Circular Choosability via Combinatorial Nullstellensatz

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Abstract

A (p,q)-list assignment L of a graph G assigns to each vertex v of G a set $L(v) \subseteq \{0, 1, \ldots, p-1\}$ of permissible colors. We say G is L-(p,q)-colorable if G has a (p,q)-coloring h such that $h(v) \in L(v)$ for each vertex v. The circular list chromatic number $\chi_{c,l}(G)$ of a graph G is the infimum of those real numbers t for which the following holds: For any p, q, for any (p,q)-list assignment L with $|L(v)| \geq tq$, G is L-(p,q)-colorable. We prove that if G has an orientation D which has no odd directed cycles, and L is a (p,q)-list assignment of G such that for each vertex v, $|L(v)| = d^+_D(v)(2q-1)+1$, then G is L-(p,q)-colorable. This implies that if G is a bipartite graph, then $\chi_{c,l}(G) \leq 2\lceil \text{mad}(G)/2 \rceil$, where $\text{mad}(G)$ is the maximum average degree of a subgraph of G. We further prove that if G is a connected bipartite graph which is not a tree, then $\chi_{c,l}(G) \leq \text{mad}(G)$.

Keywords: circular choosability, combinatorial Nullstellensatz, orientation, polynomial.

Mathematical Subject Classification: 05C15

1 Introduction

This paper studies circular choosability of graphs, which is a combination of two variations of graph coloring: circular coloring and list coloring. Circular coloring of a graph, introduced by Vince [7], is defined as follows: Suppose $G = (V, E)$ is a graph and $p \geq q$ are positive integers. Take the set $Z_p = \{0, 1, \ldots, p-1\}$ as the set of colors. For two colors $i, j \in Z_p$, the distance between i and j is $|i-j|_p = \min\{|i-j|, p-|i-j|\}$. One may view the elements of Z_p as p points evenly spaced on a circle of perimeter p. Then $|i-j|_p$ is the length of the shorter arc of the circle between points i and j.

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A (p,q)*-coloring* of G is a mapping $h : V \to Z_p$ such that colors assigned to adjacent vertices have distance at least q, i.e., for any edge xy of G, $|h(x) - h(y)|_p \geq q$. The *circular chromatic number* $\chi_c(G)$ of G is defined as

$$\chi_c(G) = \inf\{p/q : \ G \text{ has a } (p,q)\text{-coloring}\}.$$

For finite graphs, the infimum in the definition is always attained and hence can be replaced by the minimum [7]. It is known that for any graph G, $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$. So $\chi_c(G)$ is a refinement of $\chi(G)$, and $\chi(G)$ is an approximation of $\chi_c(G)$. Circular chromatic number of graphs has been studied extensively in the literature. The reader is referred to [10, 12] for surveys on this subject.

List coloring of graphs, initiated independently by Vizing [8] and Erdős, Rubin and Taylor [2], is another variation of graph coloring. A *list-assignment* L of a graph G assigns to each vertex v a set $L(v)$ of permissible colors. We say G is L*-colorable* if G has a (proper) vertex-coloring h such that $h(v) \in L(v)$ for each v. We say G is k*-choosable* if G is L-colorable for any list assignment L for which $|L(v)| = k$ for every vertex v. The *list chromatic number* (or *choosability*) of G, $\chi_l(G)$, is the minimum integer k such that G is k-choosable.

List circular coloring of graphs, first studied in [11], is the circular version of list coloring. A (p,q)-list assignment is a mapping L which assigns to each vertex v of G a set $L(v) \subseteq Z_p$ of permissible colors to v. An L-(p,q)-coloring of G is a (p,q)-coloring h of G for which $h(v) \in L(v)$ for every $v \in V$. We say G is L*-*(p,q)*-colorable* if there is an L-(p,q)-coloring of G.

Suppose $G = (V,E)$ is a graph and $\ell : V \to \{0, 1, \dots, p\}$ is a mapping. We say G is ℓ*-*(p,q)*-choosable* if for every (p,q)-list assignment L with $|L(v)| = \ell(v)$, G is L-(p,q)-colorable. The question of interest is for which mappings ℓ, G is ℓ-(p,q)-choosable. We say a graph G is *circular* t*-choosable* if for any $p \geq tq$, and with $\ell(v) = \lceil tq \rceil$ for all v, G is ℓ-(p,q)-choosable. The *circular list chromatic number* of G (or the *circular choosability* of G) is defined as

$$\chi_{c,l}(G) = \inf\{t : \ G \text{ is circular } t\text{-choosable}\}.$$

Unlike in the definition of circular chromatic number, it is proved by Norine [5] that the infimum in the above definition of the circular list chromatic number of a graph is not always attained and hence cannot be replaced by the minimum.

The concept of circular list coloring of graphs is relatively new, and some basic questions remain open. It is unknown whether $\chi_{c,l}(G)$ is always a rational number for a finite graph G. It is also unknown whether $\chi_{c,l}(G)$ is bounded from above by $\alpha\chi_l(G)$ for some constant α. A necessary and sufficient condition on ℓ is given in [6] under which a forest is ℓ-(p,q)-choosable. The problem of ℓ-$(2k+1,k)$-choosability of cycles is also studied in [6], where a sharp sufficient condition is given under which a cycle C_n is ℓ-$(2k+1,k)$-choosable. The circular list chromatic numbers are known for trees, cycles, complete graphs, odd wheels, etc. It is known that graphs of maximum degree k has circular list chromatic number at most $k+1$ [11], the circular list chromatic number of planar graphs is at most 8 [3], Outerplanar graphs G of girth at least $2n+2$ are shown to have $\chi_{c,l}(G) \leq 2 + 1/n$ [9]. Circular list coloring of graphs in which the list $L(v)$ of permissible colors to each vertex is a circular consecutive set is studied in [4].

This paper studies the circular list chromatic number of graph by using combinatorial Nullstellensatz. We generalize a result of Alon and Tarsi [1] concerning choosability of graphs with an orientation containing no odd directed cycles and prove the following result: Suppose a graph G has an orientation D in which there is no odd directed cycle. If L is a (p,q)-list assignment with $|L(v)| = d^+_D(v)(2q-1)+1$ for each vertex, then G is L-(p,q)-colorable. The $q=1$ case is proved by Alon and Tarsi [1]. As a consequence of this result, every bipartite graph G has circular list chromatic number at most $2\lceil \text{mad}(G)/2 \rceil$, where $\text{mad}(G) = \max_{H \subseteq G} 2|E(H)|/|V(H)|$ is the maximum average degree of subgraphs H of G. However, for bipartite graphs, a stronger result holds. We prove in Section 4 that if G is a connected bipartite graph which is not a tree, then $\chi_{c,l}(G) \leq \text{mad}(G)$. In Section 5, several questions are asked.

2 Circular choosability and orientation

We shall need the following theorem, called the combinatorial Nullstellensatz [1], in our proofs.

Theorem 2.1. *Let F be a field and let $f(x_1, x_2, \ldots, x_n)$ be a polynomial in $F[x_1, x_2, \ldots, x_n]$. Suppose the degree of f is equal to $\sum_{j=1}^n t_j$ and the coefficient of $\prod_{j=1}^n x_j^{t_j}$ in f is nonzero. Then for any subsets $S_1, S_2, \ldots, S_n$ of F with $|S_j| = t_j + 1$, there exist $s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n$ so that*

$$f(s_1, s_2, \ldots, s_n) \neq 0.$$

For an orientation D of a graph G, we denote by $d^+_D(v)$ the out-degree of v in D. A subgraph D' of D is *eulerian* if for each vertex v, $d^+_{D'}(v) = d^-_{D'}(v)$. An eulerian subgraph D' is called an *odd eulerian subgraph* (respectively, an *even eulerian subgraph*) if D' has an odd (respectively, even) number of edges. Denote by $EE(D)$ and $EO(D)$ the number of even eulerian subgraphs and the number of odd eulerian subgraphs of D, respectively. The following result is proved by Alon and Tarsi.

Theorem 2.2. *[1] Suppose G has an orientation D for which $EE(D) \neq EO(D)$. If L is a list-assignment with $|L(v)| = d^+_D(v) + 1$ for each vertex v, then G is L-colorable.*

We first generalize Theorem 2.2 to (p,q)-list colorings. An eulerian subgraph D' of D corresponds to a mapping $k : E(D) \to \{0,1\}$ such that $\sum_{e \in E^+_D(v)} k(e) = \sum_{e \in E^-_D(v)} k(e)$. Here $E^+_D(v)$ is the set of arcs in D with v as their tails, and $E^-_D(v)$ is the set of arcs in D with v as their heads. For any positive integer q, we call a mapping $k : E(D) \to \{0, 1, \ldots, 2q-1\}$ *eulerian* (with respect to q) if for each vertex v,

$$\sum_{e \in E^+_D(v)} k(e) = \sum_{e \in E^-_D(v)} k(e).$$

Suppose k is an eulerian mapping (with respect to q) and $p \geq q$ is an integer. We assign a weight $w_{p,q}(k)$ to k as follows:

$$w_{p,q}(k) = \prod_{e \in E(D)} a_{k(e)}(p,q),$$

where $a_{k(e)}(p,q)$ is defined as

$$a_{k(e)}(p,q) = \sum_{J \subseteq \{-q+1,\ldots,q-1\}, |J|=k(e)} \prod_{j \in J} (-e^{2\pi i j/p}).$$

Here $i = \sqrt{-1}$ and $e^{2\pi j/p}$ is a complex number for any j. However, for any integers $k(e), p, q$, the number $a_{k(e)}(p,q)$ is real, because it is easy to check that the conjugate of $a_{k(e)}(p,q)$ is

$$\overline{a_{k(e)}(p,q)} = \sum_{J \subseteq \{-q+1,\ldots,q-1\}, |J|=k(e)} \prod_{j \in J} (-e^{-2\pi i j/p}) = a_{k(e)}(p,q).$$

If $q = 1$, then $w_{p,1}(k) = (-1)^m$, where $m = |\{e \in E(D) : k(e) = 1\}|$. So $EE(D) \neq EO(D)$ is equivalent to $\sum w_{p,1}(k) \neq 0$, where the summation is taken over all eulerian mappings k.

Theorem 2.3. *Suppose a graph G has an orientation D for which $\sum w_{p,q}(k) \neq 0$, where the summation is taken over all eulerian mappings k with respect to q. If L is a (p,q)-list assignment with $|L(v)| = d_D^+(v)(2q-1)+1$, then G is L-(p,q)-colorable.*

Proof. Assume G has vertices $v_1, v_2, \ldots, v_n$. Consider the polynomial $f(x_1, x_2, \ldots, x_n) \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ defined as

$$f(x_1, x_2, \ldots, x_n) = \prod_{(v_j, v_{j'}) \in D} \prod_{t=-q+1}^{q-1} (x_j - e^{2\pi i t/p} x_{j'}).$$

Let $\phi : Z_p \to \mathbb{C}$ be defined as $\phi(k) = e^{2\pi i k/p}$. It is obvious that a mapping $h : V \to Z_p$ is a (p,q)-coloring of G if and only if

$$f(\phi(h(v_1)), \phi(h(v_2)), \ldots, \phi(h(v_n))) \neq 0.$$

Thus with $S_i = \{\phi(a) : a \in L(v_i)\}$, the graph G is L-(p,q)-colorable if and only if there exist $s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n$ such that $f(s_1, s_2, \ldots, s_n) \neq 0$.

Let $t_j = d_D^+(v_j)(2q-1)$ for $j = 1, 2, \ldots, n$. Then the degree of f is $|E|(2q-1) = \sum_{j=1}^n t_j$. To prove Theorem 2.3 by using Theorem 2.1, it suffices to show that the coefficient of $\prod_{j=1}^n x_j^{t_j}$ is nonzero.

For an arc $(v_j, v_{j'})$ of D and for $0 \le k \le 2q-1$, let a_k be the coefficient of $x_j^{2q-1-k} x_{j'}^k$ in $\prod_{t=-q+1}^{q-1}(x_j - e^{2\pi it/p}x_{j'})$. Clearly, a_k is determined by the integer k (independent of the arc $(x_j, x_{j'})$) and

$$a_k = \sum_{J \subseteq \{-q+1,\dots,q-1\}, |J|=k} \prod_{j \in J} (-e^{2\pi ij/p}).$$

Thus if $k(e) = k$, then a_k is equal to $a_{k(e)}$ defined above.

It is easy to see that a mapping $k : E(D) \to \{0, 1, \dots, 2q-1\}$ makes a contribution to the coefficient of $\prod_{j=1}^n x_j^{t_j}$ in $f(x_1, x_2, \dots, x_n)$ if and only if k is eulerian, and the coefficient of $\prod_{j=1}^n x_j^{t_j}$ in f is equal to

$$\sum_{k \text{ is eulerian}} \prod_{e \in D} a_{k(e)}.$$

I.e., the coefficient of $\prod_{j=1}^n x_j^{t_j}$ in $f(x_1, x_2, \dots, x_n)$ is equal to $\sum_{k \text{ is eulerian}} w_{p,q}(k)$. By our assumption, $\sum_{k \text{ is eulerian}} w_{p,q}(k) \ne 0$. Hence G is L-(p,q)-colorable. □

Theorem 2.4. *Suppose G is a graph and D is an orientation of G which contains no odd directed cycle. Let L be a (p,q)-list assignment for G such that $|L(v)| = d_D^+(v)(2q-1)+1$ for each vertex v. Then G is L-(p,q)-colorable.*

Proof. Without loss of generality, we may assume that G is connected. By Theorem 2.3, it suffices to show that $\sum_{k \text{ is eulerian}} w_{p,q}(k) \ne 0$. Given an eulerian mapping $k : E(D) \to Z^{\ge 0}$, we construct a multi-digraph D_k on the vertex set V of G, with each arc $e = (v_j, v_{j'})$ of D replaced by $k(e)$ parallel arcs from v_j to $v_{j'}$. Then D_k is an eulerian digraph, as $d_{D_k}^+(v_j) = d_{D_k}^-(v_j)$ for each vertex v_j. Each directed cycle of D_k corresponds to a directed cycle of D. Since D has no directed cycle of odd length, D_k has no directed cycle of odd length. Thus $|E(D_k)|$ is even, i.e., $\sum_{e \in D} k(e)$ is even.

If $d_D^+(v) \le 1$ for each vertex v, then either G is a tree, or G is an even cycle, say of length $2n$. In the former case, there is only one eulerian mapping k defined as $k(e) = 0$ for all e. Hence

$$\sum_{k \text{ is eulerian}} \prod_{e \in D} a_{k(e)} = \prod_{e \in D} a_0 = 1$$

as $a_0 = 1$. In the later case, there are $2q$ eulerian mappings, defined as $k_j(e) = j$ for $j = 0, 1, \dots, 2q-1$. Thus

$$\sum_{k \text{ is eulerian}} \prod_{e \in D} a_{k(e)} = \sum_{j=0}^{2q-1} (a_j)^{2n}.$$

As all the a_j's are real numbers, and $a_0 = 1$, $\sum_{k \text{ is eulerian}} \prod_{e \in D} a_{k(e)} \ge 1$ in this case.

It remains to consider the case that there is at least one vertex v with $d_D^+(v) \ge 2$. Since $|L(v)| = d_D^+(v)(2q-1)+1$, we must have $p \ge 4q-1$. We need the following lemma, whose proof is given in the next section.

Lemma 2.5. *Suppose $p \geq 2q$ are positive integers such that either $q = p/2$ or for some positive integer d, $q - 1 \leq (2^d - 1)p/2^{d+1}$ and $2^{d-1}|p$. Then for any $0 \leq k \leq 2q - 1$, $(-1)^k a_k > 0$.*

Suppose k is an eulerian mapping. Then $\sum_{e \in D} k(e)$ is even, and hence

$$\prod_{e \in D} a_{k(e)} = \prod_{e \in D} (-1)^{k(e)} a_{k(e)}.$$

By Lemma 2.5 (with $d = 1$), $\prod_{e \in D} a_{k(e)} = \prod_{e \in D} (-1)^{k(e)} a_{k(e)} > 0$ for any eulerian mapping k. Therefore

$$\sum_{k \text{ is eulerian}} \prod_{e \in D} a_{k(e)} > 0.$$

This completes the proof of Theorem 2.4. □

Corollary 2.6. *Suppose G has an orientation D which has no directed cycle of odd length. If $k = \max_{v \in V} d_D^+(v)$, then $\chi_{c,l}(G) \leq 2k$.*

3 Proof of Lemma 2.5

In this section, p is a fixed positive integer. Let $w = e^{2\pi i/p}$. For an integer k and a subset $Q \subseteq Z_p$, let

$$S_k(Q) = \sum_{J \subseteq Q, |J| = k} \prod_{j \in J} w^j.$$

As a convention, we let $S_k(Q) = 0$ if $k > |Q|$ and $S_0(Q) = 1$ for any Q. For an integer $1 \leq q \leq p/2$, let $I_q = \{-q+1, -q+2, \ldots, -1, 0, 1, \ldots, q-1\}$, where arithmetic in Z_p is modulo p. Lemma 2.5 can be re-stated as follows:

Lemma 3.1. *If $q = p/2$ or for some positive integer d, $q - 1 \leq (2^d - 1)p/2^{d+1}$ and $2^{d-1} \mid p$, then for $0 \leq k \leq 2q - 1$, $S_k(I_q) > 0$.*

Proof. For $Q \subseteq \{0, 1, 2, \ldots, p-1\}$, let

$$f_Q(x) = \sum_{k=0}^{|Q|} S_k(Q) x^{|Q|-k} = \prod_{j \in Q} (x + w^j).$$

If $Q, R \subseteq \{0, 1, \ldots, p-1\}$ are disjoint, then it follows from the definition that

$$f_Q(x) f_R(x) = f_{Q \cup R}(x). \tag{3.1}$$

Note that $x^p - 1 = \prod_{j=0}^{p-1}(x - w^j)$. If $p = 2q$ then $\prod_{j=0}^{p-1}(x - w^j) = \prod_{j=0}^{p-1}(x + w^j)$. As $I_q = \{0, 1, \ldots, p-1\} \setminus \{p/2\}$. Hence

$$f_{I_q}(x) = (x^p - 1)/(x - 1) = \sum_{j=0}^{p-1} x^j.$$

It follows that $S_k(I_q) = 1 > 0$ for every $0 \le k \le 2q-1$.

Assume for some positive integer d, $q - 1 \le (2^d - 1)p/2^{d+1}$ and $2^{d-1} \mid p$. We need to show that for any $0 \le k \le 2q-1$, $S_k(I_q) > 0$. For an integer $0 \le m < d-1$, and $j \in \{1, 2, \dots, p-1\}$ such that $\frac{p}{2^{m+1}} \nmid j$, we define the *snowflake* $A_m(j)$ *of order* m inductively as follows.

$$\begin{aligned} A_0(j) &= \{j, -j\} \\ A_m(j) &= A_{m-1}(j) \cup A_{m-1}(p/2^m - j). \end{aligned}$$

It can be proved by induction that for $m \ge 0$,

$$A_m(j) = \{tp/2^m \pm j : t = 0, 1, \dots, 2^m - 1\}.$$

Because $\frac{p}{2^{m+1}} \nmid j$, $A_{m-1}(j) \cap A_{m-1}(p/2^m - j) = \emptyset$ and $A_m(j)$ has cardinality 2^{m+1}.

Note that for $j \ne 0$, we have

$$f_{A_0(j)}(x) = (x + \omega^j)(x + \omega^{-j}) = x^2 + (2\cos\frac{2\pi j}{p})x + 1. \tag{3.2}$$

We prove by induction on m that for $1 \le m < d$,

$$f_{A_m(j)}(x) = x^{2^{m+1}} - (2\cos\frac{2^{m+1}\pi j}{p})x^{2^m} + 1. \tag{3.3}$$

We consider the base case $m = 1$ and the induction step simultaneously. By (3.1) we have

$$\begin{aligned} f_{A_m(j)}(x) &= f_{A_{m-1}(j)}(x) f_{A_{m-1}(p/2^m - j)}(x) \\ &= (x^{2^m} - (2\cos\frac{2^m \pi j}{p})x^{2^{m-1}} + 1) \\ &\quad \times (x^{2^m} - (2\cos\frac{2^m \pi (p/2^m - j)}{p})x^{2^{m-1}} + 1) \\ &= x^{2^{m+1}} + 2(1 - 2(\cos\frac{2^m \pi j}{p})^2)x^{2^m} + 1 \\ &= x^{2^{m+1}} - (2\cos\frac{2^{m+1}\pi j}{p})x^{2^m} + 1, \end{aligned}$$

where we use (3.2) when $m = 1$ and the induction hypothesis for $m > 1$.

We say a snowflake $A_m(j)$ is *proper* if $m = 0$ and $1 \le j \le p/4$, or $m > 0$ and $p/2^{m+2} \le j < p/2^{m+1}$. By (3.2) and (3.3), for a proper snowflake A of order m, we have $S_k(A) \ge 0$ for all $k \ge 0$ and

$$S_{2^{m+1}}(A), S_0(A) > 0.$$

For every $j \in I_q - \{0\}$, choose $m(j)$ so that the snowflake $A_{m(j)}(j)$ is defined, $A_{m(j)}(j) \subseteq I_q$ and subject to these conditions, $m(j)$ is maximum. Denote $A_{m(j)}(j)$ by

$A(j)$ for brevity. Note that the snowflake $A(j)$ is proper, as otherwise the snowflake $A_{m(j)+1}(j)$ is defined and lies in I_q, in contrary to the choice of $m(j)$. For every $j, h \in I_q$, we have $A(j) = A(h)$ or $A(j) \cap A(h) = \emptyset$. Therefore there exists $J \subseteq I_q$ such that the snowflakes $A(j)$ for $j \in J$ are pairwise disjoint and $\cup_{j\in J} A(j) = I_q$ -$\{0\}$. Therefore we have

$$f_{I_q}(x) = (x+1) \prod_{j \in J} f_{A(j)}(x)$$

and thus $f_{I_q}(x)$ is a polynomial with non-negative coefficients. It remains to show that all the coefficients of $f_{I_q}(x)$ are positive. By (3.4), it suffices to show that if $S = \{A(j) : j \in J\}$ contains a snowflake of order m, then it contains a snow flake of order $m-1$. Note that if S contains a snowflake of order m, then $q-1 > (2^m-1)p/2^{m+1}$ and $A_{m-1}((2^m-1)p/2^{m+1}) \in S$ as the number $(2^m-1)p/2^{m+1}$ is contained in no snowflake of order m and $A_{m-1}((2^m-1)p/2^{m+1}) \subseteq I_q$. This completes the proof of Lemma 2.5. □

4 Bipartite graphs

It is known [1] that any graph G has an orientation D which has maximum out-degree $\lceil \mathrm{mad}(G)/2 \rceil$. If G is a bipartite graph, then any orientation of G has no odd directed cycle. By Corollary 2.6, we have the following corollary:

Corollary 4.1. *If G is a bipartite graph, then $\chi_{c,l}(G) \le 2\lceil \mathrm{mad}(G)/2 \rceil$.*

In this section, we prove a strengthening of Corollary 4.1.

Theorem 4.2. *Suppose G is a bipartite graph. Let G' be obtained from G by replacing each edge with $2q-1$ parallel edges. Let D' be an orientation of G'. Let L be a (p,q)-list assignment for G such that $|L(v)| = d^+_{D'}(v)+1$ for each vertex v. If $p = 2q$ or for some positive integer d, $q-1 \le (2^d-1)p/2^{d+1}$ and $2^{d-1} \mid p$, then G is L-(p,q)-colorable.*

Proof. The proof is basically the same as the proof of Theorem 2.3. Let D be an arbitrary orientation of G. Consider the polynomial

$$f(x_1, x_2, \ldots, x_n) = \prod_{(v_j, v_{j'}) \in D} \prod_{k=-q+1}^{q-1} (x_j - e^{2\pi i k/p} x_{j'}).$$

Let $S_i = \{\phi(a) : a \in L(v_i)\}$, where ϕ is defined as in the proof of Theorem 2.3. It suffices to prove that there exist $s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n$ such that $f(s_1, s_2, \ldots, s_n) \ne 0$.

Let $t_j = d^+_{D'}(v_j)$ for $j = 1, 2, \ldots, n$. Then the degree of f is $|E(D')| = \sum_{j=1}^n t_j$. By using Theorem 2.1, it suffices to show that the coefficient of $\prod_{j=1}^n x_j^{t_j}$ is nonzero.

For $0 \le k \le 2q-1$, let a_k be defined as in the proof of Theorem 2.3. For each vertex v of G, let $\xi(v) = d^+_{D'}(v) - d^+_D(v)(2q-1)$. We call a mapping $k : E(D) \to \{0, 1, \ldots, 2q-1\}$

compatible with ξ if for each vertex v,

$$\sum_{e \in E_D^+(v)} k(e) - \sum_{e \in E_D^-(v)} k(e) = \xi(v).$$

Then a mapping k makes a contribution of $\prod_{e \in D} a_{k(e)}$ to the coefficient of $\prod_{j=1}^n x_j^{t_j}$ in $f(x_1, x_2, \ldots, x_n)$ if and only if k is compatible with ξ, and the coefficient of $\prod_{j=1}^n x_j^{t_j}$ in f is equal to

$$\sum_{k \text{ is compatible with } \xi} \prod_{e \in D} a_{k(e)}.$$

Given a mapping $k : E(D) \to Z^{\geq 0}$ compatible with ξ, we construct a multi-digraph D_k on the vertex set V of G, with each arc $e = (v_j, v_{j'})$ of D replaced by $k(e)$ parallel arcs from v_j to $v_{j'}$. Then D_k is a digraph with the property that $d^+_{D_k}(v) - d^-_{D_k}(v) = \xi(v)$. Let A be one partite set of the bipartite graph G. Let $||\xi|| = \sum_{v \in A} \xi(v)$. Then the total number of edges in the digraph D_k is

$$\sum_{v \in A} (d^+_{D_k}(v) + d^-_{D_k}(v)) \cong ||\xi|| \pmod 2.$$

I.e., $\sum_{e \in E(D)} k(e) \cong ||\xi|| \pmod 2$. It follows from Lemma 2.5 that for any mapping $k : E(D) \to Z^{\geq 0}$ compatible with ξ,

$$(-1)^{||\xi||} \prod_{e \in D} a_{k(e)} > 0.$$

Hence

$$\sum_{k \text{ is compatible with } \xi} \prod_{e \in D} a_{k(e)} \neq 0.$$

□

Corollary 4.3. *For any connected bipartite graph G which is not a tree, $\chi_{c,l}(G) \leq \mathrm{mad}(G)$.*

Proof. Let $r = \mathrm{mad}(G)$. It suffices to prove that for any $\epsilon > 0$, for the mapping ℓ defined as $\ell(v) = \lceil (r + \epsilon) q \rceil$, G is ℓ-(p, q)-choosable.

Observe that if for some positive integer k, for $\ell'(v) = \lceil (r + \epsilon) kq \rceil$, G is ℓ'-(kp, kq)-choosable, then G is ℓ-(p, q)-choosable. Indeed, if L is an ℓ-(p, q)-list assignment, then L' defined as $L'(v) = \cup_{j \in L(v)} \{kj, kj + 1, \ldots, kj + k - 1\}$ is a (kp, kq)-list assignment satisfying $|L'(v)| \geq \ell'(v)$. If G is ℓ'-(kp, kq)-choosable, then G is a L'-(kp, kq)-colorable. If h' is an L'-(kp, kq)-coloring of G, then $h(v) = \lfloor h'(v)/k \rfloor$ is an L-(p, q)-coloring of G.

Thus we may assume that either $p = 2q$ or there is a positive integer d, $q - 1 \leq (2^d - 1)p/2^{d+1}$ and $2^{d-1} \mid p$ and q is sufficiently large. For a positive integer q, let $G(q)$ be obtained from G by replacing each edge of G with $2q - 1$ parallel edges.

It is obvious that $\mathrm{mad}(G(q)) = \mathrm{mad}(G)(2q-1)$. It is known [1] that $G(q)$ has an orientation D' with $d^+_{D'}(v) \leq \lceil \mathrm{mad}(G(q))/2 \rceil$ for each vertex v. It follows from Theorem 4.2 that G is L-(p,q)-colorable, provided that $|L(v)| = d^+_{D'}(v) + 1$ (note that since G is connected and is not a tree, $r \geq 2$, we have $p \geq 2q$). If $q \geq (2 - r/2)/\epsilon$, then $d^+_{D'}(v) + 1 < r(2q-1)/2 + 2 < (r+\epsilon)q$. Hence G is ℓ-(p,q)-choosable, with $\ell(v) = \lceil (r+\epsilon)q \rceil$. □

Corollary 4.4. *If G is a connected bipartite planar graph of girth g and G is not a tree, then $\chi_{c,l}(G) < 2g/(g-2)$.*

Proof. It follows from Euler formula that G have $\mathrm{mad}(G) < 2g/(g-2)$. □

5 Some remarks and open problems

Theorem 2.4 can be viewed as the circular version of the following result proved by Alon and Tarsi.

Theorem 5.1. *[1] Suppose D is an orientation of a graph G which has no odd directed cycles. If L is a list assignment which assigns to each vertex v a set $L(v)$ of $d^+_D(v)+1$ colors, then G is L-colorable. I.e., G has a coloring h with $h(v) \in L(v)$ for each vertex v.*

Theorem 2.3 is the circular version of Theorem 2.2. However, it seems more difficult to check the condition in Theorem 2.3. For a directed graph D, the condition $EE(D) \neq EO(D)$ is equivalent to $\sum_{k \text{ is eulerian}} w_{p,1}(k) \neq 0$ for any p. Currently we do not know any directed graph D for which $EE(D) \neq EO(D)$ and yet $\sum_{k \text{ is eulerian}} w_{p,q}(k) = 0$ for some p, q. A natural question is whether there are such directed graphs.

Question 5.2. *Is it true that if D is a digraph with $EE(D) \neq EO(D)$, then for any p, q, $\sum_{k \text{ is eulerian}} w_{p,q}(k) \neq 0$?*

One intuitive explanation of the difference between list coloring and list (p,q)-coloring might be as follows: In list coloring, one of the color in $L(v)$ is used to color v, the other colors are "killed" by the neighbours of v. Coloring one "critical" neighbour u of v kills one color of $L(v)$, as the color assigned to u cannot be used by v. In a list (p,q)-coloring, again, one color is used by v, but coloring one "critical" neighbour u of v kills $2q-1$ colors in $L(v)$, as the $2q-1$ colors close to the color of u cannot be used by v. We do not know what neighbours of v are critical (or more likely, the neighbours of v are all "fractional" critical), however, maybe a neighbour u of v which is critical in list (p,q)-coloring is also critical in list coloring. The comparison of Theorem 2.4 and Theorem 5.1 seems to support such an intuition. If this intuition is correct in general, the following question has a positive answer.

Question 5.3. *Suppose G is a graph and $l : V(G) \to Z^{\geq 0}$ is a mapping. Assume G is L-colorable for any list-assignment L with $|L(v)| = l(v)$. Is it true that G is L'-(p,q)-colorable for any (p,q)-list assignment L' with $|L'(v)| = (l(v)-1)(2q-1)+1$?*

A positive answer to Question 5.3 would imply that for any graph G, $\chi_{c,l}(G) \leq 2\chi_l(G)$. We remark that the following question asked in [11] remains open:

Is there a constant α such that for any graph G, $\chi_{c,l}(G) \leq \alpha\chi_l(G)$? If such a constant exists, what is the smallest α?

A positive answer to either Question 5.3 or Question 5.2 implies a positive answer to the following question:

Question 5.4. *Assume G has an orientation D for which $EE(D) \neq EO(D)$. Assume L is a (p,q)-list assignment such that for each vertex v, $|L(v)| = d_D^+(v)(2q-1)+1$. Is it true that G is L-(p,q)-colorable?*

A positive answer to Question 5.3 also implies that every 2-choosable graph is circular 2-choosable. Denote by $\theta_{a,b,c}$ the graph consisting of three internally disjoint paths connecting u and v, where the lengths of the three paths are a,b,c, respectively. It is known [2] that a connected graph G is 2-choosable if and only if the heart of G (i.e., the graph obtained from G by repeatedly deleting degree 1 vertices) is K_1 or an even cycle or $\theta_{2,2,2k}$. To prove that every 2-choosable graph is circular 2-choosable, it suffices to show that K_1, even cycles and $\theta_{2,2,2k}$ are circular 2-choosable for every positive integer k. The graph K_1 is trivial and even cycles are settled in [5]. The only case remain unsolved is $\theta_{2,2,2k}$. The best known upper bound for the circular list chromatic number of $\theta_{2,2,2k}$ is obtained in [3], namely, $\chi_{c,l}(\theta_{2,2,2k}) \leq 2.5$. We can improve this bound a little bit.

First of all, it is proved in [5] that $\chi_{c,l}(K_{2,4}) = 2$, which implies that $\chi_{c,l}(\theta_{2,2,2}) = 2$. By Corollary 4.3, for $k \geq 2$, $\chi_{c,l}(\theta_{2,2,2k}) \leq 2(2k+4)/(2k+3) \leq 16/7$. So $\chi_{c,l}(\theta_{2,2,2k}) \leq 16/7$ for any $k \geq 1$.

Conjecture 5.5. *For any $k \geq 1$, $\chi_{c,l}(\theta_{2,2,2k}) = 2$.*

Even if Conjecture 5.5 is true, neither the problem of characterizing all graphs G with $\chi_{c,l}(G) = 2$ nor the problem of characterizing circular 2-choosable graphs is solved. This is due to the fact that graphs G with $\chi_{c,l}(G) = 2$ need not be circular 2-choosable.

Conjecture 5.6. *For any $k \geq 1$, $\theta_{2,2,2k}$ is circular 2-choosable.*

Conjecture 5.6 is stronger than Conjecture 5.5. If this conjecture is true, we do have a characterization of all circular 2-choosable graphs. The following theorem confirms the $k = 1$ case of Conjecture 5.6.

Theorem 5.7. *For any (p,q)-list assignment L of $K_{2,3}$ with $L(x) = 2q$ for every vertex x, $K_{2,3}$ is L-(p,q)-colorable.*

Proof. We consider Z_p as a set of points on a circle of perimeter p and that the arithmetic is modulo p. The interval $[a,b]_p$ is defined as $[a,b]_p = \{a, a+1, \ldots, b-1, b\}$. In particular generally $[a,b]_p \neq [b,a]_p$.

For $a \in Z_p$, let $B_{p,q}(a) = [a-q+1, a+q-1]_p = \{a-q+1, a-q+2, \ldots, a+q-1\}$. When p is clear from the context, we write $B_q(a)$ for $B_{p,q}(a)$. For $a, b \in Z_p$, let $B_q(a,b) = B_q(a) \cup B_q(b)$.

Let the two parts of $K_{2,3}$ be $\{u_1, u_2\}$ and $\{v_1, v_2, v_3\}$. Let L be a (p,q)-list assignment of $K_{2,3}$ with $|L(x)| = 2q$ for each vertex x. If there exist $a \in L(u_1)$ and $b \in L(u_2)$ such that for $j = 1,2,3$, $L(v_j) \not\subseteq B_q(a,b)$, then color u_1 by color a, color u_2 by color b, and color each v_i with an arbitrary color in $L(v_i) \setminus B_q(a,b)$, we obtain an L-(p,q)-coloring of $K_{2,3}$.

Assume that for each $a \in L(u_1), b \in L(u_2)$, there is an $j = f(a,b) \in \{1,2,3\}$ such that $L(v_j) \subseteq B_q(a,b)$. Note that this implies that $L(u_1) \cap L(u_2) = \emptyset$, and hence $p \geq 4q$.

If $L(v_1) = L(v_2)$, then we can find an L-(p,q)-coloring $K_{2,3} - \{v_1\}$ (which exists because even cycles are circular 2-choosable), and then color v_1 the same color as v_2 to obtain an L-(p,q)-coloring of $K_{2,3}$. Thus we assume that $L(v_j) \neq L(v_{j'})$ if $j \neq j'$.

We say an interval $[a,b]_p$ of Z_p is *clean* if for some $j \in \{1,2\}$, $a, b \in L(u_j)$ and $[a,b]_p \cap L(u_{3-j}) = \emptyset$. Let $[a_0,b_0]_p, [a_1,b_1]_p, \ldots, [a_{2k-1}, b_{2k-1}]_p$ be all the maximal clean intervals of Z_p, and without loss of generality, assume that $a_1, a_3, \ldots, a_{2k-1} \in L(u_1)$ and $a_0, a_2, \ldots, a_{2k-2} \in L(u_2)$.

For distinct $i, j \in \{0,1,\ldots,2k-1\}$, $L(u_1) \cup L(u_2) \subseteq [a_{i+1}, b_j]_p \cup [a_{j+1}, b_i]_p$ (here we let $a_{2k} = a_0$). Therefore $4q = |L(u_1)| + |L(u_2)| = |L(u_1) \cup L(u_2)| \leq |[a_{i+1}, b_j]_p| + |[a_{j+1}, b_i]_p|$. The following lemma is proved in [5].

Lemma 5.8. *Assume $p \geq 4q - 3$. If $a, b, c, d \in Z_p$ in this cyclic order, and $|[b,c]_p| + |[d,a]_p| \geq 2q + 1$, then*

$$|B_q(a,b) \cap B_q(c,d)| \leq 2q - 1.$$

It follows from Lemma 5.8 that $|B_q(b_i, a_{i+1}) \cap B_q(b_j, a_{j+1})| \leq 2q - 1$. Hence $f(b_i, a_{i+1}) \neq f(b_j, a_{j+1})$, which implies that $k = 1$. Without loss of generality, assume $f(b_0, a_1) = 1$ and $f(b_1, a_0) = 2$.

Let $a_0 = t_1, t_2, \ldots, t_{2q} = b_0$ be all the elements of $L(u_2)$ and let $a_1 = s_1, s_2, \ldots, s_{2q} = b_1$ be all the elements of $L(u_1)$. Since $|[a_1, s_i]_p| + |[t_i, b_0]_p| \geq 2q + 1$, by Lemma 5.8, $|B_q(b_0, a_1) \cap B_q(t_i, s_i)| \leq 2q - 1$. Hence $f(t_i, s_i) \neq f(b_0, a_1) = 1$. Similarly, $f(t_i, s_i) \neq 2$. So $f(t_i, s_i) = 3$ for $i = 1, 2, \ldots, 2q$.

Lemma 5.9. *Suppose $a_1, a_2, \ldots, a_{4q}$ are distinct elements of Z_p appear in this cyclic order. Then*

$$\cap_{j=1}^{2q} B_{p,q}(a_j, a_{2q+j}) = \emptyset.$$

Proof. We prove the lemma by induction on p. If $p = 4q$, then we may assume that $a_i = i - 1$ for $i = 1, 2, \ldots, 4q$, and it is easy to see that $\cap_{j=1}^{2q} B_{p,q}(a_j, a_{2q+j}) = \emptyset$. Assume $p > 4q$. Then there is $j \in Z_p \setminus \{a_1, \ldots, a_{4q}\}$. Let $\phi : Z_p \to Z_{p-1}$ be defined as $\phi(t) = t$ if $t < j$ and $\phi(t) = t - 1$ if $t \geq j$. It is easy to verify that $t \in B_{p,q}(x)$ implies that $\phi(t) \in B_{p-1,q}(\phi(x))$. Thus $t \in \cap_{j=1}^{2q} B_{p,q}(a_j, a_{2q+j})$ implies that $\phi(t) \in \cap_{j=1}^{2q} B_{p-1,q}(\phi(a_j), \phi(a_{2q+j}))$. By induction hypothesis, $\cap_{j=1}^{2q} B_{p-1,q}(\phi(a_j), \phi(a_{2q+j})) = \emptyset$. Therefore $\cap_{j=1}^{2q} B_{p,q}(a_j, a_{2q+j}) = \emptyset$. ∎

It follows from this lemma that $L(v_3) \not\subseteq \cap_{j=1}^{2q} B_q(s_j, t_j)$, in contrary to the earlier conclusion that $f(t_i, s_i) = 3$ for $i = 1, 2, \ldots, 2q$. ∎

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