Circular chromatic number of Kneser graphs

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Abstract

This paper proves that for any positive integer n, if m is large enough, then the reduced Kneser graph $\mathrm{KG}_2(m,n)$ has its circular chromatic number equal its chromatic number. This answers a question of Lih and Liu [J. Graph Theory, 2002]. For Kneser graphs, we prove that if $m \geq 2n^2(n-1)$, then $\mathrm{KG}(m,n)$ has its circular chromatic number equal its chromatic number. This provides strong support for a conjecture of Johnson, Holroyd and Stahl [J. Graph Theory, 26(1997), 137-145].

Suppose $m \geq 2n$ are positive integers. We denote by $[m]$ the set $\{1, 2, \cdots, m\}$, and denote by $\binom{[m]}{n}$ the collection of all n-subsets of $[m]$. The *Kneser graph* $\mathrm{KG}(m,n)$ has vertex set $\binom{[m]}{n}$, in which $A \sim B$ if and only if $A \cap B = \emptyset$. It was conjectured by Kneser [3] in 1955 and proved by Lovász [5] in 1978 that $\chi(\mathrm{KG}(m,n)) = m - 2n + 2$.

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A subset S of $[m]$ is called 2-*stable* if $2 \leq |x-y| \leq m-2$ for distinct elements x and y of S. The *reduced Kneser graph* $\mathrm{KG}_2(m,n)$ is the subgraph of $\mathrm{KG}(m,n)$ induced by all 2-stable n-subsets. It was proved by Schrijver [6] that $\chi(\mathrm{KG}_2(m,n)) = \chi(\mathrm{KG}(m,n))$ and every proper subgraph of $\mathrm{KG}_2(m,n)$ has a smaller chromatic number.

Given positive integers $k \geq 2d$, a (k,d)-*coloring* of a graph G is a mapping c from the vertex set $V(G)$ to the set $\{0,1,\ldots,k-1\}$ such that $d \leq |\, c(x) - c(y) \,| \leq k-d$ whenever x and y are adjacent vertices. The *circular chromatic number* $\chi_c(G)$ (also known as the "star chromatic number" [7]) is defined to be the infimum of k/d such that G has a (k,d)-coloring. It is known [7, 8] that for any graph G, $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$, and hence $\chi(G) = \lceil \chi_c(G) \rceil$. So $\chi_c(G)$ is a refinement of $\chi(G)$, and $\chi(G)$ is an approximation of $\chi_c(G)$.

The circular chromatic number of Kneser graphs has been studied by Johnson, Holroyd, and Stahl [2]. They proved that $\chi_c(\mathrm{KG}(m,n)) = \chi(\mathrm{KG}(m,n))$ if $m \leq 2n+2$ or $n = 2$, and conjectured that equality holds for all Kneser graphs.

Conjecture 1 [2] *For all $m \geq 2n+1$, $\chi_c(\mathrm{KG}(m,n)) = \chi(\mathrm{KG}(m,n))$.*

The circular chromatic number of reduced Kneser graphs has been studied by Lih and Liu [4]. They proved that $\chi_c(\mathrm{KG}_2(m,2)) = \chi(\mathrm{KG}_2(m,2))$ if $m \geq 4$ and $m \neq 5$. Note that $\chi_c(\mathrm{KG}_2(2n+1,n)) = 2 + \frac{1}{n} < 3 = \chi(\mathrm{KG}_2(2n+1,n))$. Lih and Liu [4] asked the following question:

Question 1 [4] *Given a positive integer $n > 1$, does there exist a number $t(n)$ such that the equality $\chi_c(\mathrm{KG}_2(m,n)) = \chi(\mathrm{KG}_2(m,n))$ holds for all $m \geq t(n)$?*

This paper gives a positive answer to Question 1 and provides support for Conjecture 1.

Lemma 1 *Suppose G is an n-vertex graph with $\chi(G) = k$. If for some $s \geq 1$, any independent set X of size $\geq \frac{sn}{(s+1)k}$ is contained in a unique maximal independent set, then $\chi_c(G) = p/q$ for some $q \leq s$. In particular, if any independent set of size $\geq \frac{n}{2k}$ is contained in a unique maximal independent set, then $\chi_c(G) = \chi(G)$.*

Proof. Assume to the contrary that $\chi_c(G) = p/q$ such that $q \geq s+1$ and $(p,q) = 1$. Let c be a (p,q)-coloring of G, and let $X_i = c^{-1}(i)$. It is well-known (see Lemma 1.3

of [8]) that for each i, $X_i \neq \emptyset$ and there exist $x \in X_i, y \in X_{i+q}$ such that $x \sim y$. (In this proof, the summation in the indices are modulo p). For $i = 0, 1, \cdots, p-1$, let

$$Y_i = X_i \cup X_{i+1} \cup \cdots \cup X_{i+q-2}.$$

Then each Y_i is an independent set, and

$$\sum_{i=0}^{p-1} |Y_i| = (q-1)n$$

as each vertex of G is contained in $q-1$ of the Y_i's. Therefore for some i,

$$|Y_i| \geq \frac{(q-1)n}{p} = \frac{q-1}{q}\frac{n}{p/q} \geq \frac{s}{s+1}\frac{n}{k}.$$

By our assumption, Y_i is contained in a unique maximal independent set. However, $Y = Y_i \cup X_{i-1}$ and $Y' = Y_i \cup X_{i+q-1}$ are both independent sets containing Y_i, so each can be extended to a maximal independent set, say Y^* and Y'^*. Since $Y \cup Y'$ is not independent (as X_{i-1} contains a vertex which is adjacent to a vertex of X_{i-1+q}), Y^* and Y'^* are distinct maximal independent sets, which is a contradiction. ∎

Theorem 1 *For any fixed positive integer n, if m is large enough, then* $\chi_c(\mathrm{KG}_2(m,n)) = \chi(\mathrm{KG}_2(m,n))$.

Proof. We denote by V the vertex set of $\mathrm{KG}_2(m,n)$. By Lemma 1, it suffices to show that any independent set X of $\mathrm{KG}_2(m,n)$ of size $\geq |V|/2(m-2n+2)$ is contained in a unique maximal independent set (when m is sufficiently large). The vertex set V has cardinality $\binom{m-n-1}{n-1}\frac{m}{n} = \Omega(m^n)$. (Each 2-stable n-subset of $[m]$ containing 1 corresponds to an integral solution of the equation $x_1 + x_2 + \cdots + x_n = m$ with $x_i \geq 2$. So there are $\binom{m-n-1}{n-1}$ 2-stable n-subsets of $[m]$ containing 1). Thus $|V|/2(m-2n+2) = \Omega(m^{n-1})$. It was proved by Hilton and Milner [1] that if X is an independent set of $\mathrm{KG}(m,n)$ of size

$$\binom{m-1}{n-1} - \binom{m-n-1}{n-1} + 2,$$

then

$$\cap_{A \in X} A = \{i\},$$

for some $i \in [m]$. Note that

$$\binom{m-1}{n-1} - \binom{m-n-1}{n-1} + 2 = O(m^{n-2}).$$

Therefore, there exists an integer $t(n)$ such that if $m \geq t(n)$, then

$$\binom{m-1}{n-1} - \binom{m-n-1}{n-1} + 2 \leq |V|/2(m-2n+2).$$

If X is an independent set of $\mathrm{KG}_2(m,n)$ of size $\geq |V|/2(m-2n+2)$, then X is an independent set of $\mathrm{KG}(m,n)$ (as $\mathrm{KG}_2(m,n)$ is an induced subgraph of $\mathrm{KG}(m,n)$) and

$$|X| \geq |V|/2(m-2n+2) \geq \binom{m-1}{n-1} - \binom{m-n-1}{n-1} + 2.$$

Hence

$$\cap_{A \in X} A = \{i\},$$

for some $i \in [m]$. Any independent set Y containing X also has the property

$$\cap_{A \in Y} A = \{i\}.$$

Therefore X is contained in the unique maximal independent set $Y = \{A \in V : i \in A\}$ of $\mathrm{KG}_2(m,n)$. ∎

Theorem 2 *For any positive integer n, if $m \geq 2n^2(n-1)$, then $\chi_c(\mathrm{KG}(m,n)) = \chi(\mathrm{KG}(m,n))$.*

Proof. By the result of Johnson, Holroyd, and Stahl [2], we only need to consider the case that $n \geq 3$. Assume that $n \geq 3$ and $m \geq 2n^2(n-1)$. By the proof of Theorem 1, for the equality $\chi_c(\mathrm{KG}(m,n)) = \chi(\mathrm{KG}(m,n))$ to hold, it suffices that

$$\binom{m-1}{n-1} - \binom{m-n-1}{n-1} + 2 \leq \frac{\binom{m}{n}}{2(m-2(n-1))}.$$

Or equivalently,

$$(1 - \frac{m-n-1}{m-1} \cdots \frac{m-2n+1}{m-n+1})\frac{m-2(n-1)}{m} \leq \frac{1}{2n} - \frac{2(m-2(n-1))}{\binom{m-1}{n-1}m}.$$

For $3 \leq n \leq 4$, the inequality can be verified by straightforward calculations. Assume $n \geq 5$. In the following, we shall use the inequality that for any $x > -1$,

$$e^{x/(x+1)} \leq 1 + x \leq e^x.$$

By using the inequality above, for $i = 1, 2, \cdots, n-1$,

$$\frac{m-n-i}{m-i} \quad \geq \quad e^{-n/(m-n-i)}.$$

As $n \geq 5$ and $m \geq 2n^2(n-1)$, easy calculation shows that for $i = 1, 2, \cdots, n-1$,

$$\frac{1}{m-n-i} + \frac{1}{m-2n+i} \leq \frac{2}{m-2n+2} - \frac{4n(n-1)}{(m-n)^2(m-2n+2)}.$$

So

$$\sum_{i=1}^{n-1} \frac{1}{m-n-i} \quad \leq \quad \frac{n-1}{m-2n+2} - \frac{2n(n-1)^2}{(m-n)^2(m-2n+2)}$$

Hence

$$\begin{aligned} \frac{m-n-1}{m-1} \cdots \frac{m-2n+1}{m-n+1} \quad &\geq \quad e^{-\sum_{i=1}^{n-1} \frac{n}{m-n-i}} \\ &\geq \quad e^{-(\frac{n(n-1)}{m-2n+2} - \frac{2n^2(n-1)^2}{(m-n)^2(m-2n+2)})}. \end{aligned}$$

Therefore,

$$\begin{aligned} &(1 - \frac{m-n-1}{m-1} \cdots \frac{m-2n+1}{m-n+1}) \frac{m-2(n-1)}{m} \\ &\leq (1 - e^{-(\frac{n(n-1)}{m-2n+2} - \frac{2n^2(n-1)^2}{(m-n)^2(m-2n+2)})}) \frac{m-2(n-1)}{m} \\ &\leq (\frac{n(n-1)}{m-2n+2} - \frac{2n^2(n-1)^2}{(m-n)^2(m-2n+2)}) \frac{m-2(n-1)}{m} \\ &= \frac{n(n-1)}{m} - \frac{2n^2(n-1)^2}{m(m-n)^2} \\ &\leq \frac{1}{2n} - \frac{2(m-2(n-1))}{\binom{m-1}{n-1} m}. \end{aligned}$$

∎

Conjecture 1 remains open for $2n+3 \leq m < 2n^2(n-1)$. Although Question 1 is settled, the following question, asked by Lih and Liu [4], remains open.

What is the minimum $t(n)$ such that for any $m \geq t(n)$, $\chi_c(\mathrm{KG}_2(m,n)) = \chi(\mathrm{KG}_2(m,n))$?

The upper bound for $t(n)$ derived from the proof of Theorem 1 is certainly too large. At present, the only known examples of reduced Kneser graphs G for which $\chi_c(G) \neq \chi(G)$ are $G = \mathrm{KG}_2(2n+1,n)$ for $n \geq 2$, which implies that $t(n) \geq 2n+2$.

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