding some further requirements, we can obtain a sufficient condition.

Theorem 3.1 *Suppose G is a graph such that for every vertex x of G, $N[x]$ is a perfect graph and $G - N[x]$ is a bipartite graph with no induced P_5 (i.e., a path with 5 vertices). Then G is circular perfect.*

It is easy to see that if G satisfies the condition of Theorem 3.1, then any induced subgraph of G satisfies that condition. Therefore to prove Theorem 3.1, it suffices to prove the following:

Theorem 3.2 *Suppose G is a graph and for every vertex x of G, $N[x]$ is a perfect graph and $V - N[x]$ is a bipartite graph which contains no induced P_5. Then $\chi_c(G) = \omega_c(G)$.*

The proof of Theorem 3.2 is quite long. The following example shows that these conditions are quite tight in some sense. Let G be the graph as depicted in Figure 3 below. It is easy to verify that $\chi_c(G) = 8/3$ and $\omega_c(G) = 5/2$. So G is not circular perfect. But G "almost satisfies" the condition of Theorems 3.2. For each vertex x of G, $N[x]$ is a star, and $G - N[x]$ is either a P_4 or a P_5.

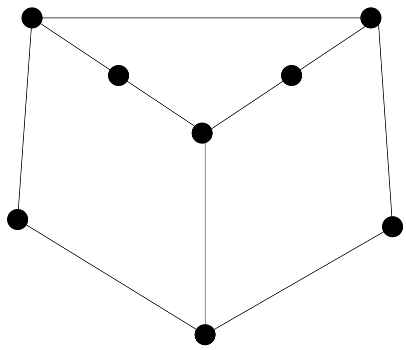

Figure 1: A non-circular perfect graph G

On the other hand, it is easy to construct circular perfect graphs G which contain vertices x such that $G - N[x]$ is not bipartite, or $G - N[x]$ is bipartite but contains induced path length greater than 5. Indeed, if $k = 2d + 1$ and $d \geq 4$, then the circular complete graph $K_{k/d}$ does not satisfy the condition of Theorem 3.1.

The next sufficient condition for a graph to be circular perfect concerns triangle free graphs. Suppose $G = (V, E)$ is a graph and x is a vertex of G. For two vertices $u, v \in V - N[x]$,

- $u \leq^x v$ means $N(u) \cap N(x) \subseteq N(v) \cap N(x)$;
- $u <^x v$ means $N(u) \cap N(x) \subset N(v) \cap N(x)$;
- $u =^x v$ means $N(u) \cap N(x) = N(v) \cap N(x)$.

Here $A \subset B$ means that A is a proper subset of B.

Definition 3.1 *Given an induced path $P_n = (p_0, p_1, \cdots, p_n)$ of $G - N[x]$, we say P_n is* badly-linked *with respect to x if one of the following holds:*

1. *There are three indices $i < j < k$ of the same parity such that $p_i \not\leq^x p_j$ and $p_k \not\leq^x p_j$.*

2. *There are three indices $i < j < k$ of the same parity such that $p_j \not\leq^x p_i$ and $p_j \not\leq^x p_k$.*

3. *There are two even indices $i < j$ and two odd indices $i' < j'$ such that $p_i \not\leq^x p_j$ and $p_{i'} \not\leq^x p_{j'}$.*

An induced path P of $G - N[x]$ is called well-linked *with respect to x if it is not badly-linked with respect to x.*

Theorem 3.3 *Suppose G is a triangle free graph such that for every vertex x of G, $G - N[x]$ is a bipartite graph with no induced C_n for $n \geq 6$, and any induced path of $G - N[x]$ is well-linked. Then G is circular perfect.*

It is easy to see that if G satisfies the condition of Theorem 3.3, then any induced subgraph of G satisfies that condition. Therefore to prove Theorem 3.3, it suffices to prove the following:

Theorem 3.4 *Suppose G is a triangle free graph. If for every vertex x of G, $V - N[x]$ is a bipartite graph which contains no induced C_n for $n \geq 6$, and every induced path of $G - N[x]$ is well-linked, then $\chi_c(G) = \omega_c(G)$.*

The proof of Theorem 3.4 is again quite complicated. Theorems 3.1 and 3.3 are used to prove an analogue of Hajós Theorem for circular chromatic number. In [14], three graph operations are given to play the role of the Hajós sum in original Hajós Theorem. Namely it was proved in [14] that if $k/d \geq 3$, then all the graphs G with $\chi_c(G) \geq k/d$ can be constructed from $K_{k/d}$ by adding edges and vertices, by identifying non-adjacent vertices and by applying the three graph operations that replace the Hajós sum. Moreover all such constructed graphs G have $\chi_c(G) \geq k/d$. The same result is proved for $2 \leq k/d < 3$ in [15], where four new graph operations are introduced in place of the Hajós sum.

4 Minimal circular imperfect graphs

A graph G is *minimal imperfect* if G is not perfect, but every induced subgraph of G is perfect. Similarly, we call a graph G *circular imperfect* if G is not circular perfect, and *minimal circular imperfect* if G is circular imperfect, but every proper induced subgraph of G is circular perfect. The strong perfect graph theorem characterizes the family of perfect graphs by determining all minimal imperfect graphs. It is natural that we would like to study minimal circular imperfect graphs. The problem of characterizing all minimal circular imperfect graphs is difficult. Thus we restricted to special classes of graphs.

First we consider norms of circular cliques. Suppose G is a graph. The *norm* of G, denoted by $\mathrm{norm}(G)$, is the spanning subgraph H of G that consists of all the edges e that are contained in a maximum clique of G. The following result, proved in [8], characterizes all circular cliques whose norms are circular imperfect, and those whose norms are minimal circular imperfect.

Theorem 4.1 *Suppose $gcd(k, d) = 1$. Then*

- $\mathrm{norm}(K_{k/d})$ *is circular imperfect if and only if* $k \not\equiv -1 \pmod{d}$ *and* $\lfloor k/d \rfloor \geq 3$.
- $\mathrm{norm}(K_{k/d})$ *is minimal circular imperfect if and only if* $k = 3d+1$ *and* $d \geq 3$.

A graph G is a (p, q)*-partitionable graph* if $|V(G)| = pq + 1$, and for each vertex v of G, $G \setminus \{v\}$ admits a partition into p cliques of cardinality q as well as a partition into q stable sets of cardinality p. A graph is *partitionable* if it is a (p, q)-partitionable graph for some $p, q \geq 2$. Partitionable graphs were introduced by Lovász [4] and Padberg [5] as a tool in the study of perfect graphs. Before the proof of the strong perfect graph theorem, it was shown by Lovász [4] and Padberg [5] that every minimal imperfect graph is a partitionable graph. Partitional graphs satisfy $\omega(G) + 1 = \chi(G)$. As circular perfect graphs satisfies $\omega(G)+1 \geq \chi(G)$, a natural question is which partitionable graphs are circular perfect. The following theorem, proved in [8], characterizes all partitionable graphs that are circular imperfect.

Theorem 4.2 *All partitionable graphs apart from circular complete graphs are circular imperfect.*

It is likely most of the partitionable graphs are not minimal circular imperfect. However, this is not proved yet.

It is not difficult to show that every outerplanar graph is circular perfect.

Theorem 4.3 *All outerplanar graphs are circular perfect.*

Outerplanar graphs is a subclass of series-parallel graphs. If we move from the class of outerplanar graphs to the the class of series-parallel graphs, then we can easily find circular imperfect graphs. It would be nice to characterize all the minimal circular imperfect series-parallel graphs. However, the task is not finished yet. It is easy to see that odd cycles and K_2 are the only circular cliques that are series-parallel graphs. Thus if a series-parallel graph G is circular perfect then the core of G is an odd cycle or K_2. It is known [6] that for every rational $r \in [2, 8/3]$, there is a series-parallel graph G with $\chi_(G) = r$. If $r \neq 2 + 1/k$, then such a series-parallel graph is circular imperfect. It remains an open problem as to determine which rational is the circular chromatic number of a minimal circular imperfect series-parallel graph. The following result can be proved by using the construction method given in [6].

Theorem 4.4 *If k/d is a rational number and there is an integer $q \geq 2$ such that $(2q + 1)d = kq - 1$, then there is a minimal circular imperfect series-parallel graph G with $\chi_c(G) = k/d$.*

Figure 2 is a minimal circular imperfect series-parallel graph with $\chi_c(G) = 13/5$.

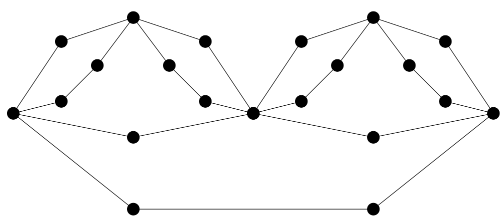

Figure 2: A minimal circular imperfect series-parallel graph G with $\chi_c(G) = 13/5$

For planar graphs, the problem of characterizing circular perfect planar graphs is more difficult. However, planar circular cliques are easy to identify: K_4, K_3 and $K_{(2k+1)/k}$ for some $k \geq 2$.

Theorem 4.5 *If a planar graph G is circular perfect then either $\chi_c(G) = 4$ or $\chi_c(G) \in \{2 + \frac{1}{k} : k = 1, 2, \cdots\}$.*

It follows from Theorem 4.5 that if G is a planar graph with $\chi(G) = 4$ but $\omega(G) = 3$, then G is circular imperfect; if $\chi(G) = 3$ and the core of G is not an odd cycle, then G is circular imperfect.

The following are two classes of minimal circular imperfect planar graphs.

Example 4.1 *Let G be the graph with vertex set*

$$V = \{x_1, x_2, \cdots, x_k, u_1, u_2, \cdots, u_{k-1}, v_1, v_2, \cdots, v_{k-1}\}$$

and edge set

$$\{x_i v_i, x_i, u_i, u_i v_i, x_{i+1} v_i, x_{i+1} u_i : i = 1, 2, \cdots, k-1\} \cup \{x_1 x_k\}.$$

Let $1 = i_1 < i_2 < \cdots < i_t = k+1$. For $j = 1, 2, \cdots, t-1$, identify $u_{i_j}, u_{i_j+1}, \cdots, u_{i_{j+1}-1}$ into a single vertex. The resulting graph is a minimal circular imperfect graph, provided that it contains no K_4.

Figure 3 is an example of such a graph.

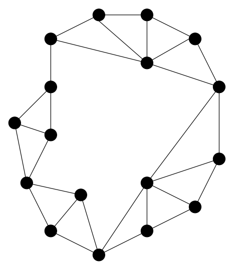

Figure 3: A minimal circular imperfect planar graph

Another class of planar graphs that are minimal circular imperfect is defined as follows:

Example 4.2 *Let G be obtained from an odd wheel by subdividing each edge into a path so that each finite face is has length $2k+1$ for some $k \geq 1$, and the infinite face has length at least $2k+1$. Then G is minimal circular imperfect.*

Figure 4 is an example of such a graph.

Suppose G is a series-parallel graph which is circular imperfect. Suppose u, x, v are three consecutive vertices of a face of G. Let H be obtained from G by identifying u, v into a single vertex. If H has the same odd girth as G, then H is circular imperfect. Usually, if G is minimal circular imperfect, H

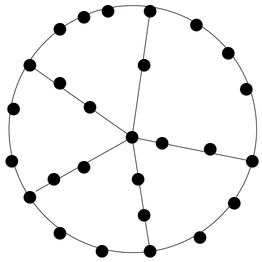

Figure 4: A minimal circular imperfect planar graph

is also minimal circular imperfect. Such minimal circular imperfect graphs may not be included in the two classes of minimal circular imperfect graphs presented above. So it seems not easy to characterize all minimal circular imperfect planar graphs.

The family of minimal circular imperfect line graphs was studied by Xu. The following result is proved in [11].

Theorem 4.6 *Suppose $G = L(H)$ is the line graph of H. Then G is minimal circular imperfect if and only if H is 3-critical, i.e., $\Delta(H) = 3$ and $\chi'(H) = 4$, and for any edge e of H, $\chi'(H - e) = 3$.*

It is a difficult problem to determine 3-critical graphs, and most likely, there is no simple characterization of such graphs. Thus this theorem suggests that there is no easy characterization of minimal circular imperfect graphs.

Line graphs is a subclass of claw-free graphs. Minimal circular imperfect claw-free graphs is studied in [9], where the following result is proved.

Theorem 4.7 *If G is a minimal circular imperfect graph and G is claw-free, then either $\alpha(G) \leq 3$ or $\omega(G) \leq 3$.*

Note that minimal imperfect graphs G have $\min\{\alpha(G), \omega(G)\} = 2$. A natural question is whether $\min\{\alpha(G), \omega(G)\} \leq c$ for some constant c for minimal circular imperfect graphs. This question was answered in the negative in [7].

Theorem 4.8 *For any integer k, there is a minimal circular imperfect graph G with $\min\{\alpha(G), \omega(G)\} > k$.*

Figure 5 below is a minimal circular imperfect graph G with $\min\{\alpha(G), \omega(G)\} = 5$. The subset $V = \{v_0, v_1, \cdots, v_{14}\}$ of $V(G)$ induces a copy of $K_{15/2}$, i.e., $v_i v_j$ is an edge if $2 \leq |i - j| \leq 13$. For simplicity, these edges are not shown in the figure.

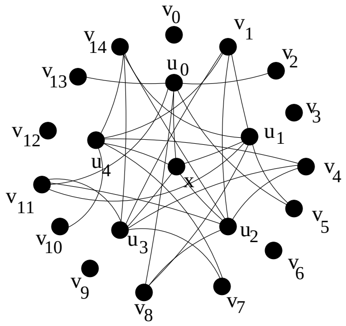

Figure 5: A minimal circular imperfect graph G with $\min\{\alpha(G), \omega(G)\} = 5$, with edges between the v_i's missing.

5 Strongly circular perfect graphs

A nice property of perfect graphs is that if G is perfect then its complement $\overline{G}$ is also perfect. This is not true for circular perfect graphs. For example, the complement of circular complete graphs are usually not circular perfect. For example, $\overline{K_{8/3}}$ is not circular perfect. Indeed, for most circular cliques $K_{k/d}$, their complements $\overline{K_{k/d}}$ are not circular perfect.

Theorem 5.1 *[3] Suppose $k \geq 2d$. Then $\overline{K_{k/d}}$ is circular perfect if and only if $d \leq 2$ or $k = 2d + 1$ or $d = 3$ and $k \equiv 0 \pmod 3$.*

A graph is called *strongly circular perfect* if both G and $\overline{G}$ are circular perfect. It follows from Theorem 5.1 that if G is a strongly circular perfect graph, then $\chi_c(G) = k/d$ for some $d \leq 2$, or $\chi_c(G) = 2 + 1/d$ for some integer d. Perfect graphs are strongly circular perfect. Odd cycles and the complements of odd cycles are also strongly circular perfect. The family of triangle free strongly circular perfect graphs is characterized by Coulnges, Pecher and Wagler [3].

A graph G is called an *interlaced odd hole* if $V(G)$ is partitioned into $(\cup_{i=0}^{2k} A_i) \cup (\cup_{i=0}^{2k} B_i)$ such that the following hold:

- $A_i \neq \emptyset$ for each i.
- If $|A_i| \geq 2$, then $|A_{i-1}| = |A_{i+1}| = 1$, where summation in the index is modulo $2k + 1$.
- If $B_i \neq \emptyset$, then $|A_i| = 1$.
- For any i, for any $v \in A_i$, $N(v) = A_{i-1} \cup A_{i+1} \cup B_i$.

- For any i, for any $v \in B_i$, $N(v) = A_i$.

Figure 6 is an example of an interlaced odd hole.

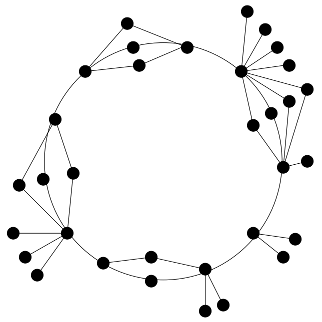

Figure 6: An interlaced odd hole

Theorem 5.2 *[3] A triangle free graph G is strongly circular perfect if and only if either G is bipartite or G is an interlaced odd hole.*

6 Some open questions

Question 6.1 *Is it true that if G is minimal circular imperfect then $\chi(G) = \omega(G) + 1$?*

Question 6.2 *For which rational number r, there is a minimal circular imperfect series-parallel graph G with $\chi_c(G) = r$?*

Question 6.3 *Characterize minimal circular imperfect series-parallel graphs.*

Question 6.4 *Suppose G is a series-parallel graph. Assume that the core of G is not an odd cycle, and for any vertex x of G, the core of $G - x$ is an odd cycle. Is it true that G is a minimal circular imperfect graphs?*

Z. Pan said the answer is no. Denote by GsH the series join of G and H, and by GpH the parallel join of G and H. Let $H = ((((P_5pP_6)s(P_5pP_6))pP_4)sP_2)pP_5$. Let $H' = P_4pP_5$. Then $L_{9,4}(H) = \{2, 7\} = L_{9,4}(H')$. Let $G = (H'sHsH')pP_1$. Then $L_{9,4}(G) = \emptyset$. So $\chi_c(G) > 9/4$. For any vertex x, $\chi_c(G - x) = 9/4 = \omega_c(G - x)$. However, G is not minimal circular imperfect because H is not circular perfect. The graph is shown in Figure 7 below.

Question 6.5 *Characterize minimal circular imperfect planar graphs.*

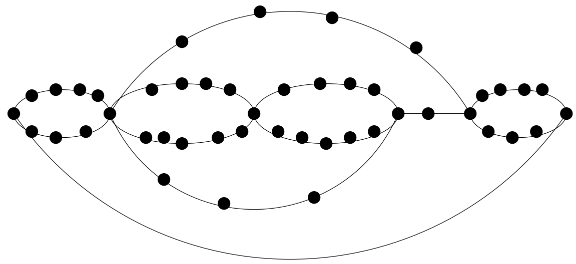

Figure 7: A counterexample to Question 6.4

References

[1] J. A. Bondy and P. Hell, *A note on the star chromatic number*, J. Graph Theory 14 (1990), 479-482.

[2] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas, *The Strong Perfect Graph Theorem*, Annals of Mathematics, (2) 164(2006), no. 1, 51-229.

[3] S. Coulonges, A. Pêcher and A.K.Wagler, *Triangle-free strongly circular perfect graphs*, LaBRI Research Report RR-1360-05, 2005.

[4] L. Lovász, *A characterization of perfect graphs*, J. Combinatorial Theory Ser. B, 13(1972), 95-98.

[5] M. W. Padberg, *Perfect zero-one matrices*, Math. Programming, 6(1974), 180-196.

[6] Z. Pan and X. Zhu. *Density of the circular chromatic numbers of series-parallel graphs*, J. Graph Theory, 46(1)(2004), 57-68.

[7] Z. Pan and X. Zhu, *Minimal circular imperfect graphs of large clique number and large independence number*, European J. Combinatorics, to appear.

[8] A. Pêcher, A. Wagler, *Three classes of minimal circular-imperfect graphs*, manuscript, 2005.

[9] A. Pêcher and X. Zhu, *Claw-free graphs that are minimal circular imperfect*, in preparation.

[10] A. Vince, *Star chromatic number*, J. Graph Theory 12 (1988), 551-559.

[11] B. Xu, *On minimally circular imperfect graphs*, manuscript, 2006.

[12] X. Zhu, *Star chromatic numbers and products of graphs*, J. Graph Theory 16 (1992), 557-569.

[13] X. Zhu, *The circular chromatic number, a survey*, Discrete Math. Vol. 229 (1-3) (2001), 371-410.

[14] X. Zhu, *An analogue of Hajós theorem for the circular chromatic number*, Proceedings of the American Mathematical Society, 129(2001), 2845-2852.

[15] X. Zhu, *An analogue of Hajós' theorem for the circular chromatic number (II)*, Graphs and combinatorics, 19(2003), 419-432.

[16] X. Zhu, *Circular perfect graphs*, J. Graph Theory, 48(3)(2005), 186-209.

[17] X. Zhu, *Perfect graphs for generalized colouring - circular perfect graphs*, AMS, DIMACS series: Proceedings of the DIMACS/DIMATIA Workshop on Graphs, Morphisms and Statistical Physics, to appear.

[18] X. Zhu, *Circular coloring and graph homomorphism*, Bulletin of the Australian Mathematical Society, Vol. 59(1999), 83-97.

[19] X. Zhu, *Recent developments in circular colouring of graphs*, Topics in Discrete Mathematics: Dedicated to Jarik Nesetril on the occasion of his 60th birthday, Algorithms and Combinatorics Series, Vol. 28, 497-550.