Density of the circular chromatic numbers of series-parallel graphs

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Abstract

Suppose G is a series-parallel graph. It was proved in [3] that either $\chi_c(G) = 3$ or $\chi_c(G) \leq 8/3$. So none of the rationals in the interval $(8/3, 3)$ is the circular chromatic number of a series-parallel graph. This paper proves that for every rational $r \in [2, 8/3] \cup \{3\}$ there exists a series-parallel graph G with $\chi_c(G) = r$.

1 Introduction

The circular chromatic number (also known as the star chromatic number) $\chi_c(G)$ of a graph G is a natural generalization of the chromatic number of a graph. There are quite a few equivalent definitions for the circular chromatic number of a graph [10, 11, 14]. In this paper we use the following definition:

Suppose $r \geq 1$ is a real number. An *r-colouring* of a graph G is a mapping $f : V(G) \to [0, r)$ such that for every edge uv of G we have $1 \leq |f(u) - f(v)| \leq r - 1$. We say G is r-colourable if there exists an r-colouring of G. In case $r = k$ is an integer, then r-colourability coincides with the ordinary vertex k-colourability. The *circular chromatic number* $\chi_c(G)$ of G is the infimum of those r for which G is r-colourable.

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It is known [14] that for a finite graph G, the infimum in the definition is always attained and hence can be replaced by the minimum. Moreover for a finite graph G, the circular chromatic number of G is always a rational.

If $r' \geq r$ and G is r-colourable then G is r'-colourable. Since for an integer $r = k$, r-colourability coincides with the ordinary k-colourability, so for any finite graph G we have

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G).$$

Therefore $\chi(G) = \lceil \chi_c(G) \rceil$. So the circular chromatic number $\chi_c(G)$ is a refinement of the chromatic number $\chi(G)$, and $\chi(G)$ is an approximation of $\chi_c(G)$.

It follows from the definition that any non-trivial graph has circular chromatic number at least 2. On the other hand, it was shown in [1, 10] that for any rational number $r \geq 2$, there exists a finite graph G of circular chromatic number r.

However if we restrict to graphs with a certain property, then the values for their circular chromatic numbers maybe restricted. The well-known Hadwiger's conjecture says that K_n-minor free graphs have chromatic number at most $n-1$. If this conjecture is true then the circular chromatic numbers of K_n-minor free graphs are also at most $n-1$. An interesting question related to this conjecture is this:

Question 1.1 *Is it true that for every rational number $2 \leq r \leq n-1$ there is a K_n-minor free graph G with $\chi_c(G) = r$?*

The question has been studied in [3, 5, 12, 13, 4], and now we have a complete solution. It is proved in [4] that every rational $r \in [2, n-1]$ is the circular chromatic number of a K_n-minor free graph if $n \geq 5$. In sharp contrast to this, the answer is negative for $n = 4$. The following result is proved in [3]:

Theorem 1.1 *If G is a K_4-minor free graph then either $\chi_c(G) = 3$ or $\chi_c(G) \leq 8/3$.*

So the interval $(8/3, 3)$ is a gap among the circular chromatic numbers of K_4-minor free graphs. A natural question is: are there other gaps among the circular chromatic numbers of K_4-minor free graphs ?

This paper answers this question: there are no other gaps. To be precise we shall prove the following result:

Theorem 1.2 *Suppose r is a rational number. If $r = 3$ or $2 \leq r \leq 8/3$ then there exists a K_4-minor free graph G with $\chi_c(G) = r$.*

2 Two terminal series-parallel graphs

The class of K_4-minor free graphs can be defined in many different ways and is referred to by different names, such as series-parallel graphs, partial 2-trees, etc., [3, 7]. We adopt the following definition of two-terminal series-parallel graphs from [3]. A two-terminal series-parallel graph $(G; x, y)$ is defined recursively as follows:

- Let $V(K_2) = \{0, 1\}$. Then $(K_2; 0, 1)$ is a two-terminal series-parallel graph.
- (The parallel construction.) Let $(G; x, y)$ and $(G'; x', y')$ be two disjoint two-terminal series-parallel graphs. Define G'' to be the graph obtained from the union of G and G' by identifying x and x' into a single vertex x'', and identifying y and y' into a single vertex y''. Then $(G''; x'', y'')$ is a two-terminal series-parallel graph.
- (The series construction.) Let again $(G; x, y)$ and $(G'; x', y')$ be two disjoint two-terminal series-parallel graphs. Define G'' to be the graph obtained from the union of G and G' by identifying y and x' into a single vertex. Then $(G''; x, y')$ is a two-terminal series-parallel graph.
- There are no other two-terminal series-parallel graphs.

A graph G is a series-parallel graph if there exist some two vertices x, y such that $(G; x, y)$ is a two-terminal series-parallel graph. For all the series-parallel graphs in the remaining part, there are always two terminals which are clearly indicated in the context. Moreover, if the series-parallel graph is denoted by G (resp. G', G'' etc.) then the two terminals are denoted by x and y (resp. x' and y', x'' and y'', etc.). The circular chromatic number of series-parallel graphs has been studied in [2, 3, 8, 9].

It is well-known [7] that a graph G is K_4-minor free if and only if every block of G is a series-parallel graph. Therefore it suffices to prove Theorem 1.2 for series-parallel graphs, i.e., we shall prove the following:

Theorem 2.1 *Suppose r is a rational number. If $r = 3$ or $2 \leq r \leq 8/3$ then there is a series-parallel graph G with $\chi_c(G) = r$.*

3 The labeling method

For a real number $r \geq 1$, an r-colouring of a graph G uses all the real numbers in the interval $[0, r)$ as colours. We view the numbers in $[0, r)$ as cyclically ordered, i.e., we identify the two ends 0 and r into a single point to obtain a circle C^r of circumference r, as depicted in Fig. 1 below. Thus an r-colouring f of G "colours" the vertices of G with points of C^r, and the points assigned to adjacent vertices must be at least unit distance apart.

As a graph with at least one edge has circular chromatic number at least 2, in the following we shall only consider r-colouring for $r \geq 2$.

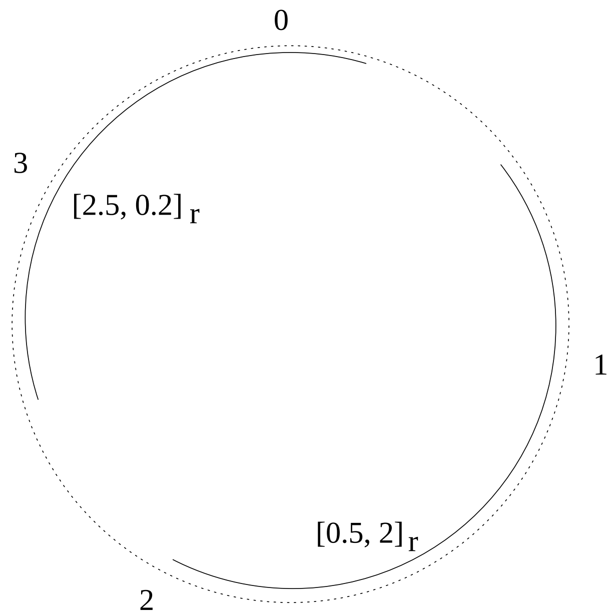

Figure 1: Illustration of the circle C^r for $r = 3.5$ and intervals of C^r

Suppose $r \geq 2$ is a real number. For any two real numbers $a, b \in [0, r)$, we denote by $[a, b]_r$ the interval of C^r goes from a to b along the clockwise (i.e., the increasing) direction. To be precise, if $a < b$ then the interval $[a, b]_r$ contains all the real numbers t such that $a \leq t \leq b$; if $a > b$ then the interval $[a, b]_r$ contains all the real numbers t such that either $a \leq t < r$ or $0 \leq t \leq b$. The case $a = b$ is ambiguous (it could be the whole circle C^r or just a single point). This case rarely occurs in this paper and for the few places it occurs, the meaning will be clear from the context. When the real number r is clear from the context, we shall write $[a, b]$ for $[a, b]_r$.

For $r = 3.5$, the intervals $[0.5, 2]$ and $[2.5, 0.2]$ are depicted in Fig. 1. The length $\ell([a, b]_r)$ of an interval $[a, b]_r$ of C^r is just the geometric length of that interval on C^r, i.e., if $a < b$ then $\ell([a, b]_r) = b - a$; if $a > b$ then

$\ell([a,b]_r) = b + r - a$.

Given a series-parallel graph G with terminals x and y. The r-label set $L_r(G)$ of G is a subset of C^r defined as follows:

$L_r(G) = \{t \in C^r :$ there is an r-colouring f of G such that $f(x) = 0$ and $f(y) = t\}$.

It follows from this definition that G is r-colourable if and only if $L_r(G) \neq \emptyset$.

For two subsets A, B of C^r, we define

$$A + B = \{t : \exists a \in A, b \in B, a + b \bmod r = t\}.$$

Lemma 3.1 *Suppose $r \geq 2$ is a real number. The r-label set $L_r(G)$ of all series-parallel graphs G can be calculated recursively as follows:*

1. *If $G = K_2$ then $L_r(G) = [1, r-1]$;*

2. *If G is obtained from G_1 and G_2 by a parallel construction then $L_r(G) = L_r(G_1) \cap L_r(G_2)$;*

3. *If G is obtained from G_1 and G_2 by a series construction then $L_r(G) = L_r(G_1) + L_r(G_2)$.*

The proof of Lemma 3.1 is straightforward and omitted.

Lemma 3.2 *Suppose $A = [a,b]_r$ and $B = [c,d]_r$. If $\ell([a,b]_r) + \ell([c,d]_r) \geq r$ then $A + B = C^r$; if $\ell([a,b]_r) + \ell([c,d]_r) < r$ then $A + B = [a+c, b+d]$, where the summations are carried out modulo r.*

The proof of Lemma 3.2 is also straightforward and omitted.

Note that if $A = \emptyset$ then for any set B, $A + B = \emptyset$.

It is well-known that series-parallel graphs have circular chromatic numbers at most 3. Moreover, the only series-parallel graphs with circular chromatic number 3 are those that contain triangles (see [3]), which we are not interested in this paper. So in the remaining of this paper, we assume that $2 \leq r < 3$. Denote by P_i the path of length i (i.e., the path with i edges).

Example 3.1 *Suppose $2 \leq r < 3$. Let C_5 be the pentagon, and let G be the graph as depicted in Fig. 2. Then*

- $L_r(P_2) = [2, r-2]$.

- $L_r(P_3) = [3-r, 2r-3]$.

- *If* $r < 5/2$ *then* $L_r(C_5) = \emptyset$, *if* $5/2 \leq r < 3$ *then* $L_r(C_5) = [2, 2r-3] \cup [3-r, r-2]$.

- *If* $r < 8/3$ *then* $L_r(G) = \emptyset$, *if* $8/3 \leq r < 3$ *then*

  $$L_r(G) = [2, 3r-6] \cup [5-r, 2r-3] \cap [3-r, 2r-5] \cup [6-2r, r-2].$$

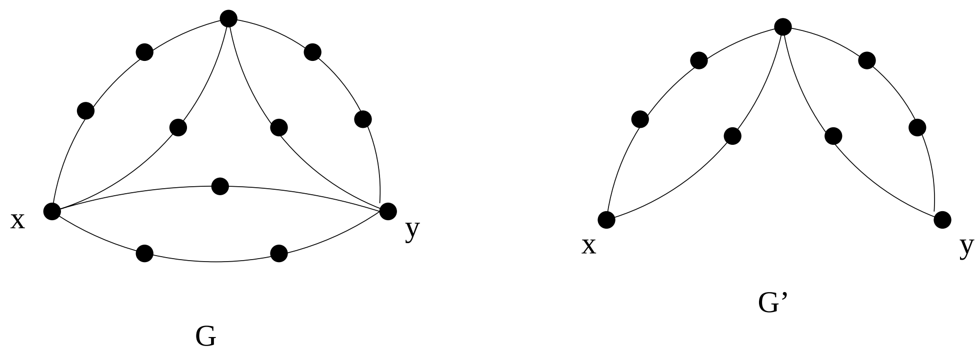

Figure 2: The graph G and G' for Example 3.1

Proof. The path P_2 is obtained from two copies of K_2 by a series construction. By Lemma 3.1,

$$L_r(P_2) = L_r(K_2) + L_2(K_2) = [1, r-1] + [1, r-1] = [2, r-2].$$

The path P_3 is obtained from P_2 and K_2 by a series construction. So

$$L_r(P_3) = L_r(P_2) + L_r(K_2) = [3, 2r-3].$$

The graph C_5 is obtained from P_2 and P_3 by a parallel construction. So

$$L_r(C_5) = L_r(P_2) \cap L_r(P_3).$$

Thus if $r < 5/2$ then $L_r(C_5) = \emptyset$ and if $r \geq 5/2$ then

$$L_r(C_5) = [2, 2r-3] \cup [3-r, r-2].$$

Now to construct the graph G as depicted in Fig. 2, we first do a series construction with two copies of C_5 to obtain the graph G' as depicted in Fig.

2, then do a parallel construction with C_5 and G'. By using Lemmas 3.1 and 3.2, when $r \geq 5/2$ we have

$$\begin{aligned} L_r(G') &= L_r(C_5) + L_r(C_5) \\ &= [4-r, 3r-6] \cup [6-2r, 2r-4] \cup [5-r, 2r-5]. \end{aligned}$$

Now if $r < 8/3$, then $L_r(G) = L_r(G') \cap L_r(C_5) = \emptyset$, if $r \geq 8/3$ then

$$\begin{aligned} L_r(G) &= L_r(G') \cap L_r(C_5) \\ &= [2, 3r-6] \cup [5-r, 2r-3] \cap [3-r, 2r-5] \cup [6-2r, r-2]. \end{aligned}$$

∎

It follows from the discussion above that $\chi_c(C_5) = 5/2$ and $\chi_c(G) = 8/3$ for the graph G as depicted in Fig. 2.

4 Proof of Theorem 2.1

The triangle K_3 is a series-parallel graph with $\chi_c(K_3) = 3$, K_2 is a series-parallel graph with $\chi_c(K_2) = 2$, and the example in Section 3 shows that there is a series-parallel graph G with $\chi_c(G) = 8/3$. In this section, for each fraction $p/q \in (2, 8/3)$ we shall construct a series-parallel graph G with $\chi_c(G) = p/q$.

The construction of these graphs is by induction on the denominator q. For each reduced fraction $p/q \in (2,3)$ we define the parents of p/q as follows: We construct a table in the following way. In the first row we write 2/1 and 3/1. For $n = 2, 3, \cdots$ we use the rule: Form the nth row by copying the $(n-1)$st in order, but insert the fraction $(a+a')/(b+b')$ between the consecutive fractions a/b and a'/b' if $b + b' \leq n$. For example, the second row is

$$2/1,\ 5/2,\ 3/1,$$

the third row is

$$2/1,\ 7/3,\ 5/2,\ 8/3,\ 3/1,$$

the fourth row is

$$2/1,\ 9/4,\ 7/3,\ 5/2,\ \ 8/3, 11/4,\ \ 3/1.$$

and the fifth row is

$$2/1,\ 11/5,\ 9/4,\ 7/3,\ 12/5,\ 5/2,\ 13/5,\ 8/3, 11/4,\ 14/5,\ 3/1.$$

Suppose the fraction p/q is constructed in the process above as $p/q = (a+a')/(b+b')$. We shall call the fraction $a/b, a'/b'$ the *lower parent* and the *upper parent* of p/q, respectively. We denote the lower parent of p/q by $p_l(p/q)$ and denote the upper parent of p/q by $p_u(p/q)$.

The nth row of the table constructed above is called the *Farey sequence of order* n [6]. It is well-known (see page 298 of [6]) that the nth row contains all the fractions in the set $[2,3]$ with denominator not exceeding n, and if a/b and a'/b' are two consecutive fractions of the nth row then $a'b - ab' = 1$. So each of the fractions $p/q \in [2,3]$ has a unique lower parent and a unique upper parent, except for $2/1$ and $3/1$ which have no parents.

Lemma 4.1 *Suppose $a/b < a'/b'$ are two consecutive fractions of the nth row for some n. If $b < b'$, then $(a'-2)/(b'-1) \le (a-2)/(b-1)$. Moreover, $(a+a'-2)/(b+b'-1) \le (a'-2)/(b'-1)$.*

Proof. Since $a/b < a'/b'$ and $b < b'$ we have

$$b'(a'/b'-2) - b(a/b-2) > 0.$$

So

$$b'(a'/b'-2) - b(a/b-2) = a' - 2b' - a + 2b \ge 1.$$

As $a'b - ab' = 1$, we have

$$a' - 2b' - a + 2b \ge a'b - ab'.$$

Therefore

$$(a'-2)(b-1) = a'b - a' - 2b + 2 \le ab' - 2b' - a + 2 = (a-2)(b'-1),$$

i.e.,

$$(a'-2)/(b'-1) \le (a-2)/(b-1).$$

For the moreover part, since $a/b > 2$ we have

$$2b - a < 0 < a'b - ab' = 1.$$

Therefore $ab' - a \le a'b - 2b$, which implies that

$$ab' + a'b' - 2b' - a - a' + 2 \le a'b + a'b' - a' - 2b - 2b' + 2,$$

i.e.,

$$(a+a'-2)/(b+b'-1) \le (a'-2)/(b'-1).$$

∎

Corollary 4.1 *If $p_l(p/q) = a/b$ and $p_u(p/q) = a'/b'$ then $(p-2)/(q-1) \leq (a-2)/(b-1)$ and $(p-2)/(q-1) \leq (a'-2)/(b'-1)$.*

Proof. If $p_l(p/q) = a/b$ then $a/b < p/q$ are two consecutive fractions of the nth row for some n and $b < q$. The second half follows from the moreover part of Lemma 4.1. ∎

Lemma 4.2 *Suppose $p/q \in (2, 8/3]$, $p_l(p/q) = a/b$ and $p_u(p/q) = a'/b'$. Then there is a series-parallel graph $G_{P/q}$ such that if $a/b \leq r < \min\{8/3, (p-2)/(q-1)\}$ then*

$$L_r(G_{p/q}) = [p-3-(q-2)r, (q-1)r-p+3],$$

if $r < a/b$ then

$$L_r(G_{p/q}) = \emptyset.$$

Proof. We shall prove Lemma 4.2 by induction on q. If $q = 2$, i.e., $p/q = 5/2$, then $a/b = 2/1$. Let $G_{5/2} = P_2$. Then $L_r(G_2) = [2, r-2]$ for $2 \leq r < 8/3$. If $p/q = 8/3$ then $a/b = 5/2$. Let $G_{8/3}$ be the graph as depicted in Fig. 3. It is straightforward to verify (by using Lemmas 3.1 and 3.2) that for $5/2 \leq r < 8/3$ $L_r(G_4) = [5-r, 2r-5]$.

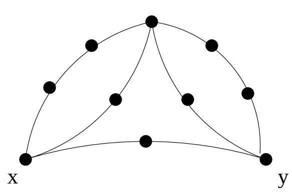

Figure 3: The graph $G_{8/3}$

Suppose $q \geq 3$ and $p/q \neq 8/3$, and Lemma 4.2 is true for all the fractions in $p'/q'(2, 8/3]$ with $q' < q$.

First we consider the case that $a/b = 2/1$. Then $p/q = (2h+1)/h$ for some integer $h \geq 3$. Let $G_{(2h+1)/h} = P_{2h-2}$. By Lemma 3.1 and Lemma 3.2, it is straightforward to verify that $L_r(G_{(2h+1)/h}) = \emptyset$ for $r < 2$ and $L_r(G_{(2h+1)/h}) = [2h-2-(h-2)r, (h-1)r-2h+2]$ for $2 \leq r < (2h-1)/(h-1)$.

In the remaining we assume that $p_l(p/q) = a/b, p_u(p/q) = a'/b' \in (2, 8/3]$. By the induction hypothesis, there is a graph $G_{a/b}$ such that if $p_l(a/b) \leq r < (a-2)/(b-1)$ then

$$L_r(G_{a/b}) = [a-3-(b-2)r, (b-1)r-a+3],$$

if $r < p_l(a/b)$ then

$$L_r(G_{a/b}) = \emptyset.$$

There is a graph $G_{a'/b'}$ such that if $p_l(a'/b') \leq r < (a'-2)/(b'-1)$ then

$$L_r(G_{a'/b'}) = [a' - 3 - (b'-2)r, (b'-1)r - a' + 3],$$

if $r < p_l(a'/b')$ then

$$L_r(G_{a'/b'}) = \emptyset.$$

We construct the graph $G_{p/q}$ as follows: Let X be the series parallel graph obtained from $G_{a/b}$ and P_3 by a parallel construction, and let Y be obtained from two copies of X by a series construction, let Z be obtained from Y and P_2 by a parallel construction. Then $G_{p/q}$ is obtained from $G_{a'/b'}$ and Z by a series construction. The structure of $G_{p/q}$ is as depicted in Fig. 4 below.

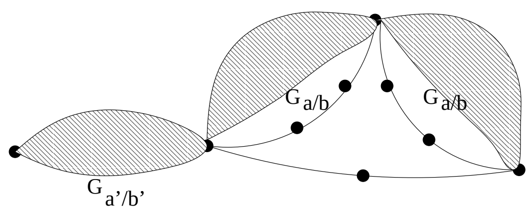

Figure 4: The graph $G_{p/q}$

First we prove that if $r < a/b$ then $L_r(X) = \emptyset$ (and hence $L_r(G) = \emptyset$, as the sum or intersection of an emptyset with any other set is an emptyset). If $r < p_l(a/b)$ then by the induction hypothesis, $L_r(G_{a/b}) = \emptyset$, hence $L_r(X) = \emptyset$. If $p_l(a/b) \leq r < a/b$ then by the induction hypothesis,

$$L_r(G_{a/b}) = [a - 3 - (b-2)r, (b-1)r - a + 3].$$

So

$$L_r(X) = L_r(G_{a/b}) \cap L_r(P_3) = [a-3-(b-2)r, (b-1)r-a+3] \cap [3-r, 2r-3].$$

Now it is straightforward to verify that $L_r(X) = \emptyset$ (as $r < a/b$).

Now we assume that $a/b \leq r < \min\{8/3, (p-2)/(q-1)\}$. By the induction hypothesis, for $a/b \leq r < (a-2)/(b-1)$,

$$L_r(G_{a/b}) = [a - 3 - (b-2)r, (b-1)r - a + 3].$$

By Corollary 4.1, $(p-2)/(q-1) \leq (a-2)/(b-1)$, hence

$$\min\{8/3, (p-2)/(q-1)\} \leq \min\{8/3, (a-2)/(b-1)\}.$$

Therefore for $a/b \le r < \min\{8/3, (p-2)/(q-1)\}$, we have

$$L_r(G_{a/b}) = [a - 3 - (b-2)r, (b-1)r - a + 3]$$

and hence

$$L_r(X) = [3 - r, (b-1)r - a + 3] \cup [a - 3 - (b-2)r, 2r - 3].$$

By Lemma 3.1

$$\begin{aligned} L_r(Y) &= L_r(X) + L_r(X) \\ &= [6 - 2r, (2b-2)r - 2a + 6] \cup [2a - 6 - 2br + 3r, 3r - 6] \\ &\quad \cup [a - (b-1)r, br - a]. \end{aligned}$$

Since $r < \min\{8/3, (a-2)/(b-1)\}$, straightforward calculation shows that

$$L_r(Z) = L_r(Y) \cap L_r(P_2) = [a - (b-1)r, br - a].$$

Fig. 5 below indicates the positions of the intervals of $L_r(Y)$ and $L_r(P_2)$.

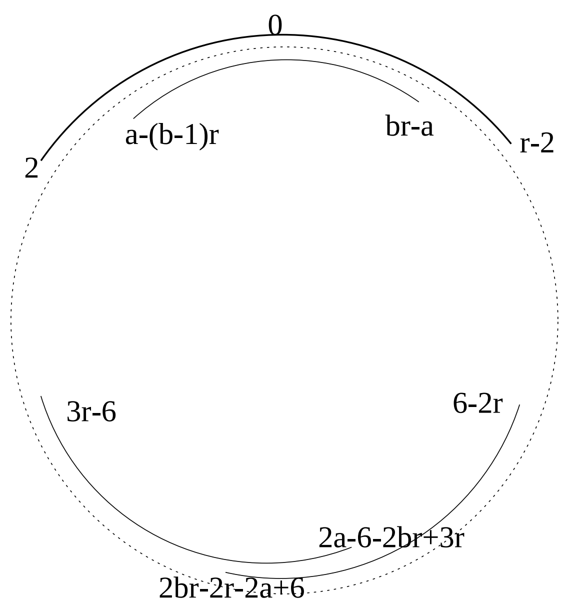

Figure 5: The positions of $L_r(Y)$ and $L_r(P_2)$

It follows from the construction of $G_{p/q}$ and Lemma 3.1 that $L_r(G_{p/q}) = L_r(G_{a'/b'}) + L_r(Z)$. Suppose $s/t = p_l(a'/b')$. By the induction hypothesis, for $s/t \le r < \min\{8/3, (a'-2)/(b'-1)\}$,

$$L_r(G_{a'/b'}) = [a' - 3 - (b'-2)r, (b'-1)r - a' + 3].$$

It follows from the construction of the Farey sequence of order n that $s/t \leq a/b$ (as $s/t, a'/b'$ are two consecutive elements of row m for some $m \leq n$). By Corollary 4.1, $\min\{8/3, (p-2)/(q-1)\} \leq \min\{8/3, (a'-2)/(b'-1)\}$. Therefore for $a/b \leq r < \min\{8/3, (p-2)/(q-1)\}$, we also have

$$L_r(G_{a'/b'}) = [a' - 3 - (b'-2)r, (b'-1)r - a' + 3].$$

Hence

$$\begin{aligned} L_r(G_{p/q}) &= L_r(Z) + L_r(G_{a'/b'}) \\ &= [a - (b-1)r, br - a] + [a' - 3 - (b'-2)r, (b'-1)r - a' + 3] \\ &= [(a+a') - 3 - (b+b'-2)r, (b+b'-1)r - (a+a') + 3]. \end{aligned}$$

This completes the proof of Lemma 4.2. ∎

Proof of Theorem 2.1: Suppose $p/q \in (2, 8/3)$. Let $G_{p/q}$ be the graph constructed in the proof of Lemma 4.2. Let $H_{p/q}$ be the graph obtained from $G_{p/q}$ and P_3 by a parallel construction. We shall show that $\chi_c(H_{p/q}) = p/q$.

If $r < p/q$, then either $r < p_l(p/q)$ and hence $L_r(G_{p/q}) = \emptyset$ which implies that $L_r(H_{p/q}) = \emptyset$, or $p_l(p/q) \leq r < p/q$ and

$$L_r(G_{p/q}) = [p - 3 - (q-2)r, (q-1)r - p + 3].$$

In the latter case we have

$$\begin{aligned} L_r(H_{p/q}) &= L_r(G_{p/q}) \cap L_r(P_3) \\ &= [p - 3 - (q-2)r, (q-1)r - p + 3] \cap [3 - r, 2r - 3] \\ &= \emptyset. \end{aligned}$$

If $r = p/q$, then

$$\begin{aligned} L_r(H_{p/q}) &= L_r(G_{p/q}) \cap L_r(P_3) \\ &= [p - 3 - (q-2)r, (q-1)r - p + 3] \cap [3 - r, 2r - 3] \\ &= \{3 - r, 2r - 3\}. \end{aligned}$$

Therefore $H_{p/q}$ is p/q-colourable, but not r-colourable for any $r < p/q$. So $\chi_c(H_{p/q}) = p/q$. This completes the proof of Theorem 2.1.

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