Circular chromatic index of graphs of maximum degree 3

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Abstract

This paper proves that if G is a graph (parallel edges allowed) of maximum degree 3, then $\chi'_c(G) \leq 11/3$ provided that G does not contain H_1 or H_2 as a subgraph, where H_1 and H_2 are obtained by subdividing one edge of K_2^3 (the graph with three parallel edges between two vertices) and K_4, respectively. As $\chi'_c(H_1) = \chi'_c(H_2) = 4$, our result implies that there is no graph G with $11/3 < \chi'_c(G) < 4$. It also implies that if G is a 2-edge connected cubic graph, then $\chi'(G) \leq 11/3$.

1 Introduction

Graphs considered in this paper may have parallel edges but no loops. Given a graph $G = (V, E)$, and positive integers $p \geq q$, a (p, q)-coloring of G is a mapping $f : V \to \{0, 1, \cdots, p-1\}$ such that for every edge $e = xy$ of G, $q \leq |f(x) - f(y)| \leq p - q$. The *circular chromatic number* $\chi_c(G)$ of G is defined as

$$\chi_c(G) = \inf\{p/q : G \text{ has a } (p, q)\text{-coloring}\}.$$

It is known [4, 6] that for any graph G, the infimum in the definition is always attained and

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G).$$

For a graph $G = (V, E)$, the *line graph* $L(G)$ of G has vertex set E, in which $e_1 \sim e_2$, if e_1 and e_2 have an end vertex in common. The *circular chromatic index* $\chi'_c(G)$ of G is defined as

$$\chi'_c(G) = \chi_c(L(G)).$$

Recall that the *chromatic index* $\chi'(G)$ of G is defined as $\chi'(G) = \chi(L(G))$. So we have

$$\chi'(G) - 1 < \chi'_c(G) \leq \chi'(G).$$

If G is connected and $\Delta(G) = 2$, then G is either a cycle or a path. This implies that either $\chi'_c(G) = 2$ or $\chi'_c(G) = 2 + \frac{1}{k}$ for some positive integer k. Since graphs G with $\Delta(G) \geq 3$ have $\chi'_c(G) \geq 3$, 'most' of the rational numbers in the interval $(2, 3)$ are not the circular chromatic index of any graph. The following question was asked in [6]:

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Figure 1: (a): The graph H_1, (b): The graph H_2.

Question 1.1 *For which rational $r \geq 3$, there is a graph G with circular chromatic index r? In particular, is it true that for any rational $r \geq 3$, there is a graph G with $\chi'_c(G) = r$?*

If $3 < \chi'_c(G) < 4$, then G has maximum degree 3. It is well-known that the Four Color Theorem is equivalent to the statement that every 2-edge connected cubic planar graph G has $\chi'_c(G) = 3$. For nonplanar 2-edge connected cubic graphs, Jaeger [2] (see also page 197 of [3]) proposed the following conjecture (Petersen Coloring Conjecture):

Conjecture 1.2 *If G is a 2-edge connected cubic graph, then one can color the edges of G, using the edges of the Petersen graph as colors, in such a way that any three mutually adjacent edges of G are colored by three edges that are mutually adjacent in the Petersen graph.*

Since the Petersen graph has circular chromatic index 11/3, Conjecture 1.2 would imply that every 2-edge connected cubic graph G has $\chi'_c(G) \leq 11/3$. The following two open problems are proposed in [6]:

Question 1.3 *Prove that if G is a 2-edge connected cubic planar graph, then $\chi'_c(G) < 4$, without using the Four Color Theorem.*

Question 1.4 *Are there any 2-edge connected cubic graph G with $\chi'_c(G) = 4$?*

This paper proves the following result:

Theorem 1.5 *Let H_1 and H_2 be the graphs as shown in Figure 1. If G is graph of maximum degree 3 and G does not contain H_1 or H_2 as a subgraph, then $\chi'_c(G) \leq 11/3$.*

It is easy to verify that $\chi'_c(H_1) = \chi'_c(H_2) = 4$. Since graphs G with $\Delta(G) \geq 4$ have $\chi'_c(G) \geq 4$, we have the following corollary:

Corollary 1.6 *There is no graph G with $11/3 < \chi'_c(G) < 4$.*

Corollary 1.6 answers the second part of Question 1.1 in the negative.

To prove Theorem 1.5, it suffices to consider 2-edge connected graphs. Indeed, if a graph G is not 2-edge connected, say e is a cut edge of G, then either e is a hanging edge, i.e., incident to a degree 1 vertex, or e is a cut vertex in $L(G)$. In the latter case, $\chi_c(L(G)) = \max\{\chi_c(B) : B \text{ is a block of } L(G)\}$. If e is a hanging edge of G, then e has degree at most 2 in $L(G)$, and hence any $(11,3)$-coloring of $L(G) - e$ can be extended to a $(11,3)$-coloring of $L(G)$. In the remainder of this paper, we assume that G is 2-edge connected and hence has minimum degree at least 2. It is easy to see that if G is 2-edge connected and has maximum degree at most 3, then G cannot contain H_1 or H_2 as a *proper* subgraph. Therefore Theorem 1.5 is equivalent to the following:

Theorem 1.7 *Suppose G is 2-edge connected and has maximum degree 3. If $G \neq H_1, H_2$, then $\chi'_c(G) \leq 11/3$.*

Theorem 1.7 implies the following corollary, which answers Questions 1.3 and 1.4.

Corollary 1.8 *The circular edge chromatic number of every 2-edge connected cubic graph G is less than or equal to 11/3.*

2 Cubic graphs of girth at least 4

The remainder of the paper is devoted to the proof of Theorem 1.7. In this section, we consider triangle free cubic graphs. First we prove a lemma which is needed in our proof.

Suppose c is a k-coloring of a graph $G=(V,E)$ with colors $0,1,\cdots,k-1$. If xy is an edge of G and $c(y)=c(x)+1 \pmod{k}$, then we say $\vec{xy}$ is a *tight arc with respect to* c. Let A be the set of tight arcs, and let $D_c(G)=(V,A)$, which is a directed graph with vertex set V. It is known [1, 6] that if there is a k-coloring c of G for which $D_c(G)$ is acyclic, then $\chi_c(G)<k$. The following lemma is a strengthening of this result.

Lemma 2.1 *Let c be a k-coloring of a graph G with colors $0,1,\cdots,k-1$, where $k>2$. If $D_c(G)$ is acyclic and each directed path of $D_c(G)$ contains at most n vertices of color $k-1$, then $\chi_c(G)\leq k-\frac{1}{n+1}$.*

Proof. Let $p=k(n+1)-1$ and $q=n+1$. It suffices to give an (p,q)-coloring for G. For each vertex v of G, let $l(v)$ be the maximum number of vertices with color $k-1$ on a directed path of $D_c(G)$ which ends in v, without considering v itself. We claim that the coloring c' defined as

$$c'(v)=(c(v)q+l(v)) \bmod p$$

is a proper (p,q)-coloring of G. Consider two adjacent vertices u and v. If $2\leq|c(u)-c(v)|\leq k-2$, then since both $l(u)$ and $l(v)$ are less than q, we have $q\leq|c'(u)-c'(v)|\leq p-q$. If $c(u)-c(v)=1$, then $\vec{vu}$ is a tight arc and hence $l(u)\geq l(v)$. So we have $q\leq|c'(u)-c'(v)|\leq p-q$. Finally, if $c(u)=0$ and $c(v)=k-1$, then $\vec{vu}$ is a tight arc and $l(u)\geq l(v)+1$. Again we have $q\leq|c'(u)-c'(v)|\leq p-q$. ■

Suppose c is a k-edge coloring of G and $e=xy$ is an edge of G. The two arcs $\vec{xy}$ and $\vec{yx}$ are called arcs corresponding to e. We say an arc $\vec{xy}$ is *unblocked with respect to* c, if there is a directed walk $W=(e_1,e_2,\cdots,e_n,e,e'_1,e'_2,\cdots,e'_m)$ in $D_c(L(G))$ such that (i) $c(e_1)=c(e'_m)=k-1$, and (ii) $e_n=x'x$ and $e'_1=yy'$. The arc $\vec{xy}$ is *blocked with respect to* c if no such directed walk exists. An edge $e=xy$ is said to be *blocked in the direction $x\to y$ with respect to* c, if the arc $\vec{xy}$ is blocked. An edge $e=xy$ is *completely blocked with respect to* c, if both arcs $\vec{xy}$ and $\vec{yx}$ are blocked. Given a partial k-edge coloring c' of G (i.e., c' colors a subset of edges of G), we say an arc $\vec{xy}$ is unblocked with respect to c', if c' can be extended to a k-edge coloring c of G such that $\vec{xy}$ is unblocked with respect to c. If no such extension exists, then we say $\vec{xy}$ is blocked with respect to c'. Similarly, we say an edge e is completely blocked with respect to c', if both arcs $\vec{xy}$ and $\vec{yx}$ are blocked with respect to c'.

Theorem 2.2 *If G is a cubic graph of girth at least 4 and has a perfect matching, then $\chi'_c(G)\leq 11/3$.*

Proof. By Lemma 2.1 it suffices to prove that there exists a 4-edge coloring ϕ of G such that $D_\phi(L(G))$ is acyclic and each directed path of $D_\phi(L(G))$ contains at most two vertices (i.e., two edges of G) which are colored by 3.

Let M be a perfect matching of G. Then $G-M$ is a collection of cycles. A 4-edge coloring of G is called a *valid coloring* with respect to M, if the following hold:

- All the M-edges (an edge in M is called an M-edge) are colored by color 0.

- The edges of any even cycle C of $G-M$ are colored by colors 1 and 2.

- The edges of any odd cycle C of $G-M$ are colored by colors 1 and 2, except one edge which is colored by color 3.

Let c' be a partial 4-edge coloring of G which can be extended to a valid 4-edge coloring of G with respect to M. We are interested in the blocked directions of the M-edges with respect to c'. Suppose $e=xy$ is an M-edge, and C and C' are (not necessarily distinct) cycles of $G-M$ such that $x\in V(C)$ and $y\in V(C')$. If $\vec{xy}$ is an unblocked arc with respect to c', then we say $\vec{xy}$ is an *input* of C' and an *output* of C with respect to c'.

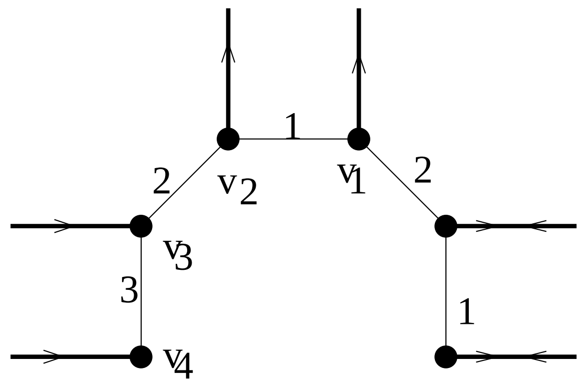

Figure 2: The blocked directions of M-edges incident to C with respect to c_C.

Let C be a cycle of $G - M$, and let c_C be the partial edge coloring of G which is the restriction of a valid coloring c to $M \cup C$. If C is an even cycle, then it is easy to see that every edge $e \in M$ incident to C is completely blocked with respect to c_C. If C is an odd cycle of $G - M$, then Figure 2 shows the blocked directions of the M-edges incident to C with respect to c_C.

In Figure 2, a thick edge indicates an M-edge. An arrow on an M-edge indicates a blocked direction of that edge. An M-edge with opposite arrows is completely blocked. Since G has girth at least 4, the four vertices v_1, v_2, v_3, v_4 as indicated in Figure 2 are distinct. Note that an M-edge e incident to C is completely blocked with respect to c_C, unless e is incident to one of the vertices v_1, v_2, v_3, v_4, which are the vertices on a path whose edges are colored by colors $1, 2, 3$. So there are at most 4 M-edges incident to C that are not completely blocked. An M-edge incident to C could be a chord of C. If an M-edge e incident to v_1, v_2, v_3, v_4 is a chord of C, then e could be completely blocked. We will discuss this case later in more detail. If an M-edge e incident to C is not completely blocked with respect to c_C, then exactly one direction of e is blocked.

For a valid 4-edge coloring c of G, let $\phi(c)$ be the total number of not completely blocked M-edges. Let $\psi(c)$ be the number of not completely blocked M-edges that are chords of cycles of $G - M$.

Claim 2.3 *Suppose c is a valid 4-edge coloring of G (with respect to a perfect matching M). If $G - M$ has a cycle C which has an input as well as an output, then there is a valid 4-edge coloring c^* of G for which $\phi(c^*) + \psi(c^*) < \phi(c) + \psi(c)$.*

Proof. Assume C is a cycle of $G - M$ which has an input as well as an output with respect to a valid 4-edge coloring c. Then C is an odd cycle and the M-edges incident to C contributes at least 2 to the summation $\phi(c) + \psi(c)$. We shall construct a valid 4-edge coloring c^* of G such that each M-edge not incident to C contributes the same amount to $\phi(c^*) + \psi(c^*)$ and $\phi(c) + \psi(c)$. However, the M-edges incident to C contributes at most 1 to the summation $\phi(c^*) + \psi(c^*)$.

Uncolor the edges of C to obtain a partial 4-edge coloring c' of G. The valid 4-edge coloring we shall construct is an extension of c'. It is obvious that for any valid 4-edge coloring c^* of G which is an extension of c', each M-edge not incident to C contributes the same amount to $\phi(c^*) + \psi(c^*)$ and $\phi(c) + \psi(c)$. So we only need to make sure that the M-edges incident to C contribute at most 1 to the summation $\phi(c^*) + \psi(c^*)$.

First we consider the case that C has no chord. As each M-edge e incident to C is incident to another cycle of $G - M$, at least one direction of e is blocked with respect to c'. Since C is an odd cycle and C has an input and an output with respect to c, it is easy to see that there are four consecutive vertices v_1, v_2, v_3, v_4 of C such that with respect to the partial edge coloring c', the M-edges incident to v_1, v_2 have a common blocked direction (i.e., either both are blocked in the direction towards C or both are blocked in the direction away from C), and the M-edges incident to v_3, v_4 have an opposite blocked direction. Depending on which directions of the four edges are blocked, there are four cases as depicted in Figure 3.

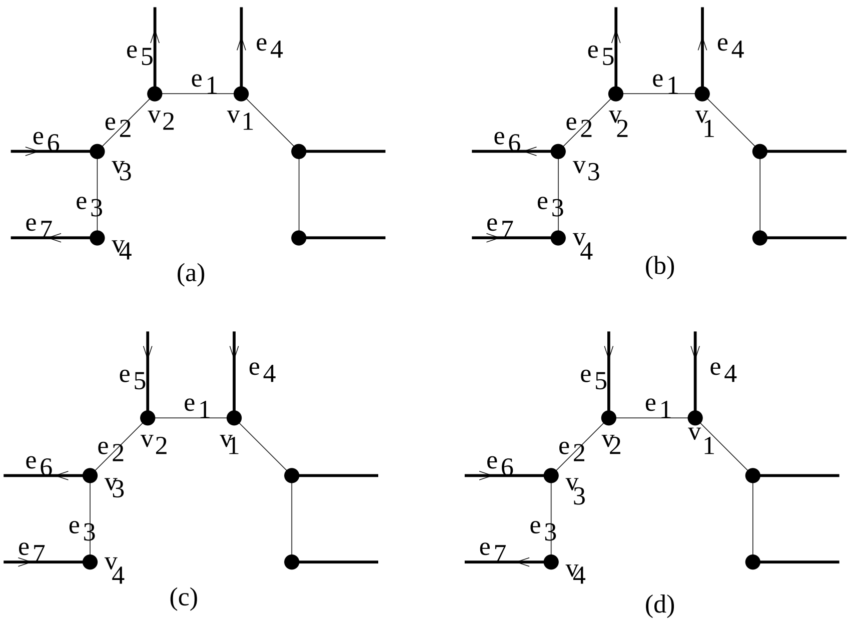

Figure 3: The blocked directions of M-edges incident to the uncolored cycle C of $G - M$

We use the following convention to interpret Figure 3 and the figures in the remaining of the paper: An M-edge without an arrow could be completely blocked, or blocked in one direction, or unblocked in both directions. An M-edge with one arrow means that the indicated direction of that edge is blocked, but the other direction of that edge could be blocked or unblocked. An M-edge with a pair of opposite arrows means that edge is completely blocked.

Consider the case indicated in Figure 3 (a) and 3 (b). We extend c' to a valid 4-edge coloring c^* of G by letting $c^*(e_1) = 3$, $c^*(e_2) = 2$, $c^*(e_3) = 1$ (the other edges of C are colored by 1 and 2 alternately). It is easy to verify that in the case indicated in Figure 3(a), e_7 is the only edge which is probably not completely blocked with respect to c^*. In Figure 3(b), e_6 is the only edge which is probably not completely blocked. Thus the M-edges incident to C contributes at most 1 to the summation $\phi(c^*) + \psi(c^*)$.

For the cases in Figure 3(c) and 3(d), let $c^*(e_1) = 1, c^*(e_2) = 2, c^*(e_3) = 3$. Then the M-edges incident to C contributes at most 1 to the summation $\phi(c^*) + \psi(c^*)$.

Next we consider the case that C has a chord.

Since C is an odd cycle, there is an M-edge incident to C which is not a chord of C. So there is a vertex v_2 of C which is incident to a chord of C and a neighbour v_1 of v_2 in C is not incident to a chord of C. Let v_3, v_4 be the vertices of C following v_1, v_2 (as shown in Figure 4).

Assume the M-edges incident to v_3, v_4 are not chords of C and have a common blocked direction, as shown in Figure 4(a) or 4(b). In the case as shown in Figure 4(a), extend c' to c^* by letting $c^*(e_1) = 1, c^*(e_2) = 2, c^*(e_3) = 3$ (and color the other edges of C alternately by colors 1 and 2). In the case as shown in Figure 4(b), extend c' to c^* by letting $c^*(e_1) = 3, c^*(e_2) = 2, c^*(e_3) = 1$. In any case, it is easy to verify that all the chords of C are completely blocked, and there is at most one M-edge incident to C which is not completely blocked.

Assume the M-edges incident to v_3, v_4 have opposite blocked directions or at least one of the M-edges incident to v_3, v_4 is a chord of C. Then depending on which direction of the M-edge incident to v_1 is blocked (with respect to c'), we color the edges as in Figure 5.

In each of the colorings, it is straightforward to verify that the M-edges incident to C contribute at

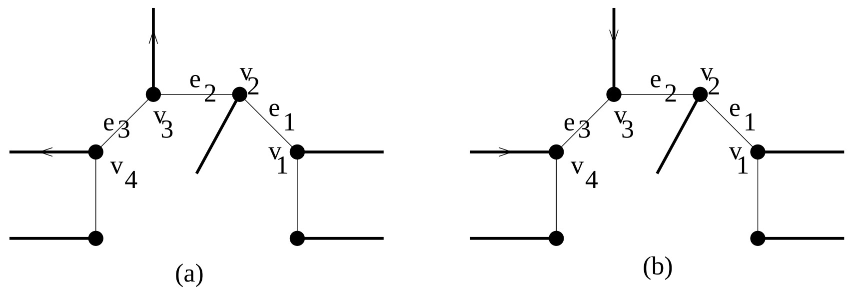

Figure 4: The M-edges incident to v_3, v_4 have a common blocked direction

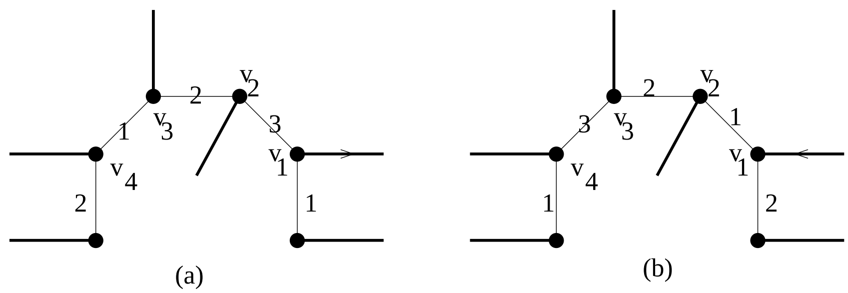

Figure 5: The M-edges incident to v_3, v_4 have an opposite blocked direction or one of the M-edges is a chord.

most 1 to the summation $\phi(c^*) + \psi(c^*)$. This completes the proof of Claim 2.3. ■

Now we choose a valid 4-edge coloring c of G such that $\phi(c) + \psi(c)$ is minimum. By Claim 2.3, no cycle C of $G - M$ has an input and an output. Since each cycle C of $G - M$ contains at most one edge of color 3, it follows that every directed path of $D_c(L(G))$ contains at most 2 vertices (i.e., edges of G) with color 3. By Lemma 2.1, $\chi_c(L(G)) = \chi'_c(G) \leq 11/3$. ■

Corollary 2.4 *If G is a 2-edge connected graph of maximum degree 3 and has girth at least 4, then $\chi'_c(G) \leq 11/3$.*

Proof. If G is cubic, then by Petersen Theorem, G has a perfect matching. Otherwise, take the disjoint union of two copies of G, say G and G'. For each degree 2 vertex x of G, connect x to the corresponding vertex x' in G' by an edge. The resulting graph G'' is cubic (as G has minimum degree 2) and is either 2-edge connected (if G has at least two degree 2 vertices), or has exactly one cut edge. In any case G'' has a perfect matching (see for example [5], page 124) and has girth at least 4. Hence $\chi'_c(G'') \leq 11/3$ by Theorem 2.2. ■

3 Proof of Theorem 1.7

We prove Theorem 1.7 by induction on the number of edges. If $|E(G)| = 3$, then it is equal to K_2^3, and has circular chromatic index 3. Assume $|E(G)| \geq 4$ and $G \neq H_1, H_2$. If G has girth at least 4, then the conclusion follows from Theorem 2.2. Thus we assume that G has a pair of parallel edges or has a triangle.

Case I: Suppose there is a pair of parallel edges between u and v. Since G is 2-edge connected and $G \neq H_1$, we conclude that u is connected to another vertex u', v is connected to another vertex v', and $u' \neq v'$. Let $G \odot uv$ be the graph obtained from G by deleting the two vertices u and v from G and adding an edge between $u'v'$. Note that this new edge may cause a multiple edge between u' and v'. If $G \odot uv \notin \{H_1, H_2\}$, then by induction hypothesis, $\chi'_c(G \odot uv) \leq 11/3$. Figure 6(a) illustrates that

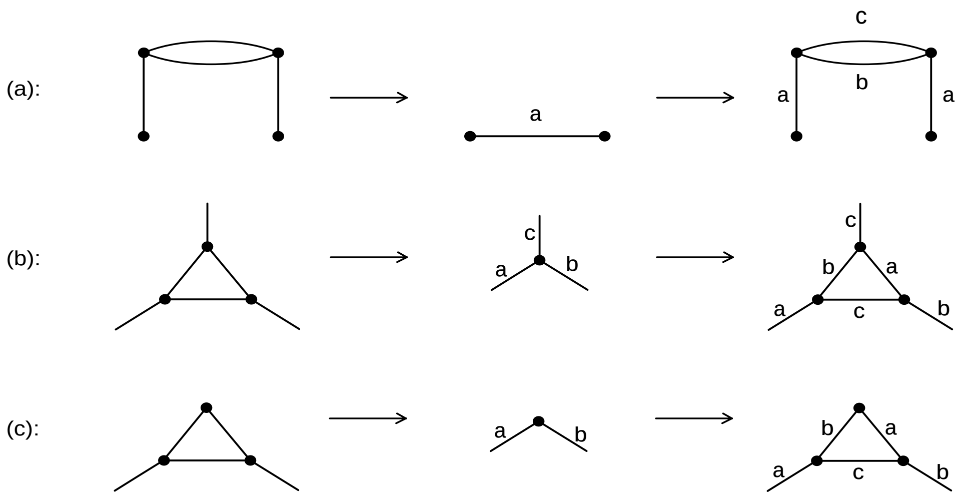

Figure 6: (a), (b), and (c) show that how a $(11/3)$-edge coloring of the new graph leads to a $(11,3)$-edge coloring of the previous one: (a): In the $(11,3)$-edge coloring of the main graph $b=(a+3)$ mod 11 and $c=(a+6)$ mod 11, (b): contracting a triangle with three vertices of degree 3, (c): after contracting a triangle with one vertex of degree 2, we can always find a color c to complete the $(11,3)$-coloring of the old graph.

a $(11,3)$-coloring of $L(G \odot uv)$ can be 'extended' to a $(11,3)$-coloring of $L(G)$. If $G \odot uv \in \{H_1, H_2\}$, then G is one of the graphs illustrated in Figure 7 or Figure 8, where a $(7,2)$-coloring of $L(G)$ is given.
Case II: Suppose G has a triangle uvw. Since G is 2-edge connected and $G \neq H_1$, there are no multiple edges in this triangle. Let $G \odot uvw$ be the graph obtained from G by contracting the triangle uvw in G to a new vertex. If $G \odot uvw \notin \{H_1, H_2\}$, then by induction hypothesis, $\chi'_c(G \odot uvw) \leq 11/3$. Figure 6(b,c) illustrates that a $(11,3)$-coloring of $L(G \odot uvw)$ can be 'extended' to a $(11,3)$-coloring of $L(G)$. If $G \odot uvw \in \{H_1, H_2\}$, then G is one of the graphs illustrated in Figure 7 or Figure 8, where a $(7,2)$-coloring of $L(G)$ is given. So in any case, $\chi'_c(G) \leq 11/3$. This completes the proof of Theorem 1.7.

Based on the result in this paper, we propose the following conjecture:

Conjecture 3.1 *For any integer $k \geq 2$, there is an $\epsilon > 0$ such that the open interval $(k-\epsilon, k)$ is a gap for circular chromatic index of graphs, i.e., no graph G has $k-\epsilon < \chi'_c(G) < k$.*

If Conjecture 3.1 is true, then let ϵ_k be the largest real number for which $(k-\epsilon_k, k)$ is a gap for the circular chromatic index of graphs. The next problem would be to determine the value of ϵ_k. For

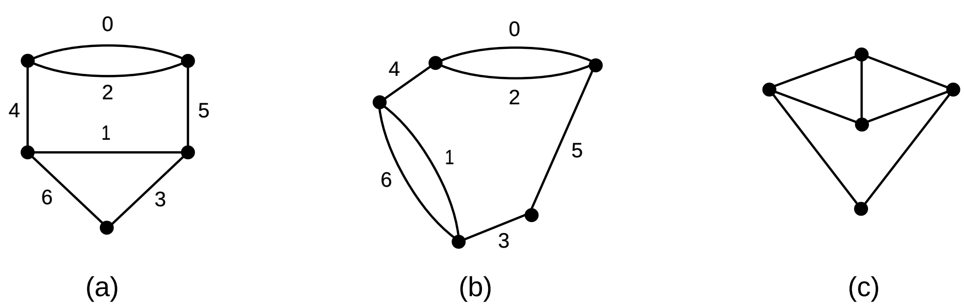

Figure 7: The graphs that can be converted to H_1 by the "$\odot$" operation. For each graph other than H_2 a $(7,2)$-edge coloring is given.

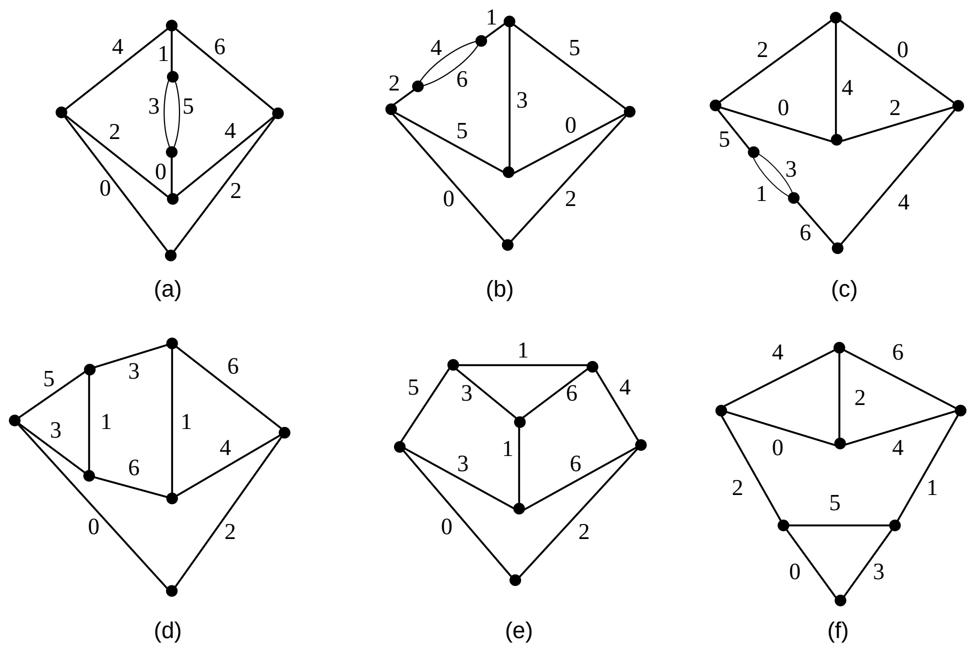

Figure 8: The graphs that can be converted to H_2 by the "$\odot$" operation. For each graph a $(7,2)$-edge coloring is given.

$k = 2, 3, 4$, Conjecture 3.1 is true and we know that $\epsilon_2 = 1, \epsilon_3 = 1/2$ and $\epsilon_4 = 1/3$. So a natural guess for ϵ_k is that $\epsilon_k = 1/(k-1)$. However, at present time, support for such a conjecture is still weak. For $k \geq 4$, we do not have natural candidate graphs G with $\chi'_c(G) = k - 1/(k-1)$.

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