Every 2-choosable graph is circular consecutive 2-choosable

Zhishi Pan[*] Xuding Zhu [†]

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Abstract

Suppose G is a graph, r is a positive real number and $S(r)$ is a circle of perimeter r. For a positive real number $t \leq r$, a (t,r)-circular consecutive colour-list assignment L is a mapping that assigns to each vertex v of G an interval $L(v)$ of $S(r)$ of length t. A circular L-colouring of G is a mapping $f : V(G) \to S(r)$ such that for each vertex v, $f(v) \in L(v)$ and for each edge uv, the distance between $f(u)$ and $f(v)$ in $S(r)$ is at least 1. A graph G is called circular consecutive t-choosable if for any $r \geq \chi_c(G)$, for any (t,r)-circular consecutive colour-list assignment L, G has a circular L-colouring. This paper proves that every 2-choosable graph is circular consecutive 2-choosable.

1 Introduction

For a positive real number r, let $\cong_r$ be the equivalence relation on $\mathbb{R}$ defined as $x \cong_r y$ if $x - y$ is a multiple of r. Let $S(r) = \mathbb{R}/\cong_r$, which is viewed as a circle of perimeter r. Elements of $S(r)$ are equivalence classes of $\cong_r$. However, for convenience, any real number x is used to denote the equivalence class in $S(r)$ containing x. In particular, for $x, y \in S(r)$, $x + y$ and $x - y$ are defined. For an element of $S(r)$ (which is an equivalence class), we usually use the unique member in the interval $[0, r)$ of the equivalence class as its

[*]Department of Applied Mathematics, National Sun Yat-sen University. Grant number: NSC95-2811-M-110-008

[†]Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, Taiwan 80424, and National Center for Theoretical Sciences. Grant number: NSC95-2115-M-110-013-MY3

representative. In this sense, we may also view $S(r)$ as obtained from the interval $[0, r]$ by identifying 0 and r into a single point.

For $x \in \mathbb{R}$, we denote by $[x]_r$ the real number which is the remainder of x upon division by r. When real numbers stand for elements of $S(r)$, they are cyclically ordered, but not linearly ordered. However, real numbers themselves are linearly ordered. In particular, we will frequently need to compare the magnitude of $[x]_r$ with that of another real number. It should be clear from the context whether a real number stands for an element of $S(r)$ or not.

For $a, b \in S(r)$, $[a, b]_r$ is the interval of $S(r)$ from a to b along the "increasing direction". To be precise, $[a, b]_r = \{t \in S(r) : [t-a]_r \leq [b-a]_r\}$. For example, if $r = 4$, then $[2, 3]_r = [2, 3]$ and $[3, 2]_r = [3, 4) \cup [0, 2]$. The interval $(a, b)_r$ is defined similarly. The *length* of the interval $[a, b]_r$ is equal to $[b - a]_r$. The *distance* $|a - b|_r$ between a and b is the length of the shorter interval in $S(r)$ connecting a and b, i.e.,

$$|a - b|_r = \min\{[b - a]_r, [a - b]_r\}.$$

Suppose $G = (V, E)$ is a graph. A *circular r-colouring* of G is a mapping $f : V(G) \to S(r)$ such that for any edge uv of G, $|f(u) - f(v)|_r \geq 1$. The *circular chromatic number* $\chi_c(G)$ of G is defined as

$$\chi_c(G) = \inf\{r : G \text{ has a circular } r\text{-colouring}\}.$$

The circular chromatic number of a graph is a refinement of the chromatic number of a graph. It provides an accurate model for many periodical scheduling problems, and has been studied extensively in the literature. Readers are referred to [11, 12] for surveys on this subject.

Given a graph G and a positive real number r, a *$(\star, r)$-circular colour-list assignment* for G is a function L that assigns to each vertex v of G a set $L(v)$ which is the union of disjoint closed intervals of $S(r)$. If for each vertex v of G, the sum of the lengths of the disjoint intervals of $L(v)$ is equal to t, then L is called a *(t, r)-circular colour-list assignment*. Suppose L is a $(\star, r)$-circular colour-list assignment for a graph G. A *circular L-colouring* of G is a circular r-colouring f of G such that $f(v) \in L(v)$ for each vertex v of G. A graph G is called *circular t-choosable* if for any r and for any (t, r)-circular colour-list assignment L, G has a circular L-colouring. The *circular choosability* $ch_c(G)$ of G (also called the *circular list chromatic number* of G

and denoted by $\chi_{c,l}(G)$) is defined in [13] as

$$ch_c(G) = \inf\{t : \ G \text{ is circular } t\text{-choosable}\}.$$

Circular list colouring of graphs has been studied in [2, 5, 6, 8, 13]. Recall that a graph G is *k-choosable* if for any colour list assignment L which assigns to each vertex x of G a set $L(x)$ of k positive integers, there is a colouring $f : V(G) \rightarrow \mathbb{N}$ such that $f(x) \in L(x)$ for every vertex x, and $f(x) \neq f(y)$ for every edge xy. The *choosability* or the *list chromatic number* $ch(G)$ of G is the least integer k for which G is k-choosable. It is shown in [13] that $ch(G) \leq \lceil ch_c(G) \rceil$, and for any $\epsilon > 0$, there are graphs G for which $ch_c(G) > 2\ ch(G) - \epsilon$. However, it is unknown if $ch_c(G) \leq \alpha\ ch(G)$ for some constant α.

The circular consecutive choosability of a graph is a variation of circular choosability of a graph, introduced in [3]. A $(\star, r)$-circular colour-list assignment L of G is called a *$(\star, r)$-circular consecutive colour-list assignment* if for each vertex v, $L(v)$ consists of a single closed interval of $S(r)$. If $L(v)$ has length t for each vertex v, then L is called a *(t, r)-circular consecutive colour-list assignment* of G (abbreviated as (t, r)-CCCL assignment). We say G is *circular consecutive (t, r)-choosable* if G is circular L-colourable for any (t, r)-CCCL assignment L of G.

Definition 1. *Suppose $r \geq \chi_c(G)$. The circular consecutive choosability of G with respect to r is defined as*

$$ch^r_{cc}(G) = \inf\{t : G \text{ is circular consecutive } (t, r)\text{-choosable}\}.$$

The circular consecutive choosability of G is defined as

$$ch_{cc}(G) = \sup\{ch^r_{cc}(G) : r \geq \chi_c(G)\}.$$

Equivalently, $ch_{cc}(G)$ is the infimum of those t such that for any $r \geq \chi_c(G)$, G is circular consecutive (t, r)-choosable.

One may view the vertices of G as jobs to be scheduled periodically with period r. Adjacent vertices represent jobs that cannot be carried out simultaneously (so the starting time of the two jobs need to be far apart). Each job x has a time interval $L(x)$ of length t in which it can be started. To find an schedule for the jobs is then to find an L-colouring of the graph G.

Another motivation for the study of circular choosability of graphs is the application in inductive proofs for upper bounds of the circular chromatic number of graphs. To prove a graph G is circular r-colourable, one may find an induced subgraph H of G, find a circular r-colouring f of $G - H$ (by induction hypothesis), then extend f to a circular r-colouring of H to obtain a circular r-colouring of G. In the extension, the colours available to vertices of H are restricted. Thus we are facing with a circular list colouring problem. Such techniques have been used in the study of the circular chromatic number of planar graphs of large girth in many papers. In the inductive proof described above, if a vertex x of H is adjacent to one coloured vertex in G, the set of available colours to x is an interval of $S(r)$. In this sense, circular consecutive choosability of graphs may also be used in certain inductive proofs of the circular colourability of graphs.

Circular consecutive choosability is also a generalization of the consecutive choosability of a graph introduced by Waters [10]. A graph G is defined in [10] to be consecutive t-choosable, if for any L which assigns to each vertex v of G an interval $L(v)$ of $\mathbb{R}$ of length t, there is a colouring $f : V(G) \to \mathbb{R}$ such that for each vertex v, $f(v) \in L(v)$ and for each edge uv of G, $|f(u) - f(v)| \geq 1$. We may view $\mathbb{R}$ as a circle of infinite perimeter. Thus consecutive t-choosability of a graph is the same a circular consecutive (t, ∞)-choosable. Indeed, it is shown in [3] that if G is an n-vertex graph, then for $r \geq n^2 + 1$, then G is consecutive t-choosable if and only if G is circular consecutive (t, r)-choosable.

The parameter $ch_{cc}(G)$ was studied in [3] and [7]. It was shown in [3] that if G is a graph on n vertices, then

$$\chi(G) - 1 \leq ch_{cc}(G) \leq 2\chi_c(G)(1 - 1/n) - 1.$$

The value of $ch_{cc}(G)$ for complete graphs, trees, even cycles and balanced complete bipartite graphs are determined there. Upper and lower bounds for $ch_{cc}(G)$ are given for some other graphs. In particular, it was shown in [7] that for $k \geq 1$, k-choosable graphs G have $ch_{cc}(G) \leq k + 1 - 1/k$ and this upper bound is tight for $k \geq 3$.

The question of finding the tight upper bound for $ch_{cc}(G)$ for 2-choosable graphs remained an open problem. The bound that k-choosable graphs have $ch_{cc}(G) \leq k+1-1/k$ is not tight for $k = 2$. It was shown in [6] that bipartite graphs G have $ch_c(G) \leq \mathrm{mad}(G)$, where $\mathrm{mad}(G) = \max\{2|E(H)|/|V(H)| : H \text{ is a subgraph of } G\}$. It follows from the characterization of 2-choosable graphs (see Section 2), a connected n-vertex 2-choosable graph G is bipartite and has $\mathrm{mad}(G) \leq 2(n+1)/n$. So if G is a connected 2-choosable n-vertex graph, then $ch_{cc}(G) \leq ch_c(G) \leq 2 + 2/n$. It is proved in [4] that the theta graph $\theta_{2,2,4}$ (see definition in Section 2) is circular consecutive 2-choosable. As a consequence, any 2-choosable graph G on at most 8 vertices has $ch_{cc}(G) \leq 2$. This implies that 2-choosable graphs G have $ch_{cc}(G) \leq 20/9$ [4]. It was conjectured in [7] that the tight upper bound should be 2, i.e., 2-choosable graphs G have $ch_{cc}(G) \leq 2$. This paper confirms this conjecture. As a consequence, for any 2-choosable graph G, the circular consecutive choosability $ch_{cc}(G)$ of G can be determined in linear time.

2 The theta graphs

Choosability of graphs was first studied in [1], where 2-choosable graphs are characterized. Given a graph G, the *heart* of G is the graph H obtained from G by repeatedly deleting degree 1 vertices. For positive integers a, b, c, the theta graph $\theta_{a,b,c}$ is the graph obtained from three disjoint paths $P_1 = (x_0, x_1, \ldots, x_a), P_2 = (y_0, y_1, \ldots, y_b)$ and $P_3 = (z_0, z_1, \ldots, z_c)$ by identifying x_0, y_0, z_0 into a single vertex and identifying x_a, y_b, z_c into a single vertex. The following result was proved in [1].

Theorem 2. *A connected graph G is 2-choosable if and only if the heart of G is K_1 or an even cycle or $\theta_{2,2,2n}$ for some $n \geq 1$.*

It was conjectured in [7] that every 2-choosable graph is circular consecutive 2-choosable. We shall confirm this conjecture.

Theorem 3. *If G is 2-choosable, then $ch_{cc}(G) \leq 2$.*

It is easy to see that a graph G is circular consecutive 2-choosable if and only if the heart of G is circular consecutive 2-choosable. The graphs K_1 and even cycles are known [3] to be circular consecutive 2-choosable. To prove Theorem 3, it remains to show that for any positive integer n, the graph $\theta_{2,2,2n}$ is consecutive circular 2-choosable.

Theorem 4. *Let $G = \theta_{2,2,2n}$ with $V(G) = \{u, v, x_1, x_2, \cdots, x_{2n+1}\}$ and $E(G) = \{x_1u, x_1v, x_{2n+1}u, x_{2n+1}v\} \cup \{x_jx_{j+1} : j = 1, 2, \cdots, 2n\}$. Let $r \geq 2$ and $l : V(G) \to S(r)$ be an arbitrary mapping and let $L(x) = [l(x), l(x)+2]_r$. Then G is circular L-colourable.*

It is known [3] that if G contains a cycle, then $ch_{cc}(G) \geq 2$. If G is an n-vertex tree, then $ch_{cc}(T) = 2(1-\frac{1}{n})$. Thus for a connected 2-choosable graph G, to determine $ch_{cc}(G)$, it is sufficient to know if G has a cycle (if G has a cycle, then $ch_{cc}(G) = 2$), and in case G has no cycle, we need to know the number of vertices (if G has no cycle and n vertices, then $ch_{cc}(G) = 2 - \frac{2}{n}$). Thus $ch_{cc}(G)$ can be determined in linear time.

The remaining of the paper is devoted to the proof of Theorem 4.

3 Circular list colouring of paths

To prove Theorem 4, we need to find possible colours assigned to x_1 and x_{2n+1} in a circular L-colouring of the long path $(x_1, x_2, \cdots, x_{2n+1})$. In this section, we present a result concerning consecutive circular list colouring of a path, which will be crucial for our proof of Theorem 4.

The following lemma concerning circular list colouring of trees is proved in [8] (the formulation is different from the one stated in [8]).

Lemma 5. *Let $r \geq 2$. Suppose T is a tree and L is a circular colour-list assignment with respect to r that assigns to each vertex x of T a union of closed intervals of $S(r)$ of total length $\ell(x)$. If for each subtree T' of T, $\sum_{x \in V(T')} \ell(x) \geq 2(|V(T')| - 1)$ (and a single vertex is assigned at least one point, which is an interval of length 0), then T has a circular L-colouring.*

In particular, we have the following corollary.

Corollary 6. *Suppose $P = (p_0, p_1, \cdots, p_k)$ is a path and $r \geq 2$ is a real number. Suppose each vertex p_j is assigned a closed interval $L(p_j)$ of $S(r)$. If for each $j = 1, 2, \cdots, k-1$, $L(p_j)$ is an interval of length 2, and the sum of the lengths of $L(p_0)$ and $L(p_k)$ is at least 2, then P has a circular L-colouring.*

Corollary 6 is used in our proofs. However, in many cases, we need to know more about possible colours that can be assigned to p_0 and p_k in a circular L-colouring of P. It turns out that if the colour list assignment L satisfies certain requirements (which will shown to be true in our case), we can say more about the possible colours assigned to p_0 and p_k in a circular L-colouring of P.

Theorem 7. *Suppose* $2 < r < 4$ *and* $k \geq \lceil 2/(r-2) \rceil$ *and* $P_k = (p_0, p_1, \cdots, p_k)$ *is a path of length* k*. Let* $l : P_k \to S(r)$ *be any mapping such that* $|l(p_i) - l(p_{i+1})|_r \geq 1$ *for* $0 \leq i \leq k-1$*. Let* $L(p_i) = [l(p_i), l(p_i) + 2]_r$*. Then the following hold:*

1. *There exists a point* $t \in L(p_0)$ *such that for any* $t' \in L(p_k)$ *there is a circular* L*-coloring* f *of* P_k *with* $f(p_0) = t$ *and* $f(p_k) = t'$*.*

2. *For any* $0 < \ell < 2$*, there exist an interval* $X \subseteq L(p_0)$ *of length* ℓ *and an interval* $Y \subseteq L(p_k)$ *of length* $2 - \ell$*, such that for any* $t \in X$ *and for any* $t' \in Y$ *there is a circular* L*-coloring* f *with* $f(p_0) = t$ *and* $f(p_k) = t'$*.*

By taking ℓ to be real number approaching 0, we can view statement (1) as a limit case of statement (2), where a single colour is viewed as an colour interval of length 0. Nevertheless, we shall prove the two statements separately.

To prove Theorem 7, we first define some notation and present two lemmas. We say two colours $t, t' \in S(r)$ are *adjacent* if $|t - t'|_r \geq 1$. For $t \in S(r)$, denote by $N(t)$ the set of colours adjacent to t, namely $N(t) = [t+1, t-1]_r$. For a subset A of $S(r)$, let $N(A) = \cup_{t \in A} N(t)$.

Lemma 8. *Suppose* $I = [a, b]_r$ *is an interval of* $S(r)$ *of length* $\ell = [b - a]_r$*. If* $\ell \geq 2$*, then* $N(I) = S(r)$*. Otherwise* $N(I) = [a + 1, b - 1]_r$*.*

The proof of Lemma 8 is trivial and omitted.

Lemma 9. *Suppose* $2 < r < 4$*,* $a, b \in S(r)$ *and* $|a - b|_r \geq 1$*. If* $I = [s, s + \ell]_r \subseteq [b, b + 2]_r$*, then the following hold.*

1. *If* $\ell \geq r - 2$*, then there is an interval* I' *of length* $\ell - (r - 2)$ *such that* $I \subseteq [a, a + 2]_r$ *and* $I = N(I')$*.*

2. *If* $\ell \leq r - 2$*, then there is a colour* $t' \in [a, a + 2]_r$ *such that* $I \subseteq N(t')$*.*

Proof. First we observe that if $\ell = r - 2$, then by (1), there is an interval I' of $[a, a + 2]_r$ of length 0 such that $I = N(I')$. Here by an interval of length 0 we mean a single point. So in this case, the conclusions in (1) and (2) coincide.

(1): Assume $\ell \geq r - 2$. Let $s' = s - 1$ and $I' = [s', s' + \ell - (r - 2)]_r$. By Lemma 8, $N(I') = I = [s' + 1, s' + \ell + 2 - r - 1]_r = [s, s + \ell]_r = I$. Now we show that $I' \subseteq [a, a + 2]_r$.

First we show that

$$I' = [s-1, s-1+\ell-(r-2)]_r \subseteq [b-1, b+3]_r.$$

Assume $t \in [s-1, s-1+\ell-(r-2)]_r$. Then $[t-(s-1)]_r \leq \ell-(r-2)$. We need to show that $[t-(b-1)]_r \leq [b+3-(b-1)]_r = 4-r$. Observe that

$$[t-(b-1)]_r = [t-(s-1)+s-b]_r = [[t-s+1]_r + [s-b]_r]_r.$$

Because $[s, s+\ell]_r \subseteq [b, b+2]_r$, we conclude that $[s-b]_r \leq 2-\ell$. Hence

$$[[t-s+1]_r + [s-b]_r]_r = [t-s+1]_r + [s-b]_r \leq \ell-(r-2)+2-\ell = 4-r.$$

It remains to show that $[b-1, b+3]_r \subseteq [a, a+2]_r$. If $t \in [b-1, b+3]_r$, then $[t-b+1]_r \leq 4-r$. Because $1 \leq [b-a]_r \leq r-1$, we have $[b-1-a]_r \leq r-2$. It follows that

$$\begin{aligned} [t-a]_r &= [t-b+1+b-1-a]_r \\ &= [[t-b+1]_r + [b-1-a]_r]_r \leq 4-r+r-2 = 2. \end{aligned}$$

Therefore $t \in [a, a+2]_r$.

(2): Assume $\ell \leq r-2$. Let I'' be an interval contained in $[b, b+2]_r$ of length $r-2$ such that $I \subseteq I''$. Apply (1) to I'', we conclude that there exists $t \in [a, a+2]_r$ such that $I'' = N(t)$. Hence $I \subseteq N(t)$. □

Proof of Theorem 7 We first consider the case that $k = \lceil 2/(r-2) \rceil$.

(1): Let $I_k = L(p_k)$. By repeatedly applying Lemma 9, we conclude that there are intervals $I_{k-1}, I_{k-2}, \cdots, I_1$ such that

- I_j is contained in $L(p_j)$.
- I_j has length $2-(k-j)(r-2)$.
- $I_{j+1} = N(I_j)$.

Since I_1 has length $2-(k-1)(r-2) \leq r-2$, apply Lemma 9 again, there is a colour $t \in L(p_0)$ such that $I_1 \subseteq N(t)$.

For any $t' \in L(p_k) = I_k$, there are colours $c_j \in I_j$ for $j = k-1, k-2, \cdots, 1$ such that $t' \in N(c_{k-1})$ and $c_{j+1} \in N(c_j)$ for $j = k-2, k-3, \cdots, 1$ and $c_1 \in N(t)$. Let $f(p_0) = t, f(p_k) = t'$ and $f(p_j) = c_j$ for $j = 1, 2, \cdots, k-1$. Then f is a circular L-colouring of P_k satisfying the requirements of the theorem. This completes the proof of (1).

(2) Let $q = \lceil \ell/(r-2) \rceil$. Similarly as in the proof of (1), by repeatedly applying Lemma 9, we have the following:

- For $j = k, k-1, k-2, \cdots, q$, there are intervals $I_j \subseteq L(p_j)$ of length $2-(k-j)(r-2)$ and $I_{j+1} = N(I_j)$ for $j = k-1, k-2, \cdots, q$.
- For $j = 0, 1, 2, \cdots, q$, there are intervals $J_j \subseteq L(p_j)$ of length $2-j(r-2)$ with $N(J_j) = J_{j-1}$ for $j = 1, 2, \cdots, q$.

Let

$$\delta = q(r-2) - \ell \text{ and } \epsilon = (k-q)(r-2) + \ell - 2.$$

Let J'_q be a closed interval contained in $L(p_q)$ of length $2 - q(r-2) + \delta$ containing J_q, and let I'_q be a closed interval contained in $L(p_q)$ of length $2-(k-q)(r-2)+\epsilon$ containing I_q. As the sum of the lengths of J'_q and I'_q is equal to 2 and both are contained in $L(p_q)$ which is an interval of length 2, $I'_q \cap J'_q \neq \emptyset$.

Let $s \in I'_q \cap J'_q$. Since $I_q \subseteq I'_q$ and I_q has length $2-q(r-2)$, there is a colour $s' \in I_q$ such that $|s-s'|_r \leq \epsilon$. Thus $N(s)$ is an interval which is a shift of the interval $N(s')$ by a distance $|s-s'|_r \leq \epsilon$. Since $N(s') \cap I_{q+1} = N(s')$, which is an interval of length $r-2$, it follows that $I'_{q+1} = N(s) \cap I_{q+1}$ is an interval of length at least $r-2-\epsilon$. For $j = q+2, q+3, \cdots, k$, let $I'_j = N(I'_{j-1})$, then $I'_j \subseteq I_j \subseteq L(p_j)$ and has length at least $(j-q)(r-2)-\epsilon$. In particular, $I'_k \subseteq L(p_k)$ has length at least $(k-q)(r-2)-\epsilon = 2-\ell$. Similarly, let $J'_{q-1} = N(s) \cap J_{q-1}$ for $j = q-2, q-3, \cdots, 1$, let $J'_j = N(J'_{j+1})$. We have $J'_j \subseteq L(p_j)$ and J'_0 has length $q(r-2) - \delta = \ell$.

Let $X = J'_0$ and $Y = I'_k$. For $t \in X$ and $t' \in Y$, there are colours $c_j \in I'_j$ for $j = k-1, k-2, \cdots, q+1$ such that $t' \in N(c_{k-1})$ and $c_j \in N(c_{j-1})$ for $j = k-1, k-2, \cdots, q+1$. Similarly, there are colours $c_j \in J'_j$ for $j = 1, 2, \cdots, q-1$ such that $t \in N(c_1)$ and $c_{j+1} \in N(c_j)$ for $j = 1, 2, \cdots, q-2$. Then $f(p_k) = t', f(p_0) = t, f(p_q) = s$ and $f(p_j) = c_j$ for $j = 1, 2, \cdots, q-1, q+1, \cdots, k-1$ is a circular L-coloring f with $f(p_0) = t$ and $f(p_k) = t'$.

Assume Theorem 7 holds for k. To prove that it also holds for $P_{k+1} = (p_0, p_1, \cdots, p_k, p_{k+1})$, we apply the theorem to the path $(p_0, p_1, \cdots, p_k)$ to obtain the required sets X and Y, and then let $Y' = Y + l(p_{k+1}) - l(p_k) = \{t + l(p_{k+1}) - l(p_k) : t \in Y\}$. Then X, Y' are the required sets for statement (2). Statement (1) is proved in the same way. □

4 Proof of Theorem 4

Assume Theorem 4 is not true. Let n be the smallest integer for which there is a real number $r \geq 2$, a $(2, r)$-CCCL assignment L of G, such that G is not circular L-colourable. We shall derive some properties of the list assignment L that eventually lead to a contradiction.

It is known [3] that we only need to consider those $2 \le r < 4$. Also it is known [7] that $\theta_{2,2,2}$ is circular consecutive 2-choosable. In the following, we assume that $2 \le r < 4$ and $n \ge 2$.

If $r \le 2 + 2/n$, then $L(x_1) \cap L(x_3) \cap \cdots \cap L(x_{2n+1}) \ne \emptyset$. Let $t \in L(x_1) \cap L(x_3) \cap \cdots L(x_{2n+1})$. Let $f(x_{2j+1}) = t$ for $j = 0, 1, \cdots, n$. For $w \in \{u, v, x_2, x_4, \cdots, x_{2n}\}$, let $f(w)$ be any colour from the nonempty set $L(w) - (t-1, t+1)_r$. Then f is an L-colouring of G. In the following, we assume that $r > 2 + 2/n$.

Lemma 10. *For any $j \in \{2, 3, \cdots, 2n-1\}$, $l(x_j)$ and $l(x_{j+1})$ are adjacent, i.e., $|l(x_j) - l(x_{j+1})|_r \ge 1$.*

Proof. Assume to the contrary that there exists an index $j \in \{2, 3, \cdots, 2n-1\}$ such that $|l(x_j) - l(x_{j+1})|_r < 1$. Delete two vertices x_j, x_{j+1} and add an edge $x_{j-1}x_{j+2}$. The resulting graph G' is $\theta_{2,2,2(n-1)}$. By the minimality of G, there exists a circular L-coloring f for G'. We shall extend f to a circular L-colouring of G, by finding appropriate colours for x_j and x_{j+1}.

Let $a = f(x_{j-1})$ and $b = f(x_{j+2})$. If $b \in L(x_j)$, then let $f(x_j) = b$ and let $f(x_{j+1})$ be any colour from the non-empty set $L(x_{j+1}) - (b-1, b+1)_r$. Then f is a circular L-coloring of G. Thus we may assume that $b \notin L(x_j)$. Similarly, we may assume that $a \notin L(x_{j+1})$.

Since $r < 4$, either $a + 1 \in L(x_j)$ or $a - 1 \in L(x_j)$. By symmetry, we may assume that $a + 1 \in L(x_j)$. Similarly, either $b + 1 \in L(x_{j+1})$ or $b - 1 \in L(x_{j+1})$. If $b+1 \in L(x_{j+1})$, then let $f(x_j) = a+1$ and $f(x_{j+1}) = b+1$. Then f is a circular L-colouring of G. Thus we may assume that $b + 1 \notin L(x_{j+1})$ and hence $b-1 \in L(x_{j+1})$. Moreover, we may also assume that $a-1 \notin L(x_j)$, for otherwise, by letting $f(x_j) = a - 1$ and $f(x_{j+1}) = b - 1$ we obtain a circular L-colouring of G. Let $f(x_j) = l(x_j) + 2$ and $f(x_{j+1}) = l(x_j) + 1$. We shall show that f is a circular L-colouring of G.

Since $a + 1 \in L(x_j)$ and $a - 1 \notin L(x_j)$, it follows that $[a-1, a]_r \subseteq [l(x_j)+2, a]_r$ and $[a, a+1]_r \subseteq [a, l(x_j)+2]_r$. Hence $[a-(l(x_j)+2)]_r \ge 1$ and $[l(x_j)+2-a]_r \ge 1$. I.e., $|f(x_j) - f(x_{j-1})|_r \ge 1$. Since $|l(x_j) - l(x_{j+1})|_r < 1$, it follows that $l(x_j) + 1 \in L(x_{j+1})$. I.e., $f(x_{j+1}) \in L(x_{j+1})$. By definition, $|f(x_j) - f(x_{j+1})|_r = 1$. Since $b \notin L(x_j)$, we have $|b - (l(x_j) + 1)|_r \ge 1$. I.e., $|f(x_{j+1}) - f(x_{j+2})|_r \ge 1$. This proves that f is indeed a circular L-colouring of G. □

Let $l(x_1) = a$, $l(x_{2n+1}) = b$, $l(u) = c$ and $l(v) = d$. Without loss of generality, we may assume that

$$c \in L(v) = [d, d+2]_r.$$

Lemma 11. *Under assumption as above, we have $d \notin [c, c+2]_r$.*

Proof. Assume to the contrary that $c \in [d, d+2]_r$ and $d \in [c, c+2]_r$. By our assumption, $r \geq 2 + 2/n$. By Theorem 7, there is a colour $t \in L(x_2)$ such that for any $t' \in L(x_{2n})$, there is a circular L-colouring f of the path $(x_2, x_3, \cdots, x_{2n})$ with $f(x_2) = t$ and $f(x_{2n}) = t'$.

We construct a circular L-colouring c of G as follows: Let $c(x_2) = t$, and let $c(x_1) \in L(x_1)$ be any colour adjacent to t. Since $c \in [d, d+2]_r$ and $d \in [c, c+2]_r$, we have

$$[c, c+2]_r \cap [d, d+2]_r = [c, d+2]_r \cup [d, c+2]_r.$$

As $N([c, d+2]_r) = [c-1, d+3]_r$ and $N([d, c+2]_r) = [d-1, c+3]_r$, it implies that

$$N([c, d+2]_r \cup [d, c+2]_r) = S(r).$$

In particular, $c(x_1) \in N([c, c+2]_r \cap [d, d+2]_r)$. Let $s \in [c, c+2]_r \cap [d, d+2]_r$ be a colour adjacent to t and let $t* \in L(x_{2n+1})$ be any colour adjacent to s. Let $c(u) = c(v) = s$ and let $c(x_{2n+1}) = t*$. Let $t' \in L(x_{2n})$ be any colour adjacent to $t*$. By the previous paragraph, c can be extended to a circular L-colouring of the path $(x_2, x_3, \cdots, x_{2n})$. □

Lemma 12. *$N([c, d+2]_r) \cup (N(c+2) \cap N(d)) = S(r)$.*

Proof. By definition, $N([c, d+2]_r) = [c+1, d+1]_r$. Since $d \notin [c, c+2]_r$, we have $N(c+2) \cap N(d) = [c+3, c+1]_r \cap [d+1, d-1]_r = [d+1, c+1]_r$. □

Proof of Theorem 4 Assume first that $b \notin (c+1, d-1)_r$. By Theorem 7, there is a colour $t \in L(x_1)$ such that for any $t' \in L(x_{2n+1})$, there is a circular L-colouring f of the path $(x_1, x_2, x_3, \cdots, x_{2n+1})$ with $f(x_1) = t$ and $f(x_{2n+1}) = t'$. We construct a circular L-colouring c of G as follows: Let $c(x_1) = t$. If $[c, d+2]_r \cap N(t) \neq \emptyset$ then let $c(u) = c(v) = s$ for some $s \in [c, d+2]_r \cap N(t)$, let $c(x_{2n+1}) = t'$, where $t' \in L(x_{2n+1})$ is any colour adjacent to s. By the choice of t, c can be extended to a circular L-colouring of the path $(x_1, x_2, x_3, \cdots, x_{2n+1})$. If $[c, d+2]_r \cap N(t) = \emptyset$, then $t \notin N([c, d+2]_r)$. By Lemma 12, t is adjacent to both $c+2$ and d. In this case, let $c(u) = c + 2, c(v) = d$. Since $b \notin (c + 1, d - 1)_r$, it follows that $[b, b + 2]_r \cap [d + 1, c + 1]_r \neq \emptyset$. Let $t' \in [b, b + 2]_r \cap [d + 1, c + 1]_r$. Then$t'$ is adjacent to both $c + 2$ and d. Let $c(x_{2n+1}) = t'$. By the choice of t, c can be extended to a circular L-colouring of the path $(x_1, x_2, x_3, \cdots, x_{2n+1})$.

Assume next that $b \in (c+1, d-1)_r$. Then $[b, b+2]_r \cap (d+1, c+1)_r = \emptyset$. This implies that for any $t \in [b, b + 2]_r$, $N(t) \cap [c, d + 2]_r \neq \emptyset$. By Theorem 7, there is a colour $t \in L(x_{2n+1})$ such that for any $t' \in L(x_1)$, there is a circular L-colouring f of the path $(x_1, x_2, x_3, \cdots, x_{2n})$ with $f(x_1) = t'$ and $f(x_{2n+1}) = t$.

Let $s \in [c, d + 2]_r \cap N(t)$ be a colour adjacent to t and let $t' \in L(x_1)$ be any colour adjacent to s. Let $c(u) = c(v) = s$ and let $c(x_1) = t'$ and $c(x_{2n+1}) = t$. Then c can be extended to a circular L-colouring of the path $(x_1, x_2, x_3, \cdots, x_{2n+1})$. This completes the proof of Theorem 4.

5 An open question

It is known [3, 4, 7] that there are graphs G that are not 2-choosable but have $ch_{cc}(G) = 2$. Odd cycles and $K_{2,n}$ (for $n \geq 4$) are such examples. The question of characterizing circular consecutive 2-choosable graphs was asked in [7] and remains open. As an attempt to answer this question, the authors studied in [7] the circular consecutive choosability of generalized theta graphs $\theta_{k_1,k_2,\cdots,k_n}$ (the graph obtained from n paths of lengths $k_1, k_2, \cdots, k_n$ by identifying their initial ends into a single vertex, and identifying their terminal ends into a single vertex). It was proved that for $n \geq 0$, $ch_{cc}(\theta_{2,2,2,2n+1}) \geq 2 + 1/(n + 5)$ and $ch_{cc}(\theta_{2,2,2,2n+8}) \geq 2 + 2/(4n + 21)$ and $ch_{cc}(\theta_{2,2,2,2,4}) \geq 2+1/8$. However, the question that which generalized theta graphs G have $ch_{cc}(G) = 2$ remains largely open. In particular, the circular consecutive choosability of theta graphs $\theta_{2,2,2n+1}$ remains an open question. We can show that $ch_{cc}(\theta_{2,2,1}) = 2$. This particular case seems too weak to support a conjecture that $ch_{cc}(\theta_{2,2,2n+1}) = 2$ for all n. We pose it as a question:

Question 13. *Is it true that for any positive integer n, the theta graph $\theta_{2,2,2n+1}$ is circular consecutive 2-choosable?*

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