Bounds on circular consecutive choosability

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Abstract

The circular consecutive choosability $ch_{cc}(G)$ of a graph G has been recently introduced in [2]. In this paper we prove upper bounds on ch_{cc} for series-parallel graphs, planar graphs and k-choosable graphs. Our bounds are tight for classes of series-parallel graphs and k-choosable graphs for $k \geq 3$. Then we study the circular consecutive choosability of generalized theta graphs. Lower bounds for the circular consecutive choosability of certain generalized theta graphs are obtained.

1 Introduction

For a positive real number r, let $S(r)$ denote the circle obtained from the interval $[0, r]$ by identifying 0 and r into a single point. For a real number t, denote by $[t]_r$ the remainder of t upon division by r. For $a, b \in S(r)$, the distance between a and b is $|a-b|_r = \min\{|a-b|, r-|a-b|\}$ and the intervals $[a, b]_r$ and $(a, b)_r$ are defined as $[a, b]_r = \{t \in S(r) : [t-a]_r \leq [b-a]_r\}$ and $(a, b)_r = \{t \in [0, r) : 0 < [t-a]_r < [b-a]_r\}$. Suppose $G = (V, E)$ is a graph. A *circular r-colouring* of G is a mapping $f : V(G) \to S(r)$ such that for any edge uv of G, $|f(u) - f(v)|_r \geq 1$. The *circular chromatic number* $\chi_c(G)$ of G is defined as

$$\chi_c(G) = \inf\{r : G \text{ has a circular } r\text{-colouring}\}.$$

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The circular chromatic number of a graph is a refinement of the chromatic number of a graph, and has been studied extensively in the literature (see [7, 8] for surveys on this subject).

Given a graph G and a positive real number r, a $(\star, r)$-*circular colour-list assignment* for G is a function L that assigns to each vertex v of G a set $L(v)$ which is the union of disjoint closed intervals of $S(r)$. If for each vertex v, the sum of the lengths of the disjoint intervals in $L(v)$ is equal to t, then L is called a (t, r)-*circular colour-list assignment.* Suppose L is a $(\star, r)$-circular colour-list assignment for a graph G. A *circular L-colouring* of G is a circular r-colouring f of G such that $f(v) \in L(v)$ for each vertex v of G. A graph G is called *circular t-choosable* if for any r and for any (t, r)-circular colour-list assignment L, G has a circular L-colouring. The *circular choosability* $ch_c(G)$ of G (also called the *circular list chromatic number* of G and denoted by $ch_c(G)$) is defined in [9] as

$$ch_c(G) = \inf\{t : \ G \text{ is circular } t\text{-choosable}\}.$$

The definition of circular t-choosable graphs given above is slightly different from the one given in [9]. In [9], the set $L(v)$ assigned to a vertex v by a circular colour-list assignment L is the disjoint union of open intervals. In this paper, $L(v)$ is the disjoint union of closed intervals, which seems to be more convenient for us. This change does affect whether a graph is circular t-choosable or not. However, as the circular choosability of G is by taking the infimum of those t for which G is circular t-choosable, the circular choosability of a graph is the same under both definitions.

The circular consecutive choosability of a graph, introduced in [2], is a variation of circular choosability of a graph and the consecutive choosability of a graph [6].

A $(\star, r)$-*circular consecutive colour-list assignment* of G is a mapping L which assigns to each vertex v of G a closed interval $L(v)$ of $S(r)$. If $L(v)$ has length t for each vertex v, then L is called a (t, r)-*circular consecutive colour-list assignment* of G. We say G is *circular consecutive (t, r)-choosable* if for any (t, r)-circular consecutive colour-list assignment L of G, G is circular L-colourable.

Observe that if $r < \chi_c(G)$, then for any $(\star, r)$-circular colour-list assignment L, G is not circular L-colourable. Therefore, for the definition to be meaningful, we restrict to real numbers $r \geq \chi_c(G)$.

Definition 1. *Suppose $r \geq \chi_c(G)$. The circular consecutive choosability of G with respect to r is defined as*

$$ch^r_{cc}(G) = \inf\{t : G \text{ is circular consecutive } (t, r)\text{-choosable}\}.$$

The circular consecutive choosability of G is defined as

$$ch_{cc}(G) = \sup\{ch^r_{cc}(G) : r \geq \chi_c(G)\}.$$

Equivalently, $ch_{cc}(G)$ is the infimum of those t such that for any $r \geq \chi_c(G)$, G is circular consecutive (t, r)-choosable.

In [2], it was shown that if G is a graph on n vertices, then

$$\chi(G) - 1 \leq ch_{cc}(G) \leq 2\chi_c(G)(1 - 1/n) - 1.$$

The values of $ch_{cc}(G)$ for complete graphs, trees, even cycles and balanced complete bipartite graphs were determined. Upper and lower bounds for $ch_{cc}(G)$ were given for some other graphs.

In this paper, we explore the relation between $ch_{cc}(G)$ and the choosability $ch(G)$ of G. We prove that if G is a k-choosable graph, then $ch_{cc}(G) \leq k + 1 - 1/k$, and if $k \geq 3$ then this upper bound is tight. A tight upper bound on $ch_{cc}(G)$ for series-parallel graphs is also presented, namely, $ch_{cc}(G) \leq 11/3$ for every series-parallel graph G. We show that if G is a planar graph, then $ch_{cc}(G) \leq 5.8$ and for each $\epsilon > 0$, there is a planar graph G with $ch_{cc}(G) > 4.7 - \epsilon$. The upper bound $ch_{cc}(G) \leq k + 1 - 1/k$ for k-choosable graphs is not tight for $k = 2$. It is conjectured that $ch_{cc}(G) \leq 2$ for 2-choosable graphs. To prove the conjecture, it amounts to show that the theta graphs $\theta_{2,2,2k}$ have circular consecutive choosability 2. More generally, one might attempt to characterize graphs with circular consecutive choosability 2. A natural class of graphs that one can study to gain intuition about this question is the class of generalized theta graphs. Suppose $P_1, P_2, \cdots, P_n$ are paths of lengths $k_1, k_2, \cdots, k_n$ respectively. The generalized theta graph $\theta_{k_1,k_2,\cdots,k_n}$ is the graph obtained from the disjoint union of $P_1, P_2, \cdots, P_n$ by identifying their initial vertices into a single vertex and their terminal vertices into a single vertex. It is proved that for any integer $n \geq 2$, $ch_{cc}(\theta_{\underbrace{2, 2, \cdots, 2}_{n}}) = 2$. However, for $n \neq 2, 4, 6$, $ch_{cc}(\theta_{2,2,2,n}) > 2$.

2 k-list colourable graphs

A graph G is called *k-choosable* if for any mapping L which assigns to each vertex v of G a set $L(v)$ of k integers, there is a mapping $f : V(G) \to \mathbb{Z}$ such that $f(v) \in L(v)$ for each $v \in V(G)$ and $f(u) \neq f(v)$ for each edge uv of G. Our first lemma bounds circular consecutive choosability of k-choosable graphs with respect to given r.

Lemma 2. *Let $k \geq 2$ be an integer and let G be a graph with $ch(G) = k$ then $ch^r_{cc}(G) \leq k + (k-1)(r - \lfloor r \rfloor)/\lfloor r \rfloor$ for every $r \geq \chi_c(G)$.*

Proof. Let $s = k+(k-1)(r-\lfloor r \rfloor)/\lfloor r \rfloor$, and let L be an s-circular consecutive list assignment of G with respect to r. For $l = 0, 1, \dots \lfloor r \rfloor - 1$ let $I_l = [lr/\lfloor r \rfloor, (l+1)r/\lfloor r \rfloor - 1]_r$ be an interval in $S(r)$. For every $v \in V(G)$ let $S(v) = \{j \mid I_j \cap L(v) \neq \emptyset\}$, then $|S(v)| \geq k$. As $ch(G) = k$ it is possible to choose $k(v) \in S(v)$ for every $v \in V(G)$ so that $k(v) \neq k(w)$ for every $vw \in E(G)$. By the choice of $S(v)$ we can choose $f(v) \in I_{k(v)} \cap L(v)$ for every $v \in V(G)$. It remains to note that for every $i, j \in \{0, 1, \dots, \lfloor r \rfloor - 1\}$, $i \neq j$ and every $x \in I_i$ $y \in I_j$ we have $|x-y|_r \geq 1$ and therefore $|f(v)-f(w)|_r \geq 1$ for every $vw \in E(G)$. □

Corollary 3. *Let $k \geq 2$ be an integer. If a graph G has list chromatic number k, then $ch_{cc}(G) \leq k + 1 - 1/k$.*

Proof. If $\chi_c(G) \leq r \leq k$ then $ch^r_{cc}(G) \leq r \leq k$. If $r \geq k$ then $ch^r_{cc}(G) \leq k + (k-1)(r - \lfloor r \rfloor)/\lfloor r \rfloor < k + (k-1)/k$ by Lemma 2. □

We shall show that for $k \geq 3$, the upper bound given in Corollary 3 is tight. For this purpose, we need an alternate definition of $ch_{cc}(G)$ given in [2].

Given positive integers $p \geq 2q$, a *(p, q)-colouring* of a graph G is a mapping $f : V(G) \to \{0, 1, \cdots, p-1\}$ such that for any edge xy of G, $q \leq |f(x) - f(y)| \leq p - q$. For any integer a, $[a]_p$ denotes the remainder of a divided by p. For $a, b \in \{0, 1, \cdots, p-1\}$, the *circular integral interval* $[a, b]_p$ is defined as

$$[a, b]_p = \{a, a+1, a+2, \cdots, b\},$$

where the additions are modulo p. Suppose G is a graph and p, q are positive integers such that $p/q \geq \chi_c(G)$, and s is a positive integer. Let $l : V(G) \to \{0, 1, \cdots, p-1\}$ be a mapping. A (p, q)-colouring f of G is *compatible with*

(l, s) if for any vertex x, $f(x) \in [l(x), l(x)+s-1]_p$. We say a graph G is *circular consecutive* (p, q)*-s-choosable* if for any mapping $l : V(G) \to \{0, 1, \cdots, p-1\}$, G has a (p, q)-colouring f which is compatible with (l, s). We define the consecutive (p, q)-choosability of G as

$$ch_{p,q}(G) = \min\{s : G \text{ is circular consecutive } (p, q)\text{-}s\text{-choosable}\}.$$

The following lemma is proved in [2].

Lemma 4. *For any graph G and for any $r = p/q \geq \chi_c(G)$,*

$$ch_{p,q}(G) = \lfloor ch^r_{cc}(G)q \rfloor + 1.$$

Now we prove a technical lemma which is later used to lower bound maximum circular consecutive choosability of graphs of fixed treewidth.

A graph G is called a *k-tree* if the vertices of G can be ordered as $v_1, v_2, \cdots, v_n$ in such a way that $\{v_1, v_2, \cdots, v_k\}$ induces a K_k, and for each $j \geq k+1$, the set $N^+(v_j) = \{v_i : i < j, v_i \sim v_j\}$ induces a K_k. The *treewidth* of a graph G is the minimum k such that G is a subgraph of a k-tree.

Lemma 5. *Let $k \geq 2, p$ and q be positive integers such that $p/q \geq k$, and let s be a positive integer. Suppose that every graph G with treewidth at most $k-1$ is circular consecutive (p, q)-s-choosable. Then there exists a non-empty family $\mathcal{S}$ of k-element subsets of $\{0, 1, \ldots, p-1\}$ such that for every $S \in \mathcal{S}$ the following conditions hold*

1. *for every distinct $x_1, x_2 \in S$ we have $q \leq |x_1 - x_2| \leq p - q$,*
2. *for every $X \subset S$ with $|X| = k-1$ and every $i \in \{0, 1, \ldots, p-1\}$ there exists $S' \in \mathcal{S}$ such that $S' = X \cup \{x_0\}$ and $x_0 \in [i, i+s-1]_p$.*

Proof. For a graph H and a (p, q)-colouring f of H let $\mathcal{S}(H, f)$ denote the family of sets of colours of cliques of size k in H. Choose a graph G of treewidth at most $k-1$ and a map $l : V(G) \to \{0, 1, \ldots, p-1\}$ so that the minimum of $|\mathcal{S}(G, f)|$ over all (p, q)-colourings f of G compatible with (l, s) is maximum. Construct the graph G' and a map $l' : V(G') \to \{0, 1, \ldots, p-1\}$ as follows: For every clique $W \subseteq V(G)$ with $|W| = k-1$ and every $i \in \{0, 1, \ldots, p-1\}$ create a vertex v^i_W of degree $k-1$ of G' that is joined by edges to vertices of W and set $l'(v^i_W) = i$. Let l' be identical to l on $V(G)$. Then G' has treewidth at most $k-1$. By the choice of G there exists a (p, q)-colouring f' of G' compatible with (l', s) such that $\mathcal{S}(G', f') = \mathcal{S}(G, f')$.

We claim that $\mathcal{S} = \mathcal{S}(G, f')$ satisfies the requirements of the lemma. Clearly $\mathcal{S}$ is non-empty. For every $S \in \mathcal{S}$ there exists a clique $U \subseteq V(G)$ such that $S = f'(U)$. Therefore the first requirement is satisfied by the definition of (p, q)-colouring. Similarly, for every $X \subset S$ with $|X| = k - 1$ there exists a clique $W \subset U$ such that $|W| = k - 1$ and $X = f'(W)$ then $S' = f'(W \cup \{v^i_W\})$ satisfies the second requirement. □

Theorem 6. *For every $k \geq 3$ and $\varepsilon > 0$ there exists a graph $G_{k,\varepsilon}$ such that $G_{k,\varepsilon}$ has treewidth at most $k - 1$ and $ch_{cc}(G_{k,\varepsilon}) > k + 1 - 1/k - \varepsilon$.*

Proof. We will show that for every positive integer n and every integer $k \geq 3$ there exists a graph $G_{k,n}$ of treewidth at most $k - 1$ that is not circular consecutive (p, q)-s-choosable, where $p = nk(k + 1) - 2, q = nk$ and $s = nk(k + 1) - n - 2$. By Lemma 4, for $r = k + 1 - 2/nk$, $ch_{cc}(G_{k,n}) \geq ch^r_{cc}(G_{k,n}) > (p-1)/q = (nk(k+1)-n-2-1)/nk = k+(k-1)/k-3/nk$. As graphs of treewidth $k-1$ are k-choosable, this implies the required lower bound on $ch_{cc}(G)$ for k-choosable graphs.

Suppose, on the contrary, that for some n and some $k \geq 3$ every graph of treewidth at most $k - 1$ is circular consecutive (p, q)-s-choosable. By Lemma 5 then there exists a family $\mathcal{S}$ of k-element subsets of $\{0, 1, \ldots, p-1\}$ satisfying the requirements of that lemma.

Choose $S = \{a_1, \ldots, a_k\} \in \mathcal{S}$ so that $a_1, \ldots, a_k$ appear in $\{0, 1, \ldots, p-1\}$ in circular order and $([a_2 - a_1]_p, [a_3 - a_2]_p, \ldots, [a_k - a_{k-1}]_p)$ is lexicographically maximum. Let $a_{k+1} = a_1$, by convention.

Consider $X = S - \{a_2\}$ and $i = a_1 + \lceil([a_3 - a_1]_p + n)/2\rceil$. Then by condition 2 in Lemma 5 there exists $S' \in \mathcal{S}$ such that $S' = X \cup \{a'_2\}$ and $a'_2 \in [i, i + s - 1]$. Note that $a'_2 \in [a_1, a_3]_p$. Otherwise $a'_2 \in [a_l, a_{l+1}]_p$ for some $l \geq 3$, and we obtain contradiction as follows,

$$\begin{aligned} p &= \sum_{j=1}^{l-1}[a_{j+1} - a_j]_p + [a'_2 - a_l]_p + [a_{l+1} - a'_2]_p + \sum_{j=l+1}^{k} [a_{j+1} - a_j]_p \\ &\geq q(l - 1 + 2 + k - l) = q(k + 1) = p + 2. \end{aligned}$$

Since $a'_2 \notin [i - n, i - 1]$, it follows that

$$|[a_3 - a'_2]_p - [a'_2 - a_1]_p| \geq n - 1.$$

Hence

$$\max\{[a_3-a'_2]_p, [a'_2-a_1]_p\} \geq ([a_3-a'_2]_p+[a'_2-a_1]_p+n-1)/2 = ([a_3-a_1]_p+n-1)/2.$$

By the choice of S, we have

$$\begin{aligned}[a_2 - a_1]_p &\geq \max\{[a_2' - a_1]_p, [a_3 - a_2']_p\} \\ &\geq ([a_3 - a_1]_p + n - 1)/2 = ([a_3 - a_2]_p + [a_2 - a_1]_p + n - 1)/2.\end{aligned}$$

Consequently,

$$[a_2 - a_1]_p \geq [a_3 - a_2]_p + n - 1.$$

By considering $X = S - \{a_l\}$ for $l \in \{3, \dots, k\}$ and $i = a_{l-1} + q + n$, and using an argument similar to the above, we deduce $[a_l - a_{l-1}]_p \geq q + n$. A contradiction follows:

$$\begin{aligned}p = \sum_{j=1}^{k} [a_{j+1} - a_j]_p &\geq [a_2 - a_1]_p + (q+n)(k-2) + [a_1 - a_k]_p \\ &\geq q + n + n - 1 + (q+n)(k-2) + q = (q+n)k - 1 = p + 1.\end{aligned}$$

□

Since graphs of treewidth at most $(k-1)$ are k-choosable, Theorem 6 shows that the bound of Corollary 3 is tight.

Corollary 7. *If G is a series-parallel graph, then $ch_{cc}(G) \leq 11/3$. For any $\epsilon > 0$, there is a series-parallel graph G with $ch_{cc}(G) > 11/3 - \epsilon$.*

3 Planar graphs

In this section we study bounds on circular consecutive choosability of planar graphs.

Theorem 8. *For every planar graph G we have $ch_{cc}(G) \leq 5.8$. For every $\varepsilon > 0$ there exists a planar graph G_ε such that $ch_{cc}(G_\varepsilon) > 4.7 - \varepsilon$.*

Proof. The upper bound follows from Theorem 6 and 5-choosability of planar graphs [5].

To obtain the lower bound it suffices to construct, for each positive integer n, a graph G_n that is not circular consecutive (p,q)-s-choosable, where $p = 200n - 1, q = 40n$ and $s = 188n - 1$.

Let K_4 be a complete graph on vertex set $\{a, b, c, d\}$. For each edge xy of K_4, we add $6n(200n-1)$ paths of length 4 (i.e., paths with 5 vertices) with every vertex in these paths joined by an edge to x and to y. The resulting graph is denoted by H_n. It is obvious that H_n is planar.

For each edge $e = xy$ of K_4, the $6n(200n - 1)$ paths in H_n joined to x, y are indexed as $P_{k,m,e}$, where $k, m \in \{0, 1, \ldots, p - 1\}$ and $50n \leq [m - k]_p < 56n$.

Claim 1. *There is a map $l : V(H_n) \to \{0, 1, \ldots, p - 1\}$ such that if f is a (p, q)-colouring of H_n compatible with (l, s), then there is an edge uv of H_n such that $56n \leq |f(u) - f(v)|_p \leq 80n - 1$.*

Proof. The map l is defined as follows: Suppose $P_{k,m,e} = (v_1, v_2, v_3, v_4, v_5)$. Then let $l(v_1) = l(v_4) = [k + m + 56n]_p$, $l(v_2) = l(v_5) = [k + 156n]_p$. Let $l(v_3)$ be arbitrary. Let $l(x)$ be arbitrary for $x \in \{a, b, c, d\}$.

Assume to the contrary of the claim, there is a (p, q)-colouring f of H_n compatible with (l, s) for which there is no edge uv with $56n \leq |f(u) - f(v)|_p \leq 80n - 1$. It is obvious that there exist an edge $e = xy$ with $x, y \in \{a, b, c, d\}$ and $50n \leq [f(x) - f(y)]_p \leq 80n - 1$. By our assumption, this implies that $50n \leq [f(x) - f(y)]_p \leq 56n - 1$.

Without loss of generality, we assume that $f(y) = 0$ and $50n \leq f(x) = m \leq 56n - 1$. Then we consider the path $P_{0,m,e} = (v_1, v_2, v_3, v_4, v_5)$. Because the five vertices of the path are adjacent to both x and y, it follows that for each v_j, $f(v_j) \in [m + 40n, m + 56n - 1]_p \cup [144n, 160n - 1]_p$. By symmetry, we may assume that $f(v_3) \in [144n, 160n - 1]_p$. As v_4 is adjacent to v_3, we have $f(v_4) \in [m + 40n, m + 56n - 1]_p$. Because f is compatible with (l, s) and $l(v_4) = [m + 56n]_p$, we conclude that

$$f(v_4) \in [m + 40n, m + 44n - 1]_p.$$

Since v_4v_5 is an edge, $f(v_5) \in [144n, 160n - 1]_p$. As f is compatible with (l, s) and $l(v_5) = [156n]_p$, we conclude that

$$f(v_5) \in [156n, 160n - 1]_p.$$

Because $50n \leq m \leq 56n - 1$, this implies that

$$56n + 2 \leq [f(v_5) - f(v_4)]_p \leq 70n - 1.$$

This completes the proof of the claim. □

For each edge $e = xy$ of H_n, we add p paths of length 4 with every vertex in these paths joined by an edge to x and to y. The resulting graph is denoted by G_n. Obviously, G_n is planar.

For an edge $e = xy$ of H_n, the paths in G_n joined to x, y are indexed as $P_{k,e}$, where $k \in \{0, 1, \dots, p-1\}$. Now we extend the map l of H_n to G_n. For each edge $e = xy$ of H_n, the images of l for vertices of the path $P_{k,e} = (v_1, v_2, v_3, v_4, v_5)$ are defined as follows: $l(v_1) = l(v_4) = [k + 108n]_p$ and $l(v_2) = l(v_5) = [k + 160n]_p$, and $l(v_3)$ is arbitrary.

Now we claim that there is no (p, q)-colouring of G_n which is compatible with (l, s). Assume to the contrary that f is a (p, q)-colouring of G_n compatible with (l, s). By Claim 1, there is an edge $e = xy$ of H_n such that $56n \leq [f(x) - f(y)]_p \leq 80n - 1$. Without loss of generality, we assume that $f(y) = 0$ and $f(x) = m$ and $56n \leq m \leq 80n - 1$. Consider the restriction of f to the path $P_{0,e}$ of G_n. Because the five vertices of the path are adjacent to both x and y, it follows that for each v_j, $f(v_j) \in [96n, 160n - 1]_p$. By symmetry, we may assume that $f(v_3) \in [128n, 160n - 1]_p$. As v_3v_4 is an edge, this forces $f(v_4) \in [96n, 120n - 1]$. Because f is compatible with (l, s) and $l(v_4) = 108n$, we conclude that $f(v_4) \in [108n, 120n - 1]_p$. This forces $f(v_5) \in [148n, 160n - 1]_p$ (as v_4v_5 is an edge). But then f is not compatible with (l, s) (as $l(v_5) = 160n$). □

4 Generalized theta graphs

If $k = 2$, then the upper bound in Theorem 6 is not tight. The so called theta graphs are used in characterizing 2-choosable graphs. For positive integers a, b, c, the theta graph $\theta_{a,b,c}$ is the graph obtained from three disjoint paths $P_1 = (x_0, x_1, \dots, x_a), P_2 = (y_0, y_1, \dots, y_b)$ and $P_3 = (z_0, z_1, \dots, z_c)$ by identifying x_0, y_0, z_0 into a single vertex and identifying x_a, y_b, z_c into a single vertex. Given a graph G, the *heart* of G is the graph H obtained from G by repeatedly deleting degree 1 vertices.

It is proved in [1] that a connected graph G is 2-choosable if and only if the heart of G is K_1 or an even cycle or $\theta_{2,2,2k}$ for some $k \geq 1$. Given a graph G, let

$$\mathrm{mad}(G) = \max\{2|E(H)|/|V(H)| : \ H \text{ is a subgraph of } G\}.$$

The following result was proved in [4].

Theorem 9. *If G is a bipartite graph, then*

$$ch_c(G) \leq \mathrm{mad}(G).$$

Since $ch_{cc}(G) \leq ch_c(G)$, for any bipartite graph G, we have $ch_{cc}(G) \leq \mathrm{mad}(G)$. As a consequence, if the heart of G is K_1 or an even cycle, then $ch_{cc}(G) \leq 2$; if the heart of G is $\theta_{2,2,2k}$ for some $k \geq 1$, then $ch_{cc}(G) \leq (4k+8)/(2k+3)$. It was proved in [4] that $ch_c(K_{2,3}) = 2$. As $K_{2,3} = \theta_{2,2,2}$, we have $ch_{cc}(\theta_{2,2,2}) \leq ch_c(\theta_{2,2,2}) = 2$. It was proved in [3] that $ch_{cc}(\theta_{2,2,4}) = 2$. Combine these results with Theorem 9, we have the following corollary, which gives a better upper bound on $ch_{cc}(G)$ for 2-choosable graphs G than that given in Theorem 6.

Corollary 10. *If G is 2-choosable, then $ch_{cc}(G) \leq 20/9$.*

A natural question is to find the tight upper bound on $ch_{cc}(G)$ for 2-choosable graphs G. It was conjectured in [4] that if G is 2-choosable, then $ch_c(G) \leq 2$. The following is a weaker conjecture:

Conjecture 11. *If G is 2-choosable, then $ch_{cc}(G) \leq 2$.*

To prove Conjecture 11, it suffices to show that $ch_{cc}(\theta_{2,2,2k}) = 2$ for every $k \geq 1$. For $k \geq 3$, the question is open. However, even if Conjecture 11 is confirmed, it does not answer the following question:

Question 12. *Which graphs G have $ch_{cc}(G) \leq 2$?*

If G has a vertex x of degree 1, then $ch_{cc}(G) \leq 2$ if and only if $ch_{cc}(G - x) \leq 2$. So a graph G is circular consecutive 2-choosable if and only if the heart of G is circular consecutive 2-choosable. There are graphs G that are not 2-choosable but are circular consecutive 2-choosable. For example, it is shown in [3] that for any odd cycle C_n, $ch_{cc}(C_n) = 2$.

Also it is easy to show that for any integer $n \geq 2$, $ch_{cc}(K_{2,n}) = 2$. Assume $2 \leq r < 4$, $V(K_{2,n}) = \{u, v\} \cup \{x_1, x_2, \ldots, x_n\}$, and L is a 2-circular consecutive colour-list assignment of G with respect to r. Then $L(u) \cap L(v) \neq \emptyset$. Let $f(u) = f(v) = t \in L(u) \cap L(v)$, and let $f(x_i) \in L(x_i) \setminus (t-1, t+1)_r$. Then f is a circular L-colouring of $K_{2,n}$.

Observe that $K_{2,3}$ is $\theta_{2,2,2}$. By finding the "theta graph" for $K_{2,n}$, we can define *generalized theta graphs* as follows. Let $n \geq 2$ and $k_1, k_2, \cdots, k_n \geq 1$ be integers and let P_i (for $i = 1, 2, \cdots, n$) be a path of length k_i. We denote by $\theta_{k_1,k_2,\cdots,k_n}$ the graph obtained from the disjoint union of $P_1, P_2, \cdots, P_n$ by identifying their initial vertices into a single vertex x and their terminal vertices into a single vertex y. So $\theta_{a,b}$ is a cycle of length $a + b$ and $\theta_{a,b,c}$ is the theta graph defined above. As mentioned above, we conjecture that for any positive integer n, the graph $\theta_{2,2,2n}$ is circular consecutive 2-choosable. On the other hand, $\theta_{\underbrace{2,2,\cdots,2}_{n}}$ is simply the graph $K_{2,n}$ and hence is also circular consecutive 2-choosable.

Question 13. *For which positive integers $k_1, k_2, \cdots, k_n$, the generalized theta graph $\theta_{k_1,k_2,\cdots,k_n}$ is circular consecutive 2-choosable?*

In the following, we provide some partial answer to this question. First we consider circular L-colourings of the graph $\theta_{2,2,2}$ for some special colour-list assignment L.

Let the three paths of length 2 in $\theta_{2,2,2}$ be (x, z_1, x'), (x, z_2, x') and (x, z_3, x'). Assume $0 < \epsilon \leq 1/2$. Let $0 < \delta \leq (1-\epsilon)/3$ and $r = 4 - \epsilon$. Let $l : V(\theta_{2,2,2}) \to [0, 4-\epsilon)$ be defined as

$$l(v) = \begin{cases} 0, & \text{if } v = x, \\ 2+2\delta, & \text{if } v = x', \\ r-1-\delta, & \text{if } v = z_1, \\ r-1+3\delta+\epsilon, & \text{if } v = z_2, \\ 1+\delta, & \text{if } v = z_3, \end{cases}$$

Lemma 14. *Let $l : V(\theta_{2,2,2}) \to [0, 4-\epsilon)$ be defined as above. Let $L(v) = (l(v), l(v)+2+\delta)_r$ for $v \in \theta_{2,2,2}$. If f is a circular L-colouring of $\theta_{2,2,2}$, then*

$$f(x) \in (0, 4\delta+\epsilon)_r \quad \text{and} \quad f(x') \in (r-\delta, 3\delta+\epsilon)_r.$$

Proof. Assume the lemma is not true and f is a circular L-colouring of $\theta_{2,2,2}$ for which $f(x) \notin (0, 4\delta+\epsilon)_r$ or $f(x') \notin (r-\delta, 3\delta+\epsilon)_r$.

First we consider the case that $f(x) \notin (0, 4\delta+\epsilon)_r$. Then $f(x) \in [4\delta+\epsilon, 2+\delta)_r$. (Refer to Figure 1 for the positions of the intervals $L(x), L(x'), L(z_1)$, $L(z_2)$ and $L(z_3)$.)

Since $L(z_2) = (r-1+3\delta+\epsilon, 1+4\delta+\epsilon)_r$, this forces $f(z_2) \in (r-1+3\delta+\epsilon, f(x)-1]_r$. As $L(x') = (2+2\delta, 3\delta+\epsilon)_r$, we must have $f(x') \in (2+2\delta, f(z_2)-1]_r$. On the other hand, we have $f(z_3) \in [f(x)+1, f(x')-1]_r$. The four colours $f(x), f(z_2), f(x'), f(z_3)$ occur in the circle $S(r)$ in this cyclic order, and every two consecutive colours have distance at least 1. This is a contradiction, because $S(r)$ has length $r = 4-\epsilon < 4$.

If $f(x') \notin (r-\delta, 3\delta+\epsilon)_r$, then $f(x') \in (2+2\delta, r-\delta]_r$. This forces $f(z_1) \in [f(x')+1, 1)_r$, which in turn forces $f(x) \in [f(z_1)+1, 2+\delta)_r$. As

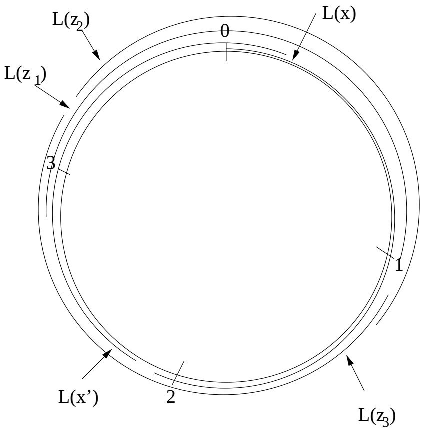

Figure 1: The intervals $L(x), L(x'), L(z_1), L(z_2), L(z_3)$

$f(z_3) \in [f(x)+1, f(x')-1]_r$ the four colours $f(x'), f(z_1), f(x), f(z_3)$ occur in $S(r)$ in this cyclic order, and every two consecutive colours have distance at least 1, which leads to the same contradiction. □

In the following, we use Lemma 14 to prove that $\theta_{2,2,2,n}$ has circular consecutive choosability greater than 2, provided that $n \neq 2, 4, 6$, and $\theta_{2,2,2,2,n}$ has circular consecutive choosability greater than 2 if $n \neq 2, 6$.

Theorem 15. *Suppose $n \geq 0$ is an integer. Then*

1. $ch_{cc}(\theta_{2,2,2,2n+1}) \geq 2 + 1/(n+5)$.

2. $ch_{cc}(\theta_{2,2,2,2n+8}) \geq 2 + 2/(4n+21)$.

Proof. Let the graph $\theta_{2,2,2,k}$ be obtained from the graph $\theta_{2,2,2}$, with vertices labeled as in Lemma 14, by adding the path $(x, y_1, y_2, \cdots, y_{k-1}, x')$.

First we show that $ch_{cc}(\theta_{2,2,2,2n+1}) \geq 2 + 1/(n+5)$ for any $n \geq 0$. It suffices to show that for any $0 < \epsilon \leq 1/2$, for $r = 4 - \epsilon$ and for $\delta = (1-\epsilon)/(n+5)$, there is a list assignment L which assigns to each vertex v an open interval of length $2 + \delta$ of $S(r)$, for which there is no circular L-colouring of $\theta_{2,2,2,2n+1}$.

Let $l : V(\theta_{2,2,2,2n+1}) \to [0, 4-\epsilon)$ be defined so that the restriction to $\theta_{2,2,2}$ is the same as in Lemma 14, and

$$l(y_j) = \begin{cases} r + (4+t)\delta + \epsilon - 1, & \text{if } j = 2t+1, \\ r - t\delta, & \text{if } j = 2t. \end{cases}$$

We shall show that there is no circular L-colouring of $\theta_{2,2,2,2n+1}$. Assume to the contrary that there is a circular L-colouring f of $\theta_{2,2,2,2n+1}$. By Lemma 14, $f(x) \in (0, 4\delta + \epsilon)_r$ and $f(x') \in (r - \delta, 3\delta + \epsilon)_r$.

Since $L(y_1) = (r + 4\delta + \epsilon - 1, 1 + 5\delta + \epsilon)_r$ and $|f(x) - f(y_1)|_r \geq 1$, we conclude that $f(y_1) \in (1, 1 + 5\delta + \epsilon)_r$. Since $L(y_2) = (r - \delta, 2)_r$ and $|f(y_1) - f(y_2)|_r \geq 1$, we have $f(y_2) \in (r - \delta, 5\delta + \epsilon)_r$. Inductively, one can show that

$$\begin{aligned} f(y_{2j+1}) &\in (1 - j\delta, 1 + (j+5)\delta + \epsilon)_r \\ f(y_{2j}) &\in (r - j\delta, (j+4)\delta + \epsilon)_r. \end{aligned}$$

In particular, $f(y_{2n}) \in (r - n\delta, (n+4)\delta + \epsilon)_r$. As $f(x') \in (r - \delta, 3\delta + \epsilon)_r$ and $(n+5)\delta + \epsilon < 1$, we conclude that $|f(x') - f(y_{2n})|_r < 1$, in contrary to the assumption that f is a circular L-colouring of $\theta_{2,2,2,2n+1}$. This completes the proof of (1).

Next we prove that $ch_{cc}(\theta_{2,2,2,2n+8}) \geq 2 + 2/(4n+21)$ for any $n \geq 0$.

Let $\epsilon = \frac{2n+6}{4n+21}$, $r = 4 - \epsilon$ and $\delta = \frac{2}{4n+21}$. Let $l : V(\theta_{2,2,2,2n+8}) \to [0, 4 - \epsilon)$ be defined so that the restriction of l to $\theta_{2,2,2}$ is as defined in Lemma 14 and

$$l(y_j) = \begin{cases} j - 2 + (3 + j)\delta + \epsilon, & \text{if } 1 \leq j \leq 7, \\ 6 + (7 + t)\delta + \epsilon, & \text{if } j = 2t \geq 8, \\ 7 - (t - 3)\delta, & \text{if } j = 2t + 1 \geq 9. \end{cases}$$

Now we shall prove that $\theta_{2,2,2,2n+8}$ has no circular L-colouring. Assume to the contrary that f is a circular L-colouring of $\theta_{2,2,2,2n+8}$. By Lemma 14, $f(x) \in (0, 4\delta + \epsilon)_r$ and $f(x') \in (r - \delta, 3\delta + \epsilon)_r$.

Similarly as in the proof of (1), we can prove by induction that for $j = 1, 2, \cdots, 7$,

$$f(y_j) \in (j, j + (j + 4)\delta + \epsilon)_r.$$

For $j = 2t \geq 8$,

$$f(y_j) \in (8 - (t - 4)\delta, 8 + (t + 8)\delta + \epsilon)_r.$$

For $j = 2t + 1 \geq 9$,

$$f(y_j) \in (7 - (t - 3)\delta, 7 + (t + 8)\delta + \epsilon)_r.$$

In particular,

$$f(y_{2n+7}) \in (7 - n\delta, 7 + (n + 11)\delta + \epsilon)_r.$$

However, it is straightforward to verify that for any $a \in (r-\delta, 3\delta+\epsilon)_r$, for any $b \in (7-n\delta, 7+(n+11)\delta+\epsilon)_r$, $|a-b|_r < 1$. This is in contrary to our assumption that f is a circular L-colouring of $\theta_{2,2,2,2n+8}$. This completes the proof of (2). □

We do not know whether $ch_{cc}(\theta_{2,2,2,2n}) > 2$ for $n = 2, 3$. The next lemma shows that $ch_{cc}(\theta_{2,2,2,2,4}) > 2$.

Theorem 16. $ch_{cc}(\theta_{2,2,2,2,4}) \geq 2 + 1/8$.

Proof. Similar to Lemma 14, we first consider circular L-colourings of $\theta_{2,2,2,2}$, which is obtained from the graph $\theta_{2,2,2}$ in Lemma 14 by adding the path (x, z_4, x'). Let $l : V(\theta_{2,2,2,2}) \to [0, 4-\epsilon)$ be defined such that the restriction of l to $\theta_{2,2,2,2} \setminus \{z_4\}$ is the same as in Lemma 14, and let $l(z_4) = r-1+\delta+\epsilon/2$.

Claim 2. *If f is a circular L-colouring of $\theta_{2,2,2,2}$ then either*

$$f(x) \in (0, 2\delta+\epsilon/2)_r, \quad \text{and} \quad f(x') \in (-\delta, 2\delta+\epsilon/2)_r$$

or

$$f(x) \in (\delta+\epsilon/2, 4\delta+\epsilon)_r, \quad \text{and} \quad f(x') \in (\delta+\epsilon/2, 3\delta+\epsilon)_r.$$

Proof. If the claim is not true, then by using Lemma 14, we conclude that one of $f(x), f(x')$ lies in the interval $(-\delta, \delta+\epsilon/2]_r$ and the other lies in the interval $[2\delta+\epsilon/2, 4\delta+\epsilon)_r$. Since z_4 is adjacent to both x and x', there is no legal colour for z_4 in the interval $L(z_4)$. This proves the claim. □

Let $l : V(\theta_{2,2,2,2,4}) \to [0, 4-\epsilon)$ be defined so that the restriction of l to $\theta_{2,2,2,2}$ is as in Claim 2 and for $j = 1, 2, 3$, $l(y_j) = j-2+(3+j)\delta+\epsilon$. We shall show that, for appropriate ϵ and δ, $\theta_{2,2,2,4}$ has no circular L-colouring. Assume to the contrary that f is a circular L-colouring of $\theta_{2,2,2,2,4}$.

Let $\epsilon = 1/2$ and let $\delta = 1/8$. By Claim 2, we have two cases.

Case 1

$$f(x) \in (0, 2\delta+\epsilon/2)_r, \quad \text{and} \quad f(x') \in (-\delta, 2\delta+\epsilon/2)_r.$$

By using the proof of Theorem 15, we can show that $f(y_3) \in (3, 3+7\delta+\epsilon)_r$. Since $\epsilon = 1/2$ and $\delta = 1/8$, straightforward calculation shows that for any $a \in (-\delta, 2\delta+\epsilon/2)_r$, for any $b \in (3, 3+7\delta+\epsilon)_r$, we have $|a-b|_r < 1$, in contrary to our assumption that f is a circular L-colouring of $\theta_{2,2,2,2,4}$.

Case 2

$$f(x) \in (\delta + \epsilon/2, 4\delta + \epsilon)_r, \quad \text{and} \quad f(x') \in (\delta + \epsilon/2, 3\delta + \epsilon)_r.$$

Observe that, in comparison with Case 1, the possible colour of $f(x)$ is "shifted to the right" by a distance of $\delta + \epsilon/2$. By using the argument as in the proof of Theorem 15, we can show that $f(y_3) \in (3+\delta+\epsilon/2, 3+7\delta+\epsilon)_r$. Again, straightforward calculation shows that for any $a \in (\delta + \epsilon/2, 3\delta + \epsilon)_r$, for any $b \in (3+\delta+\epsilon/2, 3+7\delta+\epsilon)_r$, we have $|a-b|_r < 1$, in contrary to our assumption that f is a circular L-colouring of $\theta_{2,2,2,2,4}$. □

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