Tight relation between the circular chromatic number and the girth of series-parallel graphs

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Abstract

This paper proves the sharpness of an upper bound for the circular chromatic number of series-parallel graphs of large girth.

1 Introduction

Suppose $G = (V, E)$ is a graph and $r \geq 1$ is a real number. An *r-coloring* of G is a mapping $f : V \to [0, r)$ such that for any edge xy of G, $1 \leq |f(x) - f(y)| \leq r - 1$. We say G is *r-colorable* if there exists an r-coloring of G. The circular chromatic number $\chi_c(G)$ (also called the star chromatic number) of a graph G is the infimum of those r for which G is r-colorable, i.e.,

$$\chi_c(G) = \inf\{r : \ G \text{ is } r\text{-colorable}\}.$$

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It is known [1, 13, 16] that the infimum in the definition can always be attained, and hence can be replaced by minimum. For finite graphs G, $\chi_c(G)$ is always a rational number, and for any graph G, $\chi(G)-1 < \chi_c(G) \leq \chi(G)$. So the circular chromatic number is a refinement of the chromatic number, and it contains more information about the structure of the graph.

The relation between the girth of a graph and its circular chromatic number has been studied in many papers. It was shown in [14] that for any rational number $r \geq 2$ and for any integer g, there exists a graph G of girth at least g and $\chi_c(G) = r$ (see [16] for a constructive proof of this result, and see [8, 9] for a generalization of this result). However, if restricted to H-minor free graphs G (for a fixed graph H), then it was shown in [4] that for any $\epsilon > 0$ there exists an integer $n(\epsilon)$ such that when the girth of G is at least $n(\epsilon)$ then $\chi_c(G) \leq 2 + \epsilon$. For planar graphs G, it was shown in [7] that for any $\epsilon > 0$ there exists an integer $n(\epsilon)$ such that when the odd girth of G is at least $n(\epsilon)$ then $\chi_c(G) \leq 2 + \epsilon$. In [4] and [3, 7, 17], explicit formulae for $n(1/k)$ were given. However, it seems that the bounds are far from tight.

For series-parallel graphs G, we have a better understanding of the relation between the girth and the circular chromatic number. The following result was proved in [5]:

Theorem 1.1 *If G is a series-parallel graph of girth at least $2\lfloor(3k-1)/2\rfloor$, then $\chi_c(G) \leq 4k/(2k-1)$.*

The bound on the girth given in Theorem 1.1 is sharp. The following result was proved in [2]:

Theorem 1.2 *For any $k \geq 2$, there is a K_4-minor free graph G of girth $2\lfloor(3k-1)/2\rfloor - 1$ such that $\chi_c(G) > 4k/(2k-1)$.*

However, the relation between the girth and the circular chromatic number of series-parallel graphs given in Theorem 1.1 is not tight in the following sense: there do not exist series-parallel graphs G of girth $2\lfloor(3k-1)/2\rfloor$ with $\chi_c(G)$ arbitrarily close to $4k/(2k-1)$. Indeed, the following stronger result was proved in [10]:

Theorem 1.3 *Suppose G is a series-parallel graph and $k \geq 1$ is an integer.*

1. *If G has odd girth at least $6k-1$ then $\chi_c(G) \leq 8k/(4k-1)$;*

2. *If* G *has odd girth at least* $6k+1$ *then* $\chi_c(G) \leq (4k+1)/2k$*;*

3. *if* G *is has odd girth at least* $6k+3$ *then* $\chi_c(G) \leq (4k+3)/(2k+1)$*.*

Theorem 1.3 strengthens Theorem 1.1 in two aspects: (1) replace the girth requirement with an odd girth requirement, (2) give a tighter relation between the girth and the circular chromatic number. In this paper, we prove that Theorem 1.3 is tight in the strongest sense:

Theorem 1.4 *Let* $k \geq 1$ *be an integer, and let* $\epsilon > 0$*.*

1. *There exists a series-parallel graph* G *of girth* $6k-1$ *with* $\chi_c(G) > 8k/(4k-1) - \epsilon$*;*

2. *There exists a series-parallel graph* G *of girth* $6k+1$ *with* $\chi_c(G) > (4k+1)/2k - \epsilon$*;*

3. *There exists a series-parallel graph* G *of girth* $6k+3$ *with* $\chi_c(G) > (4k+3)/(2k+1) - \epsilon$*.*

2 Labeling method

The proof of Theorem 1.4 uses the labeling method which was developed in [2, 5, 6, 10, 11, 12, 15]. First we recall the following definition from [5]. A two-terminal series-parallel graph $(G; x, y)$ is defined recursively as follows:

- Let $V(K_2) = \{0, 1\}$. Then $(K_2; 0, 1)$ is a two-terminal series-parallel graph.
- (The parallel join.) Let $(G; x, y)$ and $(G'; x', y')$ be two disjoint two-terminal series-parallel graphs. Let G'' be obtained from the union of G and G' by identifying x and x' into a single vertex x'', and identifying y and y' into a single vertex y''. Then $(G''; x'', y'')$ is a two-terminal series-parallel graph, which is called the parallel join of G and G'
- (The series join.) Let again $(G; x, y)$ and $(G'; x', y')$ be two disjoint two-terminal series-parallel graphs. Let G'' be obtained from the union of G and G' by identifying y and x' into a single vertex. Then $(G''; x, y')$ is a two-terminal series-parallel graph, which is called the series join of G and G'.

- There are no other two-terminal series-parallel graphs.

Definition 2.1 *Suppose $(G;x,y)$ is a two-terminal series-parallel graph. The* length $\ell(G;x,y)$ *of G is the distance in G between the two terminals x,y.*

If the two terminals are understood, or are of no significance, we shall write G for $(G;x,y)$, and write $\ell(G)$ for $\ell(G;x,y)$.

To prove Theorem 1.4, we need to construct series-parallel graphs of girth at least $6k-1$ (resp. $6k+1$ or $6k+3$), which are not r-colorable for $r = 8k/(4k-1)-\epsilon$ (resp. $r=(4k+1)/2k-\epsilon$ or $r=(4k+3)/(2k+1)-\epsilon$). To maintain that the constructed series-parallel graphs have girth at least $6k-1$ (resp. $6k+1$ or $6k+3$), we can only do parallel joins for pairs of graphs G and G' such that the sum of the lengths of the two graphs is at least $6k-1$ (resp. $6k+1$ or $6k+3$).

Definition 2.2 *Suppose $r \geq 2$ is a real number and $(G;x,y)$ is a two-terminal series-parallel graph. The* r-label set $L_r(G;x,y)$ *(usually denoted by $L_r(G)$) is defined as follows:*

$$L_r(G) = \{t \in [0,r) : \exists r\text{-coloring } f \text{ of } G \text{ with } f(x)=0 \text{ and } f(y)=t\}.$$

It is obvious that if G is r-colorable then there exists an r-coloring f of G such that $f(x)=0$. Therefore G is r-colorable if and only if $L_r(G) \neq \emptyset$. So to determine the circular chromatic number of a two-terminal series-parallel graph G, it amounts to determine the least r for which $L_r(G) \neq \emptyset$. (This is also the case for non-series-parallel two-terminal graphs). Such a method for determining the circular chromatic number of a graph is called the *labeling method*, which has been successfully used in many cases [2, 5, 6, 10, 11, 12].

For a real number $r \geq 2$, denote by C^r the circle obtained from the interval $[0,r)$ by identifying the two end points. For two distinct real numbers $a,b \in [0,r)$, we denote by $[a,b]_r$ the interval of C^r from a to b along the clockwise direction. To be precise, if $a<b$ then $[a,b]_r = \{t : a \leq t \leq b\}$; if $a>b$ then $[a,b]_r = \{t : a \leq t < r\} \cup \{t : 0 \leq t \leq b\}$. When r is clear from the context, we write $[a,b]$ for $[a,b]_r$. The *length*, $\ell([a,b])$, of an interval $[a,b]$ is the geometric length of $[a,b]$ on the circle C^r, i.e., $\ell([a,b]) = b-a$ if $b>a$, and $\ell([a,b]) = r+b-a$ if $b<a$.

Fig. 1 below illustrates the circle $C^{3.5}$ and some intervals on this circle.

For $r>0$ and for any real number x, denote by $x \bmod r$ the remainder of x divided by r, i.e., $x \bmod r$ is the unique real number $0 \leq x' < r$ such that

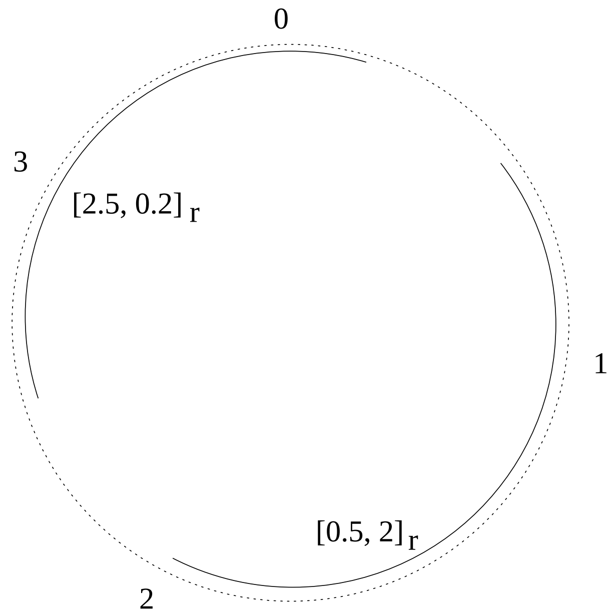

Figure 1: Illustration of the circle C^r for $r = 3.5$ and intervals of C^r

$x - x'$ is a multiple of r. Given two subsets $A, B \subseteq [0, r)$, the sum $A + B$ is defined as

$$A + B = \{a + b \bmod r : a \in A,\ b \in B\}.$$

The following lemmas are quoted from [11], and their proofs are straightforward.

Lemma 2.1 *Suppose* $A_i = [a_1, b_1]$ *for* $i = 1, 2, \cdots, t$. *If* $\sum_{i=1}^{t} \ell([a_i, b_i]) \geq r$ *then* $A_1 + A_2 + \cdots + A_t = C^r$; *if* $\sum_{i=1}^{t} \ell([a_i, b_i]) < r$ *then* $A_1 + A_2 + \cdots + A_t = [a_1 + a_2 + \cdots + a_t, b_1 + b_2 + \cdots + b_t]$, *where the summations are carried out modulo* r.

Note that if $A = \emptyset$ then for any set B, $A + B = \emptyset$.

Lemma 2.2 *If* G^* *is the series join of* G *and* G', *then*

$$L_r(G^*) = L_r(G) + L_r(G');$$

if G^* *is the parallel join of* G *and* G', *then*

$$L_r(G^*) = L_r(G) \cap L_r(G').$$

Lemma 2.3 *For any two-terminal series-parallel graph* $(G; x, y)$, $t \in L_r(G)$ *if and only if* $r - t \bmod r \in L_r(G)$.

3 Proof of Theorem 1.4

The proof of Theorem 1.4 is divided into a few lemmas. The proofs of each of these lemmas are similar. However, the differences in the calculations make it difficult to combine them into a single proof. We shall exhibit the detailed calculations for the first case, and for the other cases, we describe the constructions of the corresponding graphs and their r-label sets.

Lemma 3.1 *Suppose $m \geq 1$ is an integer, and $\epsilon > 0$. Let $r = \frac{16m-8}{8m-5} - \epsilon$. Then there exists a series-parallel graph G of girth $12m-7$ such that $\chi_c(G) > r$.*

Proof. It suffices to construct a two-terminal series-parallel graph $(G; x, y)$ of girth $12m - 7$ such that $L_r(G) = \emptyset$. For this purpose, we shall construct a sequence of two-terminal series-parallel graphs $G_0, G_1, G_2, \cdots, G_p, \cdots$ such that the label sets $L_r(G_i)$ become smaller and smaller, until it eventually becomes the empty set. To be precise, we have the following claim:

Claim 3.1 *Suppose $r < \frac{16m-8}{8m-5}$. For each integer $p \geq 0$, there exists a two-terminal series-parallel G_p of girth at least $12m - 7$ and length $6m - 4$ such that the following is true:*

- *If $r < \frac{p(8m-5)+4m-2}{p(4m-3)+2m-1}$, then $L_r(G_p) = \emptyset$;*
- *if $r > \frac{p(8m-5)+4m-2}{p(4m-3)+2m-1}$, then*

$$\begin{aligned} L_r(G_p) &= [p(8m-5) + 6m - 4 - (p(4m-3) + 3m - 3)r, \\ &\quad (p(4m-3) + 3m - 2)r - (p(8m-5) + 6m - 4)]. \end{aligned}$$

It is easy to see that the conclusion of Lemma 3.1 follows from Claim 3.1. So it remains to prove Claim 3.1. The proof will be by induction on p.

For an integer $k \geq 1$, let P_k be the path of length k. It follows from Lemma 2.2 that if $2 < r < \frac{4k}{2k-1}$, then

$$L_r(P_{2k}) = [2k - (k-1)r, kr - 2k];$$

and if $2 < r < \frac{2k+1}{k}$, then

$$L_r(P_{2k+1}) = [2k + 1 - kr, (k+1)r - (2k+1)];$$

Let $G_0 = P_{6m-4}$. Then G_0 is acyclic (and hence has infinite girth), and has length $6m-4$. If $r < 2$ then certainly $L_r(G_0) = \emptyset$. If $r > 2$, then $L_r(G_0) = [6m-4-(3m-3)r, (3m-2)r-(6m-4)]$. This proves Claim 3.1 for $p = 0$.

Suppose we have constructed G_p having the properties described in Claim 3.1. The construction of G_{p+1} is as depicted in Fig. 2 below:

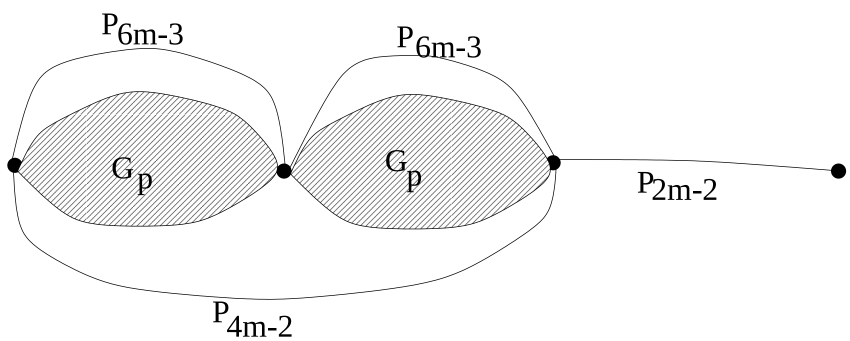

Figure 2: Construction of G_{p+1}

It is obvious that G_{p+1} has girth $12m-7$ and length $6m-4$. Now we shall calculate the r-label set of G_{p+1}. Let H be the parallel join of G_p and P_{6m-3}. By Lemma 2.2,

$$\begin{aligned} L_r(H) &= L_r(G_p) \cap L_r(P_{6m-3}) \\ &= [p(8m-5)+6m-4-(p(4m-3)+3m-3)r, \\ &\quad (p(4m-3)+3m-2)r-(p(8m-5)+6m-4)] \\ &\quad \cap [6m-3-(3m-2)r, \quad (3m-1)r-(6m-3)]. \end{aligned}$$

If $r < \frac{(p+1)(8m-5)+4m-2}{(p+1)(4m-3)+2m-1}$, then

$$(p(4m-3)+(3m-2))r-(p(8m-5)+6m-4) < 6m-3-(3m-2)r.$$

Therefore, the intersection is empty, as illustrated in Fig. 3 below (note that by Lemma 2.3, the label set is symmetric).

If $r > \frac{(p+1)(8m-5)+4m-2}{(p+1)(4m-3)+2m-1}$, then

$$\begin{aligned} L_r(H) &= [6m-3-(3m-2)r, \ (p(4m-3)+3m-2)r-(p(8m-5)+6m-4)] \\ &\cup [p(8m-5)+6m-4-(p(4m-3)+3m-3)r, \ (3m-1)r-(6m-3)]. \end{aligned}$$

Let H' be the series join of two copies of H. By Lemma 2.2,

$$L_r(H') = L_r(H) + L_r(H).$$

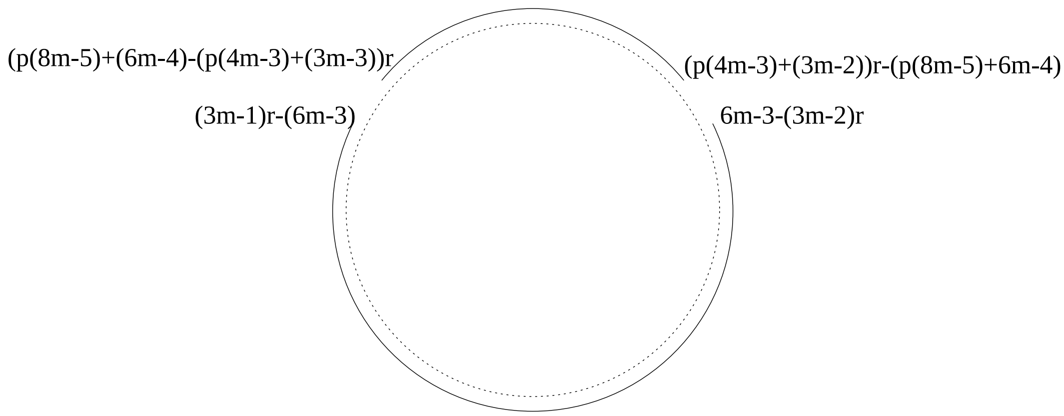

Figure 3: Illustration of $L_r(G_p) \cap L_r(P_{6m-3})$ for "small" r

Let

$$\begin{aligned}
u &= p(8m-5)+12m-7-(p(4m-3)+6m-5)r, \\
v &= (p(4m-3)+6m-4)r-(p(8m-5)+12m-7), \\
a &= 2(6m-3-(3m-2)r), \\
b &= 2((p(4m-3)+(3m-2))r-(p(8m-5)+6m-4)), \\
c &= 2(p(8m-5)+6m-4-(p(4m-3)+(3m-3))r)-r, \\
d &= 2((3m-1)r-(6m-3))-r.
\end{aligned}$$

Then

$$L_r(H') = [a,\ b] \cup [c,\ d] \cup [u,\ v].$$

Let H'' be the parallel join of H' and P_{4m-2}. Then

$$L_r(H'') = L_r(H') \cap L_r(P_{4m-2}).$$

At first glance, this intersection seems very complicated. However, the positions of the involved intervals are nicely matched, and the intersection turns out to be a single interval. Recall that

$$L_r(P_{4m-2}) = [4m-2-(2m-2)r,\ (2m-1)r-(4m-2)].$$

Using the assumption that $r < \frac{16m-8}{8m-5}$, it can be verified that

$$\begin{aligned}
a &> (2m-1)r-(4m-2), \\
c &> (2m-1)r-(4m-2), \\
v &< (2m-1)r-(4m-2).
\end{aligned}$$

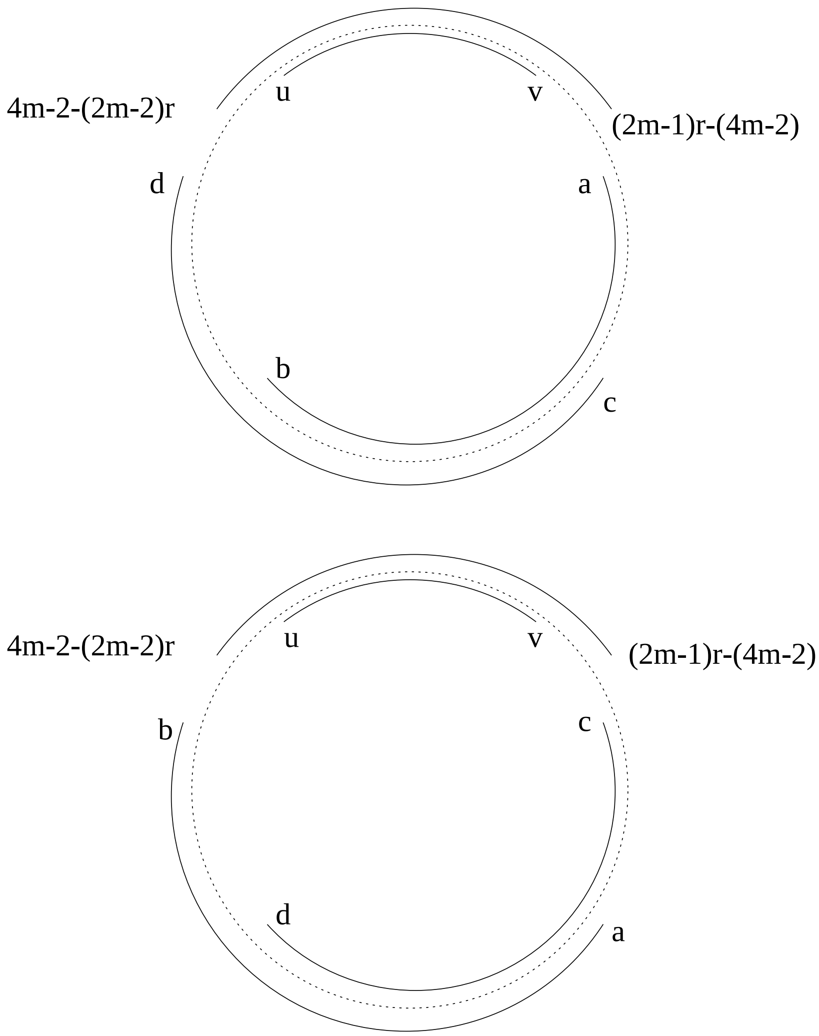

Figure 4: The relative positions of involved intervals

Therefore the relative positions of the involved intervals are as depicted in Fig. 4(a) or Fig. 4(b).

Therefore,

$$L_r(H'') \;=\; [u, v].$$

Referring to Fig. 2, we see that G_{p+1} is the series join of H'' and P_{2m-2}. Therefore,

$$L_r(G_{p+1}) \;=\; L_r(H'') + L_r(P_{2m-2}).$$

Recall that $L_r(P_{2m-2}) = [2m-2-(m-2)r,\ (m-1)r-(2m-2)]$. Using Lemma 2.1 with the values of u and v, we have

$$\begin{aligned} L_r(G_{p+1}) &= [(p+1)(8m-5)+6m-4-((p+1)(4m-3)+3m-3)r, \\ &\quad ((p+1)(4m-3)+3m-2)r-((p+1)(8m-5)+6m-4)]. \end{aligned}$$

This completes the proof of Claim 3.1, as well as the proof of Lemma 3.1. ∎

Lemma 3.2 *Suppose $m \geq 1$ is an integer, and $\epsilon > 0$. Let $r = \frac{8m-3}{4m-2} - \epsilon$. Then there exists a series-parallel graph G of girth $12m-5$ such that $\chi_c(G) > r$.*

Lemma 3.2 follows from the following claim:

Claim 3.2 *Suppose $r < \frac{8m-3}{4m-2}$. For each integer $p \geq 0$, there exists a two-terminal series-parallel G_p of girth at least $12m-5$ and length $6m-2$ such that the following is true:*

- *If $r < \frac{p(8m-3)+4m-2}{p(4m-2)+2m-1}$, then $L_r(G_p) = \emptyset$;*
- *if $r > \frac{p(8m-3)+4m-2}{p(4m-2)+2m-1}$, then*

$$\begin{aligned} L_r(G_p) &= [p(8m-3)+6m-2-(p(4m-2)+3m-2)r, \\ &\quad (p(4m-2)+3m-1)r-(p(8m-3)+6m-2)]. \end{aligned}$$

The proof of Claim 3.2 is by induction on p. For $p=0$, $G_0 = P_{6m-2}$. Suppose G_p has been constructed, then the construction of G_{p+1} is as indicated in Fig. 5.

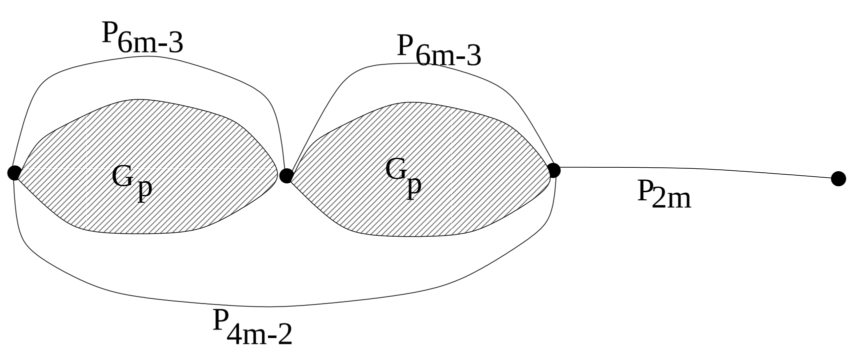

Figure 5: Construction of G_{p+1} for Claim 3.2

The detailed calculations are similar to the proof of Claim 3.1 and omitted.

Lemma 3.3 *Suppose $m \geq 1$ is an integer, and $\epsilon > 0$. Let $r = \frac{8m-1}{4m-1} - \epsilon$. Then there exists a series-parallel graph G of girth $12m-3$ such that $\chi_c(G) > r$.*

Lemma 3.3 follows from the following claim:

Claim 3.3 *Suppose $r < \frac{8m-1}{4m-1}$. For each integer $p \geq 0$, there exists a two-terminal series-parallel G_p of girth at least $12m-3$ and length $6m-2$ such that the following is true:*

- *If $r < \frac{p(8m-1)+4m-2}{p(4m-1)+2m-1}$, then $L_r(G_p) = \emptyset$;*

- *if $r > \frac{p(8m-1)+4m-2}{p(4m-1)+2m-1}$, then*

$$\begin{aligned} L_r(G_p) &= [p(8m-1)+6m-2-(p(4m-1)+3m-2)r, \\ &\quad (p(4m-1)+3m-1)r-(p(8m-1)+6m-2)]. \end{aligned}$$

The proof of Claim 3.3 is by induction on p. For $p = 0$, $G_0 = P_{6m-2}$. Suppose G_p has been constructed, then the construction of G_{p+1} is as indicated in Fig. 6.

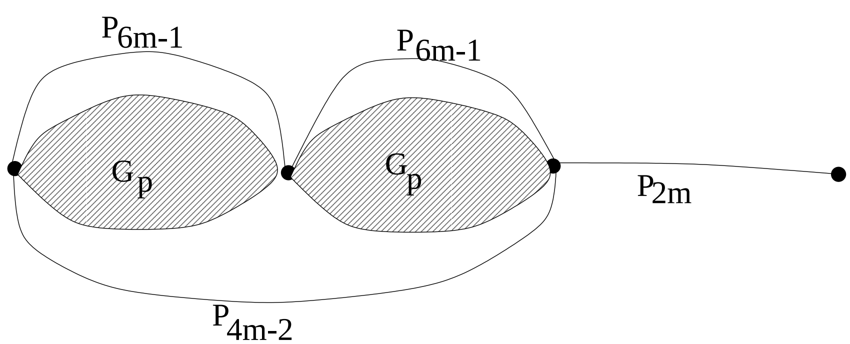

Figure 6: Construction of G_{p+1} for Claim 3.3

Lemma 3.4 *Suppose $m \geq 1$ is an integer, and $\epsilon > 0$. Let $r = \frac{16m}{8m-1} - \epsilon$. Then there exists a series-parallel graph G of girth $12m-1$ such that $\chi_c(G) > r$.*

Lemma 3.4 follows from the following claim:

Claim 3.4 *Suppose $r < \frac{16m}{8m-1}$. For each integer $p \geq 0$, there exists a two-terminal series-parallel G_p of girth at least $12m-1$ and length $6m$ such that the following is true:*

- *If* $r < \frac{p(8m-1)+4m}{p(4m-1)+2m}$*, then* $L_r(G_p) = \emptyset$*;*

- *if* $r > \frac{p(8m-1)+4m}{p(4m-1)+2m}$*, then*

$$\begin{aligned} L_r(G_p) &= [p(8m-1)+6m-(p(4m-1)+3m-1)r, \\ &\quad (p(4m-1)+3m)r-(p(8m-1)+6m)]. \end{aligned}$$

The proof of Claim 3.4 is by induction on p. For $p = 0$, $G_0 = P_{6m}$. Suppose G_p has been constructed, then the construction of G_{p+1} is as indicated in Fig. 7.

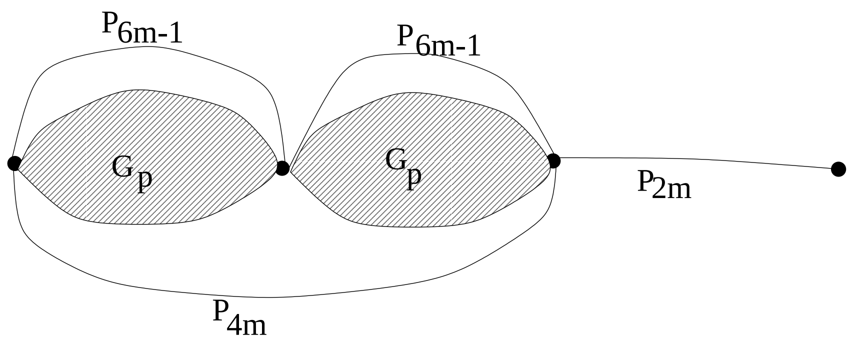

Figure 7: Construction of G_{p+1} for Claim 3.4

Lemma 3.5 *Suppose* $m \geq 1$ *is an integer, and* $\epsilon > 0$*. Let* $r = \frac{8m+1}{4m} - \epsilon$*. Then there exists a series-parallel graph* G *of girth* $12m+1$ *such that* $\chi_c(G) > r$*.*

Lemma 3.5 follows from the following claim:

Claim 3.5 *Suppose* $r < \frac{8m+1}{4m}$*. For each integer* $p \geq 0$*, there exists a two-terminal series-parallel* G_p *of girth at least* $12m+1$ *and length* $6m$ *such that the following is true:*

- *If* $r < \frac{p(8m+1)+4m}{4pm+2m}$*, then* $L_r(G_p) = \emptyset$*;*

- *if* $r > \frac{p(8m+1)+4m}{4pm+2m}$*, then*

$$\begin{aligned} L_r(G_p) &= [p(8m+1)+6m-(4pm+3m-1)r, \\ &\quad (4pm+3m)r-(p(8m+1)+6m)]. \end{aligned}$$

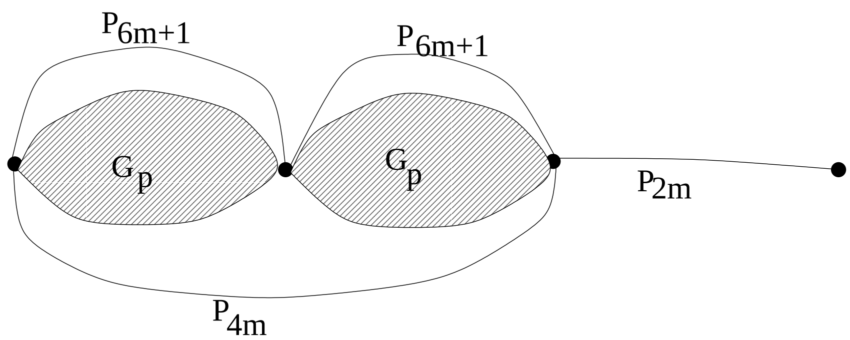

Figure 8: Construction of G_{p+1} for Claim 3.5

The proof of Claim 3.5 is by induction on p. For $p = 0$, $G_0 = P_{6m}$. Suppose G_p has been constructed, then the construction of G_{p+1} is as indicated in Fig. 8.

Lemma 3.6 *Suppose $m \geq 1$ is an integer, and $\epsilon > 0$. Let $r = \frac{8m+3}{4m+1} - \epsilon$. Then there exists a series-parallel graph G of girth $12m + 3$ such that $\chi_c(G) > r$.*

Lemma 3.6 follows from the following claim:

Claim 3.6 *Suppose $r < \frac{8m+3}{4m+1}$. For each integer $p \geq 0$, there exists a two-terminal series-parallel G_p of girth at least $12m + 3$ and length $6m + 2$ such that the following is true:*

- *If $r < \frac{p(8m+3)+4m}{p(4m+1)+2m}$, then $L_r(G_p) = \emptyset$;*
- *if $r > \frac{p(8m+3)+4m}{p(4m+1)+2m}$, then*

$$\begin{aligned} L_r(G_p) \;=\; & [p(8m+3) + 6m + 2 - (p(4m+1) + 3m)r, \\ & (p(4m+1) + 3m + 1)r - (p(8m+3) + 6m + 2)]. \end{aligned}$$

The proof of Claim 3.6 is by induction on p. For $p = 0$, $G_0 = P_{6m+2}$. Suppose G_p has been constructed, then the construction of G_{p+1} is as indicated in Fig. 9.

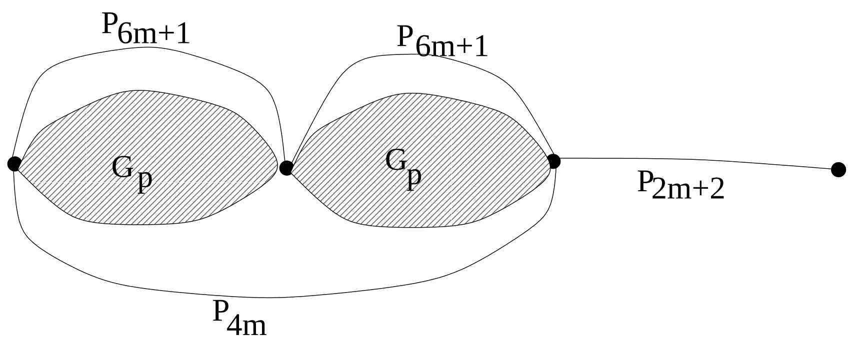

Figure 9: Construction of G_{p+1} for Claim 3.6

References

[1] J. A. Bondy and P. Hell, *A note on the star chromatic number*, J. Graph Theory **14** (1990), 479-482.

[2] C. Chien and X. Zhu, *The circular chromatic numbers of series-parallel graphs of large girth*, J. Graph Theory, 33(2000), 185-198.

[3] G. Fijavž, M. Juvan, B. Mohar* and R. Škrekovski, *Circular colorings of planar graphs with prescribed girth*, manuscript, 2001.

[4] A. Gallucio, L. Goddyn and P. Hell, *High girth graphs avoiding a minor are nearly bipartite*, J. Combinatorial Theory (B), to appear.

[5] P. Hell and X. Zhu, *The circular chromatic numbers of series-parallel graphs*, J. Graph Theory, 33(2000), 14-24.

[6] S. Liaw, Z. Pan and X. Zhu, *Construction of K_n-minor free graphs with given circular chromatic number*, manuscript, 1999.

[7] W. Klostermeyer and C. Q. Zhang, $2+\epsilon$*-coloring of planar graphs with large odd girth*, J. Graph Theory, 33(2000), 109-119.

[8] J. Nešetřil and X. Zhu, *On Sparse Graphs with Given Colorings and Homomorphisms*, J. Combinatorial Theory (B), to appear.

[9] J. Nešetřil and X. Zhu, *Construction of sparse graphs with prescribed circular colorings*, Discrete Mathematics, 233(2001), 277-291.

[10] Z. Pan and X. Zhu, *Circular chromatic number of series-parallel graphs of large odd girth*, Discrete Mathematics, to appear.

[11] Z. Pan and X. Zhu, *Density of the circular chromatic numbers of series-parallel graphs*, manuscript, 1999.

[12] Z. Pan and X. Zhu, *Construction of graphs with given circular flow number*, manuscript, 2000.

[13] A. Vince, *Star chromatic number*, J. Graph Theory **12** (1988), 551-559.

[14] X. Zhu, *Uniquely H-colorable graphs with large girth*, J. Graph Theory, **23** (1996), 33-41.

[15] X. Zhu, *Circular colouring and graph minors*, Taiwanese Journal of Mathematics, 4(2000), 643-660.

[16] X. Zhu, *Circular chromatic number, a survey*, Discrete Mathematics, 229 (1-3)(2001), 371-410.

[17] X. Zhu, *Circular chromatic number of planar graphs of large odd girth*, Electronic Journal of Combinatorics, 8(2001), #R25, 1-11.