The circular chromatic number of series-parallel graphs of large odd girth

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Abstract

Suppose G is a series-parallel graph. We prove that if G has odd girth at least $6k-1$ then $\chi_c(G) \leq 8k/(4k-1)$; if G has odd girth at least $6k+1$ then $\chi_c(G) \leq (4k+1)/2k$; if G has odd girth at least $6k+3$ then $\chi_c(G) \leq (4k+3)/(2k+1)$.

1 Introduction

Given two integers k, d such that $k \geq d$, a (k,d)-colouring of a graph G is a colouring c of the vertices of G with colours $0, 1, 2, \cdots, k-1$ such that for any two adjacent vertices x and y of G we have $d \leq |c(x) - c(y)| \leq k - d$. The *circular chromatic number* (also known as the star chromatic number) $\chi_c(G)$ of G is defined as the infimum of the ratio k/d for which there exists a (k,d)-colouring of G. Since a $(k,1)$-colouring of a graph G is just an ordinary k-colouring of G, it follows that $\chi_c(G) \leq \chi(G)$ for any graph G. On the other hand, it was shown in [12] that $\chi_c(G) > \chi(G) - 1$. Hence $\chi(G) = \lceil \chi_c(G) \rceil$. In this sense, the circular chromatic number $\chi_c(G)$ is a refinement of the chromatic number $\chi(G)$, and $\chi(G)$ is an approximation of $\chi_c(G)$. The parameter $\chi_c(G)$ has been studied extensively in the past ten years (see [20] for a survey).

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It follows from the definition that any non-trivial graph has circular chromatic number at least 2. It was shown by Vince [12] (cf. also [1] for a combinatorial proof) that if G is finite then the infimum in the definition above is always attained, and hence can be replaced by the minimum. Therefore the circular chromatic number of a finite graph is always rational. On the other hand, it was shown in [12] that for any rational number $r \geq 2$, there exists a finite graph G of circular chromatic number r.

The problem of determining the circular chromatic numbers of special classes of graphs (such as graphs embeddable on a surface, graphs of large girth, graphs forbidden certain minors, etc.) has attracted considerable recent attention [2, 5, 6, 7, 8, 16, 17, 19]. One problem of particular interest is the relation between the circular chromatic number of a graph and its girth. It was shown in [16] that for every rational $r \geq 2$ and for every g there is a graph G of girth at least g and with $\chi_c(G) = r$ (see [20] for a constructive proof). However, if we restrict to the class of H-minor free graphs (where H is any fixed finite graph) then for any $\epsilon > 0$ there is an integer g such that any H-minor free graph G of girth at least g has $\chi_c(G) \leq 2 + \epsilon$ [2, 4]. This implies that for any fixed surface S, for any $\epsilon > 0$ there is an integer g such that any graph G embeddable on S and of girth at least g has $\chi_c(G) \leq 2+\epsilon$. The proof of this result is simple (cf. [2]), however, the girth requirement derived in such a proof is usually much too large. Given an $\epsilon > 0$, it is usually very difficult to determine the smallest g such that any H-minor free graph G of girth at least g (or any G embeddable on a surface S of girth at least g) has $\chi_c(G) \leq 2 + \epsilon$.

For planar graphs, a better bound on the girth requirement was derived by Galluccio et al. It was proved [4] that for any integer $k \geq 2$ if G is a planar graph of girth at least $10k - 3$ then $\chi_c(G) \leq 2 + \frac{1}{k}$. This girth requirement seems far from sharp. For series-parallel graphs (or equivalently for K_4-minor free graphs), we do have a sharp girth requirement. It was proved by Hell and Zhu [5] that if G is a series-parallel graph with girth at least $2\lfloor(3k-1)/2\rfloor$ then $\chi_c(G) \leq 4k/(2k-1)$; and proved by Chien and Zhu [2] that for any $k \geq 2$, there is a series-parallel graph G of girth $2\lfloor(3k-1)/2\rfloor - 1$ such that $\chi_c(G) > 4k/(2k-1)$.

It is interesting to ask whether the girth requirement above can be replaced by an odd girth requirement, There are negative results as well as positive results. It was shown in [13] that there are 4-chromatic graphs that embed on the projective plane with all regions 4-sided and with all noncontractible cycles (i.e., cycles that do not bound an area homeomorphic to the disc) arbitrarily long. As observed by Robertson (personal communication), this implies that for any graph H which does not embed on the projective plane, for any g there is a 4-chromatic H-minor free graph of odd girth at least g.

On the other hand, Klostermeyer and Zhang [6] succeeded in replacing the girth requirement in the result of [4] by an odd girth requirement. They proved that if a planar graph has odd girth at least $10k-3$ then $\chi_c(G) \leq 2+\frac{1}{k}$. Very recently, this result is improved by Zhu [21], who proved that every planar graph with odd girth at least $8k-3$ has $\chi_c(G) \leq 2+\frac{1}{k}$.

In this paper, we shall prove the following result:

Theorem 1.1 *Suppose G is a series-parallel graph.*

1. *If G has odd girth at least $6k-1$ then $\chi_c(G) \leq 8k/(4k-1)$;*

2. *If G has odd girth at least $6k+1$ then $\chi_c(G) \leq (4k+1)/2k$;*

3. *If G has odd girth at least $6k+3$ then $\chi_c(G) \leq (4k+3)/(2k+1)$.*

Theorem 1.1 is a strengthening of the above mentioned result of [5]. Indeed if G is a series-parallel graph with girth at least $2\lfloor(3k-1)/2\rfloor$ then either $k=2m$ is even and G has odd girth at least $6m-1$ and hence by (1), $\chi_c(G) \leq 8m/(4m-1) = 4k/(2k-1)$, or $k = 2m+1$ is odd and G has odd girth at least $6m+3$ and hence by (3), $\chi_c(G) \leq (4m+3)/(2m+1) < (8m+4)/(4m+1) = 4k/(2k-1)$.

It was proved in [2] that for any k, there is a series-parallel graph G of girth $2\lfloor(3k-1)/2\rfloor - 1$ with $\chi_c(G) > 4k/(2k-1)$. This implies that for any k, there is a series-parallel graph of girth $6k-3$ with $\chi_c(G) > 8k/(4k-1)$; and there is a series-parallel graph of girth $6k+1$ with $\chi_c(G) > (4k+3)/(2k+1)$. So the odd girth requirements here are not only sharp, they are also sharp as girth requirements. After the submission of this paper, the authors [11] proved Theorem 1.2 below, which shows that bound on the circular chromatic number is also sharp.

Theorem 1.2 *For any $\epsilon > 0$, for any integer $k \geq 1$,*

1. *there exists a series-parallel graph G of girth $6k-1$ such that $\chi_c(G) > 8k/(4k-1) - \epsilon$;*

2. *there exists a series-parallel graph G of girth $6k+1$ such that $\chi_c(G) > (4k+1)/2k - \epsilon$;*

3. *there exists a series-parallel graph G of girth $6k+3$ such that $\chi_c(G) > (4k+3)/(2k+1) - \epsilon$.*

An interesting consequence of the result in [5] is that there is no series-parallel graph G with circular chromatic number $\chi_c(G) \in (\frac{8}{3}, 3)$. Recently it is proved in [10] that for every rational number $r \in [2,3] - (\frac{8}{3}, 3)$ there exists a series-parallel graph G with $\chi_c(G) = r$.

2 Two terminal series-parallel graphs

The class of series-parallel graphs can be defined in many different ways and is referred to by different names, such as K_4-minor free graphs, partial 2-trees, etc., [5, 9] We adopt the following definition of two-terminal series-parallel graphs from [5]. A two-terminal series-parallel graph $(G; x, y)$ is defined recursively as follows:

- Let $V(K_2) = \{0, 1\}$. Then $(K_2; 0, 1)$ is a two-terminal series-parallel graph.
- (The parallel construction.) Let $(G; x, y)$ and $(G'; x', y')$ be two disjoint two-terminal series-parallel graphs. Define G'' to be the graph obtained from the union of G and G' by identifying x and x' into a single vertex x'', and identifying y and y' into a single vertex y''. Then $(G''; x'', y'')$ is a two-terminal series-parallel graph.
- (The series construction.) Let again $(G; x, y)$ and $(G'; x', y')$ be two disjoint two-terminal series-parallel graphs. Define G'' to be the graph obtained from the union of G and G' by identifying y and x' into a single vertex. Then $(G''; x, y')$ is a two-terminal series-parallel graph.
- There are no other two-terminal series-parallel graphs.

A graph G is a series-parallel graph if there exist some two vertices x, y such that $(G; x, y)$ is a two-terminal series-parallel graph. For all the series-parallel graphs in the remaining part, there are always two terminals which are clearly indicated in the context. Moreover, if the series-parallel graph is denoted by G (resp. G', G'' etc.) then the two terminals are denoted by x and y (resp. x' and y', x'' and y'', etc.).

The *odd-length* of G, denoted by $\ell_o(G)$, is defined to be the length (number of edges) of a shortest odd path between the two terminals x and y. The *even-length* of G, denoted by $\ell_e(G)$, is defined to be the length of a shortest even path between the two terminals x and y. If there is no odd length path (resp. even length path) between x and y then we set $\ell_o(G) = \infty$ (resp. $\ell_e(G) = \infty$).

The proof of Theorem 1.1 uses induction on the number of steps in the construction of G. For the purpose of using induction, instead of proving Theorem 1.1 directly, we shall prove some stronger results.

For a positive integer n, we denote by I_n the set $\{0, 1, \cdots, n-1\}$. We shall view this set of integers as cyclically ordered, as depicted in Fig. 1 below. For $0 \le i, j \le n-1$ we denote by $[i,j]_n$ the set of integers $\{i, i+1, \cdots, j\}$, where the addition is modulo n. For example, $[2,5]_n = \{2,3,4,5\}$ and $[5,2]_n = \{5, 6, \cdots, n-1, 0, 1, 2\}$. In case the index n is clear from the context, we shall write $[i,j]$ for $[i,j]_n$.

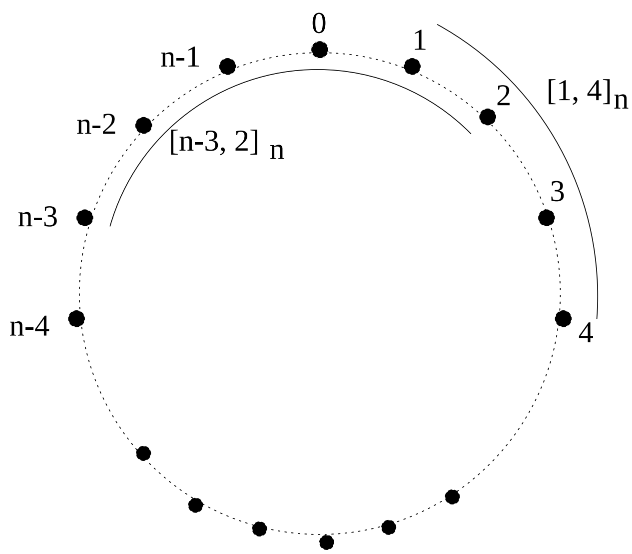

Figure 1: Illustration of the set I_n

Given two subsets A, B of I_n, the sum $A + B$ is defined as

$$A + B = \{x + y \bmod n : \ x \in A, y \in B\}.$$

We let $2A = A + A$ and $nA = (n-1)A + A$. Note that when we calculate the sum of two sets A and B, they must be both subsets of I_n for some n.

Lemma 2.1 and Lemma 2.2 are straightforward and the proofs are omitted.

Lemma 2.1 *Suppose $X = [a,b]_n$ and $Y = [a',b']_n$.*

- *If* $|X| + |Y| \geq n+1$, *then* $X + Y = I_n$;
- *If* $|X| + |Y| \leq n$, *then* $X + Y = [a+a', b+b']_n$.

Lemma 2.2 *For any positive integers* i *and* j *and for any subset* A *of* I_n *we have* $iA + jA = (i+j)A$.

In the following, we are interested in iA for A being an interval of I_n. For convenience, we let $\infty A = I_n$.

In order to prove Theorem 1.1, we shall prove the following results:

Theorem 2.1 *Suppose* G *is a series-parallel graph of odd girth at least* $6k-1$. *Let* $J = [4k-1, 4k+1]_{8k}$. *If* $\ell_o(G) = i$ *and* $\ell_e(G) = j$, *then for any* $q \in iJ \cap jJ$, *there is an* $(8k, 4k-1)$*-colouring* f *of* G *such that* $f(x) = 0$ *and* $f(y) = q$.

Theorem 2.2 *Suppose* G *is a series-parallel graph of odd girth at least* $6k+1$. *Let* $J = [2k, 2k+1]_{4k+1}$. *If* $\ell_o(G) = i$ *and* $\ell_e(G) = j$, *then for any* $q \in iJ \cap jJ$, *there is a* $(4k+1, 2k)$*-colouring* f *of* G *such that* $f(x) = 0$ *and* $f(y) = q$.

Theorem 2.3 *Suppose* G *is a series-parallel graph of odd girth at least* $6k+3$. *Let* $J = [2k+1, 2k+2]_{4k+3}$. *If* $\ell_o(G) = i$ *and* $\ell_e(G) = j$, *then for any* $q \in iJ \cap jJ$, *there is a* $(4k+3, 2k+1)$*-colouring* f *of* G *such that* $f(x) = 0$ *and* $f(y) = q$.

Lemma 2.3 *For* $J = [4k-1, 4k+1]_{8k}$, *if* $i + j \geq 6k-1$ *then* $iJ \cap jJ \neq \emptyset$. *For* $J = [2k, 2k+1]_{4k+1}$, *if* $i + j \geq 6k+1$ *then* $iJ \cap jJ \neq \emptyset$. *For* $J = [2k+1, 2k+2]_{4k+1}$, *if* $i + j \geq 6k+3$ *then* $iJ \cap jJ \neq \emptyset$.

Proof. It follows from Lemma 2.1 that for $J = [4k-1, 4k+1]_{8k}$, iJ is either equal to I_{8k} or is an interval of length $2i+1$. If $iJ = I_{8k}$ or $jJ = I_{8k}$ then of course $iJ \cap jJ \neq \emptyset$. Otherwise, since $i + j \geq 6k-1$, we have $|iJ| + |jJ| \geq 2i+1+2j+1 > 8k$. As $|iJ \cup jJ| \leq 8k$, we conclude that $iJ \cap jJ \neq \emptyset$. The remaining part of Lemma 2.3 is proved similarly. ∎

Theorem 1.1 follows from Theorems 2.1, 2.2, 2.3 and Lemma 2.3.

3 Proof of Theorem 2.1

For this section and the next section, $k \geq 1$ is a fixed integer and all the sets of integers A are regarded as subsets of I_{8k} and J stands for the set $J = [4k-1, 4k+1]_{8k}$.

Lemma 3.1 *Let $J = [4k-1, 4k+1]_{8k}$ and h be a positive integer. If $h \geq 4k$ then $hJ = I_{8k}$. If $h < 4k$ is even then $hJ = [8k-h, h]_{8k}$. If $h < 4k$ is odd then $hJ = [4k-h, 4k+h]_{8k}$.*

Lemma 3.2 *Let h_1, h_2 be positive integers. If $h_1 < 4k$ is odd, $h_2 < 4k$ is even, and $h_1 + h_2 \geq 4k$, then*

$$h_1 J \cap h_2 J = [8k - h_2, 4k + h_1]_{8k} \cup [4k - h_1, h_2]_{8k}.$$

If $h_1 \equiv h_2 \pmod 2$ and $h_1 \leq h_2$, then $h_1 J \cap h_2 J = h_1 J$.

Both Lemmas 3.1 and 3.2 follow easily from Lemma 2.1. We omit the detail calculations.

We shall prove Theorem 2.1 by induction on the number of steps in the construction of G. If $G = K_2$, then this is obviously true.

Assume that G is obtained from G_1 and G_2 by a parallel construction, and that Theorem 2.1 is true for G_1 and G_2. Let

$$\ell_o(G_1) = i,\ \ell_e(G_1) = j,\ \ell_o(G_2) = i',\ \ell_e(G_2) = j'.$$

Let

$$i'' = \min\{i, i'\}, j'' = \min\{j, j'\}.$$

Then $\ell_o(G) = i''$ and $\ell_e(G) = j''$. By Lemma 3.2, $i''J = iJ \cap i'J$ and $j''I = jJ \cap j'J$. Therefore

$$i''J \cap j''J = iJ \cap i'J \cap jJ \cap j'J.$$

Thus if $q \in i''J \cap j''J$ then $q \in iJ \cap jJ$ and $q \in i'J \cap j'J$. It follows from the induction hypothesis that there is an $(8k, 4k-1)$-colouring f of G_1 such that $f(x) = 0$ and $f(y) = q$, and an $(8k, 4k-1)$-colouring g of G_2 such that $f(x) = 0$ and $f(y) = q$. The union of f and g is an $(8k, 4k-1)$-colouring f^* of G such that $f^*(x) = 0$ and $f^*(y) = q$.

Finally we assume that G is obtained from G_1 and G_2 by a series construction, and that Theorem 2.1 is true for G_1 and G_2. We shall denote the two terminals of G_1 by x_1 and y_1, the two terminals of G_2 by x_2 and y_2. In the series construction, we identified y_1 and x_2 into a single vertex. The two terminals of G are $x = x_1$ and $y = y_2$.

Let $\ell_o(G_1) = i,\ \ell_e(G_1) = j,\ \ell_o(G_2) = i',\ \ell_e(G_2) = j'$. Suppose $\ell_o(G) = i''$ and $\ell_e(G) = j''$. Then

$$\begin{aligned} i'' &= \min\{i + j', i' + j\} \\ j'' &= \min\{i + i', j + j'\}. \end{aligned}$$

By Lemma 3.2,

$$\begin{aligned} i''J &= (i+j')J \cap (i'+j)J \\ j''J &= (i+i')J \cap (j+j')J. \end{aligned}$$

We shall show that Theorem 2.1 is true for G, i.e., for any

$$q \in i''J \cap j''J = (i+j')J \cap (i'+j)J \cap (i+i')J \cap (j+j')J,$$

there is an $(8k, 4k-1)$-colouring f of G such that $f(x) = 0$ and $f(y) = q$.

To prove this we need the following lemma:

Lemma 3.3 *Suppose i, i' are odd positive integers, j, j' are even positive integers, $i + j \geq 6k - 1$ and $i' + j' \geq 6k - 1$. Then*

$$(i+j')J \cap (i'+j)J \cap (i+i')J \cap (j+j')J \subseteq (iJ \cap jJ) + (i'J \cap j'J). \qquad (*)$$

We shall leave the proof of Lemma 3.3 to the next section. Now we shall use Lemma 3.3 to prove Theorem 2.1.

Since G has odd girth at least $6k-1$, so $i+j \geq 6k-1$ and $i'+j' \geq 6k-1$. Then it follows from Lemma 3.3 that there are colours $d \in iJ \cap jJ$ and $d' \in i'J \cap j'J$ such that $d + d' \bmod 8k = q$. By the induction hypothesis, there is an $(8k, 4k-1)$-colouring g of G_1 such that $g(x_1) = 0$ and $g(y_1) = d$ and an $(8k, 4k-1)$-colouring g' of G_2 such that $g'(x_2) = 0$ and $g'(y_2) = d'$. We define a colouring f of G by letting $f(v) = g(v)$ if $v \in V(G_1)$ and $f(v) = g'(v) + d \bmod 8k$ if $v \in V(G_2)$. Then obviously f is a $(8k, 4k-1)$-colouring of G for which $f(x) = 0$ and $f(y) = q$.

4 Proof of Lemma 3.3

First we consider the case that at least two of the integers i, i', j, j' are greater than or equal to $4k$. By Lemma 3.1, if $h \geq 4k$ then $hJ = I_{8k}$.

If $i \geq 4k$ and $j \geq 4k$, then $iJ \cap jJ = I_{8k}$ and $I_{8k} + (i'J \cap j'J) = I_{8k}$. The conclusion is certainly true. Suppose one of i, j is greater than or equal to $4k$, say $i \geq 4k$ and one of i', j' is greater than or equal to $4k$, say $j' \geq 4k$. Then the left side of $(*)$ is equal to $(i'+j)J$ (the other terms are all equal to I_{8k} by Lemma 3.1). Since $iJ \cap jJ = jJ$ and $i'J \cap j'J = i'J$, the right side of $(*)$ is equal to $(iJ \cap jJ) + (i'J \cap j'J) = jJ + i'J = (i'+j)J$. So the lemma is true. (The case $j \geq 4k$ and $j' \geq 4k$, the case $i \geq 4k$ and $i' \geq 4k$ are all similar and can be verified easily).

Next we consider the case that exactly one of the integers i, i', j, j' is at least $4k$. Assume that $i \geq 4k$ and $j, i', j' < 4k$. Then $iJ \cap jJ = jJ$.

Since $i', j' < 4k$ and $i' + j' \geq 6k - 1$, it follows that $i', j' \geq 2k$.

By Lemma 3.1 and Lemma 3.2,

$$iJ \cap jJ = [8k - j, j]$$

and

$$i'J \cap j'J = [8k - j', 4k + i'] \cup [4k - i', j'].$$

By Lemma 2.1, either

$$(iJ \cap jJ) + (i'J \cap j'J) = I_{8k}$$

or

$$(iJ \cap jJ) + (i'J \cap j'J) = [8k - (j + j'), 4k + (i' + j)] \cup [4k - (i' + j), j + j'].$$

In the former case, the lemma is certainly true. In the latter case, the lemma is also true, because

$$(i' + j)J \cap (j + j')J = [8k - (j + j'), 4k + (i' + j)] \cup [4k - (i' + j), j + j'].$$

The case that $i' \geq 4k$ and $i, j, j' < 4k$ is symmetric to the above case.

Now we assume that $j \geq 4k$ and $i, i', j' < 4k$ (again the case $j' \geq 4k$ and $i, j, i' < 4k$ is symmetric).

Similarly, because $i', j' < 4k$ and $i' + j' \geq 6k - 1$, we have $i', j' \geq 2k$. Hence

$$iJ \cap jJ = iJ = [4k - i, 4k + i].$$

By Lemma 3.2,

$$i'J \cap j'J = [8k - j', 4k + i'] \cup [4k - i', j'].$$

By Lemma 2.1, the right side of $(*)$ is either

$$(iJ \cap jJ) + (i'J \cap j'J) = I_{8k}$$

or

$$(iJ \cap jJ) + (i'J \cap j'J) = [4k - (i + j'), i + i'] \cup [8k - (i + i'), 4k + (i + j')].$$

In the former case, the lemma is certainly true. In the latter case, the lemma is also true, because by Lemma 3.2, part of the left side of $(*)$ is

$$(i + j')J \cap (i + i')J = [4k - (i + j'), i + i'] \cup [8k - (i + i'), 4k + (i + j')].$$

Finally we consider the case that all the integers i, j, i', j' are less than $4k$. Because $i+j \geq 6k-1$ and $i'+j' \geq 6k-1$, it follows that all the integers i, j, i', j' are at least $2k$. By Lemma 3.2,

$$iJ \cap jJ = [8k-j, 4k+i] \cup [4k-i, j],$$

and

$$i'J \cap j'J = [8k-j', 4k+i'] \cup [4k-i', j'].$$

By Lemma 2.1, either

$$(iJ \cap jJ) + (i'J \cap j'J) = I_{8k}$$

or

$$\begin{aligned}(iJ \cap jI) + (i'J \cap j'J) &= [8k-(j+j'), i+i'] \cup [12k-(i+j'), (i'+j)-4k] \\ &\cup [12k-(i'+j), (i+j')-4k] \cup [8k-(i+i'), j+j'].\end{aligned}$$

We shall prove that in the latter case we also have

$$(iJ \cap jJ) + (i'J \cap j'J) = I_{8k}.$$

First observe that

$$X = [8k-(j+j'), i+i'] \cup [8k-(i+i'), j+j']$$

is an interval of I_{8k} containing $4k$, and

$$Y = [12k-(i+j'), (i'+j)-4k] \cup [12k-(i'+j), (i+j')-4k]$$

is an interval of I_{8k} containing 0. Let

$$\begin{aligned} a &= \min\{8k-(j+j'), 8k-(i+i')\}, \\ b &= \max\{i+i', j+j'\}, \\ c &= \min\{12k-(i+j'), 12k-(i'+j)\}, \\ d &= \max\{(i'+j)-4k, (i+j')-4k\}. \end{aligned}$$

Then $X = [a, b]$ and $Y = [c, d]$. To prove $X \cup Y = I$, it suffices to show that $a \leq d+1$ and $c \leq b+1$.

If $i > j$ then $2i \geq 6k$ (as $i+j \geq 6k-1$). Because $i'+j' \geq 6k-1$, we have $12k \leq 2i+i'+j'+1$, i.e.,

$$12k-(i+j') \leq i+i'+1.$$

Hence $c \leq b+1$.

If $i < j$ then $2j \geq 6k$. So $12k \leq 2j + i' + j' + 1$. Therefore

$$12k - (i' + j) \leq j + j' + 1.$$

In any case we have $c \leq b+1$ (note that i, j are of different parity, so $i \neq j$).

If $i' < j'$ then $2j' \geq 6k$. So $12k \leq 2j' + i + j + 1$. Therefore

$$8k - (j + j') \leq (i + j') - 4k + 1.$$

If $i' > j'$ then $2i' \geq 6k$. So $12k \leq 2i' + i + j + 1$.

$$8k - (i + i') \leq (i' + j) - 4k + 1.$$

In any case we have $a \leq d+1$. This completes the proof of Lemma 3.3.

5 Proofs of Theorems 2.2 and 2.3

The proofs of Theorems 2.2 and 2.3 are similar to the proof of Theorem 2.1. For Theorem 2.2, we let $J = [2k, 2k+1]_{4k+1}$, the proof is indeed identical to the proof of Theorem 2.1 (copied word by word) with the understanding that J stands for $[2k, 2k+1]_{4k+1}$, $8k$ replaced by $4k+1$, $4k-1$ replaced by $2k$ and $4k+1$ replaced by $2k+1$). The only difference is that instead of Lemma 3.3, we need the following lemma:

Lemma 5.1 *Suppose i, i' are odd positive integers, j, j' are even positive integers, and $i + j \geq 6k+1$ and $i' + j' \geq 6k+1$. Then*

$$(i + j')J \cap (i' + j)J \cap (i + i')J \cap (j + j')J \subseteq (iJ \cap jJ) + (i'J \cap j'J).$$

Where $J = [2k, 2k+1]_{4k+1}$.

To prove Theorem 2.3, instead of Lemma 3.3, we need the following lemma:

Lemma 5.2 *Suppose i, i' are odd positive integers, j, j' are even positive integers, and $i + j \geq 6k+3$ and $i' + j' \geq 6k+3$. Then*

$$(i + j')J \cap (i' + j)J \cap (i + i')J \cap (j + j')J \subseteq (iJ \cap jJ) + (i'J \cap j'J).$$

Where $J = [2k+1, 2k+2]_{4k+3}$.

The proofs of Lemmas 5.1 and 5.2 are similar to the proof of Lemma 3.3, and we omit the details.

6 Circular flow number and other comments

The dual concept of the circular chromatic number of a graph is the *circular flow number* which is defined as: Suppose G is a bridgeless graph and $\vec{G}$ is an orientation of G. A flow of $\vec{G}$ is a mapping $f : E(\vec{G}) \to R$ such that for each vertex v of $\vec{G}$,

$$\Sigma_{e \in N^+(v)} f(e) = \Sigma_{e \in N^-(v)} f(e),$$

i.e., the flow that goes into a vertex is equal to the flow that goes out of the vertex. For any real number $r \geq 2$, an r-flow of $\vec{G}$ is a flow f of $\vec{G}$ such that for all e, $1 \leq |f(e)| \leq r - 1$. The circular flow number $F_c(G)$ of a graph G is the infimum of those r for which an orientation $\vec{G}$ of G has an r-flow[1]. The circular flow number can be defined for matroids. It was proved in [3] that the circular chromatic number of a graph is equal to the circular flow number of its cocyclic matroid. In particular, for a planar graph G, the circular chromatic number of G is equal to the circular flow number of its dual graph.

The dual of a series-parallel graph is still a series-parallel graph. We define the *odd edge connectivity* of a graph to be the minimum cardinality of an odd edge cut of G. As pointed out by C.Q.Zhang [14], the results of this paper can be written as results concerning the relation between the circular flow number of a series-parallel graph and its odd edge connectivity. To be precise, we have the following corollary:

Corollary 6.1 *Suppose G is a bridgeless series-parallel graph. If G has odd edge connectivity at least $6k-1$, then its circular flow number is at most $8k/(4k-1)$; if G has odd edge connectivity at least $6k+1$, then its circular flow number is at most $(4k+1)/2k$; if G has odd edge connectivity at least $6k+3$, then its circular flow number is at most $(4k+3)/(2k+1)$.*

For graphs G embeddable on a given surface, Zhang [14] proved that if the odd edge connectivity of G is "large", then the circular flow number of G is "close" to 2.

The results in this paper concerns the relation between the circular chromatic number of a K_4-minor free graph and its odd girth. As observed in Section 1, for general H-minor free graphs, large odd girth does not imply

[1] The circular flow number of a graph generalizes the flow number of a graph in the same way the circular chromatic number generalizes the chromatic number. The name "circular flow number" is only a tentative one. There are other names for the same concept: "fractional flow number" [3], "flow index" [14].

circular chromatic number close to 2. There is another condition which is "between" a girth requirement and an odd girth requirement. To state this condition, we need the concept of treewidth. The class of k-trees is defined recursively as follows: (1): K_k is a k-tree. (2) Suppose G is a k-tree and X is a k-clique of G. Let G' be obtained from G by adding a vertex v and adding edges to connect v to every vertex of X. Then G' is also a k-tree. (3) There are no other k-trees. A partial k-tree is simply a subgraph of a k-tree. The treewidth of a graph G is the smallest k such that G is a partial k-tree. The condition "between" a girth requirement and an odd girth requirement is illustrated in the following result of [8]:

Given a positive integer k and a real number $\epsilon > 0$ there is an integer g such that the following is true: if G is a graph of girth at least g and G' is a partial k-tree which admits a homomorphism (i.e., an edge-preserving vertex mapping) to G then $\chi_c(G') \leq 2 + \epsilon$.

In this result, instead of the H-minor free condition we used the condition of bounded treewidth; and instead of bounded odd girth we used the condition of admitting a homomorphism to a graph of large girth (or equivalently, one can wrap up the small even cycles while keeping the odd cycles large). It seems nontrivial to replace any one of these two conditions by a weaker condition. To be precise, we have the following two questions:

Question 6.1 *Given a finite graph H and a real number $\epsilon > 0$, does there exists an integer g such that if G is a H-minor free graph and G admits a homomorphism to a graph G' of girth at least g then $\chi_c(G) \leq 2 + \epsilon$?*

Question 6.2 *Given an integer k and a real number $\epsilon > 0$, does there exists an integer g such that any partial k-tree of odd girth at least g has $\chi_c(G) \leq 2 + \epsilon$?*

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7 Some omitted details, for the reference of the referees

Proof of Lemma 5.1: Straightforward calculation (using Lemma 2.2) shows that if $i \geq 4k$ then $iJ = I_{4k+1}$; if $i = 2p+1 < 4k$ is odd then

$$iJ = [2k-p, 2k+p+1],$$

if $i = 2q < 4k$ is even then

$$iJ = [4k+1-q, q].$$

First we consider the case that at least two of the integers i, i', j, j' are greater than or equal to $4k$.

If $i \geq 4k$ and $j \geq 4k$, then $iJ \cap jJ = I_{4k+1}$ and $I_{4k+1} + (i'J \cap j'J) = I_{4k+1}$. The conclusion is certainly true. Suppose one of i, j is greater than or equal to $4k$, say $i \geq 4k$ and one of i', j' is greater than or equal to $4k$, say $j' \geq 4k$. Then $iJ \cap jJ = jJ$ and $i'J \cap j'J = i'J$. Hence $(iJ \cap jJ) + (i'J \cap j'J) = jJ + i'J = (i'+j)J$. So the lemma is true. (The case $j \geq 4k$ and $j' \geq 4k$, the case $i \geq 4k$ and $i' \geq 4k$ are all similar and can be verified easily).

Next we consider the case that exactly one of the integers i, i', j, j' is at least $4k$. Assume that $i \geq 4k$ and $j, i', j' < 4k$. Then $iJ \cap jJ = jJ$.

Since $i', j' < 4k$ and $i' + j' \geq 6k+1$, it follows that $i', j' \geq 2k+2$.

As observed before, if $i = 2p+1 < 4k$ is odd then

$$iJ = [2k-p, 2k+p+1],$$

if $i = 2q < 4k$ is even then

$$iJ = [4k+1-q, q].$$

Let $i = 2p+1, j = 2q$ and $i' = 2p'+1, j' = 2q'$. Therefore

$$iJ \cap jJ = [4k+1-q, q]$$

and

$$i'J \cap j'J = [4k+1-q', 2k+p'+1] \cup [2k-p', q'].$$

By Lemma 2.1, either

$$(iJ \cap jJ) + (i'J \cap j'J) = I_{4k+1}$$

or

$$(iJ \cap jJ) + (i'J \cap j'J) = [4k+1-(q+q'), 2k+1+(p'+q)] \cup [2k-(p'+q), q+q'].$$

In the former case, the lemma is certainly true. In the latter case, the lemma is also true, because

$$(i'+j)J \cap (j+j')J = [4k+1-(q+q'), 2k+1+(p'+q)] \cup [2k-(p'+q), q+q'].$$

The case that $i' \geq 4k$ and $i, j, j' < 4k$ is symmetric. If $j \geq 4k$ and $i, i', j' < 4k$, the corresponding calculations are similar and omitted. (The case $j' \geq 4k$ and $i, j, i' < 4k$ is symmetric).

Finally we consider the case that all the integers i, j, i', j' are less than $4k$. Then all the integers i, j, i', j' are at least $2k+2$. Therefore

$$iJ \cap jJ = [4k+1-q, 2k+1+p] \cup [2k-p, q],$$

and

$$i'J \cap j'J = [4k+1-q', 2k+1+p'] \cup [2k-p', q'].$$

By Lemma 2.1, either

$$(iJ \cap jJ) + (i'J \cap j'J) = I_{4k+1}$$

or

$$\begin{aligned}(iJ \cap jI) + (i'J \cap j'J) &= [4k+1-(q+q'), p+p'+1] \cup [6k+1-(p+q'), (p'+q)-2k] \\ &\cup [6k+1-(p'+q), (p+q')-2k] \cup [4k-(p+p'), q+q'].\end{aligned}$$

We shall prove that in the latter case we also have

$$(iJ \cap jJ) + (i'J \cap j'J) = I_{4k+1}.$$

Observe that

$$X = [4k+1-(q+q'), p+p'+1] \cup [4k-(p+p'), q+q']$$

is an interval of I_{4k+1} containing $2k$, and

$$Y = [6k+1-(p+q'), (p'+q)-2k] \cup [6k-(p'+q), (p+q')-2k]$$

is an interval of I_{4k+1} containing 0. Let

$$\begin{aligned}a &= \min\{4k+1-(q+q'), 4k-(p+p')\}, \\ b &= \max\{p+p'+1, q+q'\}, \\ c &= \min\{6k+1-(p'+q), 6k+1-(p+q')\}, \\ d &= \max\{(p'+q)-2k, (p+q')-2k\}.\end{aligned}$$

Then $X = [a, b]$ and $Y = [c, d]$. To prove $X \cup Y = I$, it suffices to show that $a \leq d + 1$ and $c \leq b + 1$.

If $i > j$ then $2i \geq 6k + 2$ (as $i + j \geq 6k + 1$), hence $2p \geq 3k$. Because $i' + j' \geq 6k + 1$, we have $p' + q' \geq 3k$. So $2p + p' + q' + 1 \geq 6k + 1$, i.e.,

$$6k + 1 - (p + q') \leq p + p' + 1.$$

Hence $c \leq b$.

If $i < j$ then $2j \geq 6k + 2$, hence $2q \geq 3k + 1$. Also we have $p' + q' \geq 3k$ Therefore

$$6k + 1 - (p' + q) \leq q + q'.$$

So in any case we have $c \leq b$.

If $i' < j'$ then $2j' \geq 6k + 2$, hence $2q' \geq 3k + 1$. As $i + j \geq 6k + 1$, which implies that $p + q \geq 3k$, we have $6k + 1 \leq 2q' + p + q$, i.e.,

$$4k + 1 - (q + q') \leq (p + q') - 2k.$$

Hence $a \leq d$.

If $i' > j'$ then $2i' \geq 6k + 2$, hence $2p' \geq 3k$. As $p + q \geq 3k$, we have $6k \leq 2p' + p + q$, i.e.,

$$4k - (p + p') \geq (p' + q) - 2k.$$

So in any case we have $a \leq d$.

Proof of Lemma 5.2: It follows from Lemma 2.2 that if $i \geq 4k + 2$ then $iJ = I_{4k+3}$. First we consider the case that at least two of the integers i, i', j, j' are greater than or equal to $4k + 2$. Similarly as in the proof of Lemma 2.2, let $i = 2p + 1, j = 2q$ and $i' = 2p' + 1, j' = 2q'$.

If $i \geq 4k + 2$ and $j \geq 4k + 2$, then $(iJ \cap jJ) + (i'J \cap j'J) = I_{4k+3}$. The conclusion is certainly true. Suppose one of i, j is greater than or equal to $4k + 2$, say $i \geq 4k + 2$ and one of i', j' is greater than or equal to $4k + 2$, say $j' \geq 4k + 2$. Then $iJ \cap jJ = jJ$ and $i'J \cap j'J = i'J$. Hence $(iJ \cap jJ) + (i'J \cap j'J) = jJ + i'J = (i' + j)J$. So the lemma is true. (The case $j \geq 4k + 2$ and $j' \geq 4k + 2$, the case $i \geq 4k + 2$ and $i' \geq 4k + 2$ are all similar and can be verified easily).

Next we consider the case that exactly one of the integers i, i', j, j' is at least $4k+2$. Assume that $i \geq 4k+2$ and $j, i', j' < 4k+2$. Then $iJ \cap jJ = jJ$.

Since $i', j' < 4k + 2$ and $i' + j' \geq 6k + 3$, it follows that $i', j' \geq 2k + 2$. Straightforward calculation shows that if $i = 2p + 1 < 4k + 2$ is odd then

$$iJ = [2k + 1 - p, 2k + 2 + p],$$

if $j = 2q < 4k+2$ is even then

$$jJ = [4k+3-q, q].$$

Therefore

$$iJ \cap jJ = [4k+3-q, q]$$

and

$$i'J \cap j'J = [4k+3-q', 2k+2+p'] \cup [2k+1-p', q'].$$

By Lemma 2.1, either

$$(iJ \cap jJ) + (i'J \cap j'J) = I_{4k+3}$$

or

$$(iJ \cap jJ) + (i'J \cap j'J) = [4k+3-(q+q'), 2k+2+(p'+q)] \cup [2k+1-(p'+q), q+q'].$$

In the former case, the lemma is certainly true. In the latter case, the lemma is also true, because

$$(i'+j)J \cap (j+j')J = [4k+1-(q+q'), 2k+1+(p'+q)] \cup [2k-(p'+q), q+q'].$$

The case that $i' \geq 4k$ and $i, j, j' < 4k$ is symmetric. The case $j \geq 4k$ and $i, i', j' < 4k$ and the case $j' \geq 4k$ and $i, j, i' < 4k$ are similar.

Finally we consider the case that all the integers i, j, i', j' are less than $4k+2$. Then all the integers i, j, i', j' are at least $2k+2$. Therefore

$$iJ \cap jJ = [4k+3-q, 2k+2+p] \cup [2k+1-p, q],$$

and

$$i'J \cap j'J = [4k+3-q', 2k+2+p'] \cup [2k+1-p', q'].$$

By Lemma 2.1, either

$$(iJ \cap jJ) + (i'J \cap j'J) = I_{4k+3}$$

or

$$\begin{aligned} (iJ \cap jI) + (i'J \cap j'J) &= [4k+3-(q+q'), p+p'+1] \\ &\cup [6k+4-(p+q'), (p'+q)-2k-1] \\ &\cup [6k+4-(p'+q), (p+q')-2k-1] \\ &\cup [4k+2-(p+p'), q+q']. \end{aligned}$$

We shall prove that in the latter case we also have

$$(iJ \cap jJ) + (i'J \cap j'J) = I_{4k+3}.$$

Observe that

$$X = [4k+3-(q+q'), p+p'+1] \cup [4k+2-(p+p'), q+q']$$

is an interval of I_{4k+1} containing $2k+2$, and

$$Y = [6k+4-(p+q'), (p'+q)-2k-1] \cup [6k+4-(p'+q), (p+q')-2k-1]$$

is an interval of I_{4k+3} containing 0. Let

$$\begin{aligned} a &= \min\{4k+3-(q+q'), 4k+2-(p+p')\}, \\ b &= \max\{p+p'+1, q+q'\}, \\ c &= \min\{6k+4-(p'+q), 6k+4-(p+q')\}, \\ d &= \max\{(p'+q)-2k-1, (p+q')-2k-1\}. \end{aligned}$$

Then $X = [a,b]$ and $Y = [c,d]$. To prove $X \cup Y = I$, it suffices to show that $a \leq d+1$ and $c \leq b+1$.

If $i > j$ then $2i \geq 6k+4$ (as $i+j \geq 6k+3$), hence $2p \geq 3k+1$. Because $i'+j' \geq 6k+3$, we have $p'+q' \geq 3k+1$. So $2p+p'+q' \geq 6k+2$, i.e.,

$$6k+4-(p+q') \leq p+p'+2.$$

Hence $c \leq b+1$.

If $i < j$ then $2j \geq 6k+4$, hence $2q \geq 3k+2$. Also we have $p'+q' \geq 3k+1$ Therefore

$$6k+4-(p'+q) \leq q+q'+1.$$

So in any case we have $c \leq b+1$.

If $i' < j'$ then $2j' \geq 6k+4$, hence $2q' \geq 3k+2$. As $p+q \geq 3k+1$, we have $6k+3 \leq 2q'+p+q$ i.e.,

$$4k+3-(q+q') \leq (p+q')-2k-1+1.$$

Hence $a \leq b+1$.

If $i' > j'$ then $2i' \geq 6k+4$, hence $2p' \geq 3k+1$. As $p+q \geq 3k+1$, we have $6k+2 \leq 2p'+p+q$, i.e.,

$$4k+2-(p+p') \leq (p'+q)-2k-1+1.$$

So in any case we have $a \leq b+1$.