

Circular chromatic numbers of a class of distance graphs*

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March 5, 2003

Abstract

Suppose m, k, s are positive integers with $m > sk$. Let $D_{m,k,s}$ denote the set $\{1, 2, \dots, m\} \setminus \{k, 2k, \dots, sk\}$. The distance graph $G(Z, D_{m,k,s})$ has as vertex set all integers Z and edges connecting i and j whenever $|i - j| \in D_{m,k,s}$. This paper determines the circular chromatic number of all the distance graphs $G(Z, D_{m,k,s})$.

1 Introduction

Given a set D of positive integers, the *distance graph* $G(Z, D)$ has all integers as vertices, and two vertices are adjacent if and only if their difference is in D ; that is, the vertex set is Z and the edge set is $\{uv : |u - v| \in D\}$. The set D is called the *distance set*.

The problem of determining the chromatic number, fractional chromatic number and circular chromatic number of distance graphs are found to be related to T -colouring as well as problems in number theory [4], and has attracted much recent attention [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 18, 19]. The chromatic number of $G(Z, D)$ is easy to determine if $|D| = 1$ or 2 [5, 15]. The case when D contains three integers is much more complicated, and has recently been settled in [18]. For $|D| \geq 4$, it seems hopeless to completely determine the values of $\chi(G(Z, D))$. Nevertheless, for many special types of distance sets D , the chromatic numbers of the distance graphs $G(Z, D)$ have been studied. One such type of distance sets is the one defined below, which we shall study in this paper.

Given positive integers m, k, s with $m > sk$. Let $D_{m,k,s} = \{1, 2, \dots, m\} \setminus \{k, 2k, \dots, sk\}$. The chromatic number of distance graphs with distance set $D_{m,k,s}$ has been investigated in quite a few articles. Initially, the investigation has been focused on the case $s = 1$. In [8], Eggleton, Erdős and Skilton determined the chromatic numbers of graphs $G(Z, D_{m,1,1})$, and some of the chromatic numbers of graphs $G(Z, D_{m,2,1})$. For $3 \leq k < m$, only rough bounds were obtained in [8] for the chromatic numbers of $G(Z, D_{m,k,1})$. The same result for graphs $G(Z, D_{m,1,1})$ was also proven by Kemnitz and Kolberg in [11] by a different approach. In [12], Liu improved the lower bound of [8] and obtained

*Supported in part by the National Science Council under grant NSC87-2115-M110-004.

the exact values of $\chi(G(Z, D_{m,k,1}))$ for odd k . In [4], Chang, Liu and Zhu determined the chromatic number of all the graphs $G(Z, D_{m,k,1})$ (i.e., for arbitrary m and k).

For $s \geq 2$, the chromatic number of $G(Z, D_{m,k,s})$ was first studied in [13], where the case $s = 2$ was completely solved, and upper and lower bounds were obtained for $s \geq 3$. In [7], very tight upper and lower bounds were obtained. Namely it was proven in [7] that if $m \geq (s + 1)k$, then

$$\lceil (m + sk + 1)/(s + 1) \rceil \leq \chi(G(Z, D_{m,k,s})) \leq \lceil (m + sk + 1)/(s + 1) \rceil + 1.$$

Moreover, both the upper and the lower bounds are attainable. (The case that $m < (s + 1)k$ is trivial, cf. Section 2.) Thus it remained to determine for which $D_{m,k,s}$, $\chi(G(Z, D_{m,k,s}))$ attains the upper bound, and for which $D_{m,k,s}$ it attains the lower bound. This task has been accomplished very recently by Huang and Chang [10], whose result we shall cite in detail in Section 2. An interesting phenomenon is that Huang and Chang accomplished this task by investigating the circular chromatic number of such graphs.

The circular chromatic number of a graph is a natural generalization of the chromatic number of a graph, introduced by Vince [14] as the “star chromatic number.” Suppose p and q are positive integers such that $p \geq q$. A (p, q) -colouring of a graph $G = (V, E)$ is a mapping c from V to $\{0, 1, \dots, p - 1\}$ such that $q \leq |c(x) - c(y)| \leq p - q$ for any edge xy in E . The *circular chromatic number* $\chi_c(G)$ of G is the infimum of the ratios p/q for which there exists a (p, q) -colouring of G .

Note that a $(p, 1)$ -colouring of a graph G is simply an ordinary p -colouring of G . Therefore, $\chi_c(G) \leq \chi(G)$ for any graph G . On the other hand, it has been shown [14] that for all graphs G , we have $\chi(G) - 1 < \chi_c(G)$. Therefore, $\chi(G) = \lceil \chi_c(G) \rceil$. Thus two graphs with the same circular chromatic number also have the same chromatic number. However, two graphs with the same chromatic number may have different circular chromatic numbers. In this sense, $\chi_c(G)$ could be regarded as a refinement of $\chi(G)$, and $\chi(G)$ is an approximation of $\chi_c(G)$. Readers are referred to [20] for a survey on research about this parameter.

For a concrete graph G , it is usually much more difficult to determine its circular chromatic number than to determine its chromatic number. Indeed, partial information about the circular chromatic number of a graph might be enough to determine its chromatic number. This is just what was done in [10]. Some tight upper and lower bounds for the circular chromatic number of the graphs $G(Z, D_{m,k,s})$ were obtained. These bounds enable the authors to completely determine the chromatic number of these graphs. Of course, the discussion in [10] could easily be translated into a discussion of the ordinary colouring, and leads to a complete solution for the chromatic number problem. However, by discussing the circular colouring instead of the ordinary colouring, the patterns of the colourings are easier to recognize. The colouring rules for the circular chromatic number are easier to describe than that for the chromatic number. This is due to the fact that the distance graphs $G(Z, D_{m,k,s})$ have very nice structure.

In [7], the authors found sharp upper and lower bounds for the chromatic number of $G(Z, D_{m,k,s})$. The difference between upper and lower bounds is ≤ 1 . However, we failed to determine the exact values of the chromatic number of all such graphs. In the opinion of this author, one reason is that we were too much concentrated on the chromatic number. That prevented us from recognizing the colouring patterns from the point of view of circular colouring. The solution of this problem by Huang and Chang provides evidence to support the point of

view that “the circular chromatic number of a graph is a very natural concept”, sometimes, it is more natural than the concept of chromatic number.

Although Huang and Chang [10] obtained upper and lower bounds for the circular chromatic numbers of $G(Z, D_{m,k,s})$ which are sharp enough to determine the chromatic number, the exact values of the circular chromatic number of these graphs remained unknown, except for some special cases. This paper determines the circular chromatic number of all the graphs $G(Z, D_{m,k,s})$.

The circular chromatic number of graphs $G(Z, D_{m,k,s})$ has been studied in some other papers. In [4], the authors studied the circular chromatic number of $G(Z, D_{m,k,s})$ for $s = 1$. For some values of m and k , the exact values for $\chi_c(G(Z, D_{m,k,s}))$ were determined in [4]. In [3], the authors determined the circular chromatic number of $G(Z, D_{m,k,s})$ for $s = 1$ and for all m and k . The result in this paper can be viewed as a generalization of that solution to the case $s \geq 2$.

2 The main result

In this section, we state the main result and list those special cases that have already been solved. Section 3 contains the proof of all the other cases. For the completeness of this paper, we shall sketch the proofs of those results cited from other papers.

First we note that if $m < (s+1)k$, then it is straightforward to verify that the mapping $f(x) = x \bmod k$ for any $x \in Z$ defines a k -colouring of $G(Z, D_{m,k,s})$. As any consecutive k vertices in $G(Z, D_{m,k,s})$ form a clique of size k , we conclude that $\chi(G(Z, D_{m,k,s})) \leq k \leq \omega(Z, D_{m,k,s})$, which implies that

$$\chi(G(Z, D_{m,k,s})) = \chi_c(G(Z, D_{m,k,s})) = \omega(G(Z, D_{m,k,s})) = k.$$

In the remaining part of this paper, we shall assume that m, s, k are fixed positive integers such that $m \geq (s+1)k$. We shall let $G = G(Z, D_{m,k,s})$, and for $a < b$, we denote by $[a, b]$ the set $\{a, a+1, \dots, b\}$, and denote by $G[a, b]$ the subgraph of G induced by the set $[a, b]$.

The following theorem, which is the main result of this paper, determines the circular chromatic number of the distance graph G . For the rest of the paper, we let $m' = m + sk + 1$ and $d = \gcd(m', k) = \gcd(m+1, k)$.

Theorem 1

$$\chi_c(G) = \begin{cases} m'/(s+1), & \text{if } d = 1 \text{ or } d(s+1) \mid m', \\ (m'+1)/(s+1), & \text{otherwise.} \end{cases}$$

As $\chi(G) = \lceil \chi_c(G) \rceil$, we have the following corollary:

Corollary 1 [10]

$$\chi(G) = \begin{cases} \lceil m'/(s+1) \rceil + 1, & \text{if } (s+1) \mid m' \text{ and } d(s+1) \nmid m', \\ \lceil m'/(s+1) \rceil, & \text{otherwise.} \end{cases}$$

The rest of the paper is devoted to the proof of Theorem 1. The main task is to show that $\chi_c(G)$ is not smaller. The following well-known fact (cf. [20]) will be frequently used: for any finite graph H ,

$$\max\left\{\frac{|V(H)|}{\alpha(H)}, \omega(H)\right\} \leq \chi_c(H) \leq \lceil \chi_c(H) \rceil = \chi(H).$$

First we have the following lower bound for $\chi_c(G)$ obtained in [13]:

Proposition 2 [13] $\chi_c(G) \geq m'/(s+1)$.

Proof. It is straightforward to verify that $G[0, m'-1]$ has independence number $s+1$. Hence $\chi_c(G) \geq \chi_c(G[0, m'-1]) \geq |V(G[0, m'-1])|/(s+1) = m'/(s+1)$. ■

For some special cases, the exact values of $\chi_c(G)$ were determined in [3, 10, 13].

Proposition 3 [10] *If $d = 1$, then $\chi_c(G) = m'/(s+1)$.*

Proof. Suppose $x = am' + b$ where a, b are integers such that $0 \leq b < m'$. If $b \equiv ik \pmod{m'}$ where $0 \leq i < m'$, then let $f(x) = i$. It is straightforward to verify that f is an $(m', s+1)$ -colouring of G . ■

The following result was implicit in [13]:

Proposition 4 [13] *If $d(s+1) \mid m'$, then $\chi_c(G) = m'/(s+1) = \chi(G)$.*

Proof. Let $x \in Z$. There are unique integers a, b such that $x = am' + b$, and $0 \leq b < m'$. Also there are unique integers i, j such that $b = ik \pmod{m'} + j$ where $0 \leq i < m'$ and $0 \leq j < d-1$. Let $f(x) = \lfloor i/(s+1) \rfloor + j \frac{m'}{d(s+1)}$. It can be verified that f is a $\frac{m'}{s+1}$ -colouring of G . Therefore $\chi_c(G) \leq \chi(G) \leq m'/(s+1)$. Hence $\chi_c(G) = \chi(G) = m'/(s+1)$. ■

For the remaining part of this paper, we assume that $d \neq 1$ and that $d(s+1) \nmid m'$. We need to show that under this condition, $\chi_c(G) = (m'+1)/(s+1)$.

The following upper bound for the circular chromatic number of G was given in [10]:

Proposition 5 [10] $\chi_c(G) \leq (m'+1)/(s+1)$.

Proof. Let $x \in [0, m'-1]$. Then there are unique integers i, j such that $x = ik \pmod{m'} + j$, $0 \leq i \leq m'/d - 1$ and $0 \leq j < d-1$. Let $f(x) = i - j \frac{m'}{d} \pmod{m'+1}$. Extend the colouring f to the set Z by letting $f(x) = f(x-k) + 1 \pmod{m'+1}$ for $x \geq m'$ and $f(x) = f(x+k) - 1 \pmod{m'+1}$ for $x < 0$. Then it can be verified that f is a $(m'+1, s+1)$ -colouring of G . Interested readers are referred to [10] for the detail checking, where the colouring is a little different, namely, the colouring is $g(x) = f(x) + (d-1) \frac{m'}{d} \pmod{m'+1}$. ■

To complete the proof of Theorem 1, it remains to show that $\chi_c(G) \geq (m'+1)/(s+1)$ (under the assumption that $d \neq 1$ and that $d(s+1) \nmid m'$). The following special case was verified in [10]:

Proposition 6 [10] *If $(s+1) \mid m'$ but $d(s+1) \nmid m'$, then*

$$\chi_c(G[0, m'+k-1]) = (m'+1)/(s+1).$$

Proof. Let $r = m'/(s+1)$, which is an integer. First we show that $\chi_c(G[0, m'+k-1]) > r$. Assume to the contrary that $\chi_c(G[0, m'+k-1]) \leq r$. Then $\chi(G[0, m'+k-1]) \leq r$. Let f be an r -colouring of $G[0, m'+k-1]$. For $i = 0, 1, \dots, k-1$, consider the restriction of f to $G[i, i+m'-1]$.

It is easy to verify that $G[i, i+m'-1]$ has independence number $s+1$. It follows that each of the r colour classes consists of exactly $s+1$ vertices of $G[i, i+m'-1]$. Hence $f(i) = f(m'+i)$ for any integer $0 \leq i \leq k-1$.

Now, consider the colour classes of f for the graph $G[0, m' - 1]$. Let $u = \frac{m'}{d}$. Divide the vertex set of $G[0, m' - 1]$ into d subsets of the form $S_i = \{i, i + d, i + 2d, \dots, i + (u - 1)d\} \pmod{m'}$, for $i = 0, 1, \dots, d - 1$. Since each colour class has size $s + 1$, and since $f(j) = f(m' + j)$ for $j = 0, 1, \dots, k - 1$, it is not difficult to verify that each colour class must be contained in one of the sets S_i . Indeed, assume that $\{x_1, x_2, \dots, x_{s+1}\}$ is a colour class, where $x_1 < x_2 < \dots < x_{s+1}$. Then for $j = 2, 3, \dots, s + 1$, either $x_j - x_{j-1} = \ell k$ for some $1 \leq \ell \leq s$ or $x_j - x_{j-1} \geq m + 1$. Since $x_{s+1} - x_1 \leq m + sk$, it follows that there is at most one index j such that $x_j - x_{j-1} \geq m + 1$ and that $x_1 \leq k - 1$. Moreover, if $x_j - x_{j-1} \geq m + 2$, then $x_{s+1} > m' + x_1 - k$, which implies that x_{s+1} is adjacent to $m' + x_1$, contrary to the assumption that $f(x_1) = f(x_1 + m')$.

Thus we conclude that each of the sets S_i is the disjoint union of the colour classes, and hence $(s + 1) \mid u$. This implies that $d(s + 1) \mid m'$, contrary to the assumption. Therefore $\chi_c(G[0, m' - 1 + k]) > m'/(s + 1) = r$.

Next we prove that $\chi_c(G[0, m' + k - 1]) = (m' + 1)/(s + 1) = r + 1/(s + 1)$. Assume to the contrary that $\chi_c(G[0, m' + k - 1]) = p/q$, $\gcd(p, q) = 1$ and $r < p/q < r + 1/(s + 1)$. Then $p \leq |V(G[0, m' + k - 1])| = m' + k$. On the other hand, $r < p/q < r + 1/(s + 1)$ implies that $q \geq s + 2$. Then $p > (s + 2)m'/(s + 1)$. Since $m \geq (s + 1)k$, it follows that $p > m' + k$, which is a contradiction. ■

3 The complete proof of Theorem 1

In this section, we shall complete the proof of Theorem 1, by proving the following proposition:

Proposition 7 *If $d \neq 1$ and $(s + 1) \nmid m'$, then*

$$\chi_c(G[0, m' + 3k - 1]) = (m' + 1)/(s + 1).$$

Obviously, Proposition 7 implies that

$$\chi_c(G) \geq \chi_c(G[0, m' + 3k - 1]) = (m' + 1)/(s + 1),$$

provided that $d \neq 1$ and $(s + 1) \nmid m'$. Combining Proposition 7 with the results cited in Section 2, we obtain Theorem 1. We divide the the proof of Proposition 7 into a sequence of claims.

By Proposition 5, we have $\chi_c(G[0, m' + 3k - 1]) \leq \chi_c(G) \leq (m' + 1)/(s + 1)$. Assume to the contrary of Proposition 7 that

$$\chi_c(G[0, m' + 3k - 1]) = p/q < (m' + 1)/(s + 1),$$

where $\gcd(p, q) = 1$. Let c be a (p, q) -colouring of $G[0, m' + 3k - 1]$. It is well known that c is an onto colouring, i.e., the colouring uses every colour (cf. [1, 14, 17]). By Proposition 2, we know that $m'/(s + 1) \leq p/q$.

For $j = 0, 1, \dots, p - 1$, let

$$\begin{aligned} X_j &= c^{-1}(j), \\ Y_j &= X_j \cup X_{j+1} \cup \dots \cup X_{j+q-1}, \end{aligned}$$

where the indices are modulo p . For $0 \leq a < b \leq m' + 3k - 1$, let

$$\begin{aligned} X_j[a, b] &= X_j \cap [a, b], \\ Y_j[a, b] &= Y_j \cap [a, b]. \end{aligned}$$

The sets Y_j ($0 \leq j \leq p-1$) are independent sets of G . Therefore for any two vertices $u < v$ of Y_j , $v - u$ is either equal to tk for some $1 \leq t \leq s$, or $v - u \geq m+1$. Frequently, we shall consider subgraphs of $G[0, m' + 3k - 1]$ of the form $G[u, u + m']$. We shall denote this graph by G_u , and let $x_{u,i} = |X_i[u, u + m']|$ and $y_{u,i} = |Y_i[u, u + m']|$.

Suppose u and v are nonadjacent vertices of $G[0, m' + 3k - 1]$. We call (u, v) a *regular pair* if $v \equiv u \pmod{d}$ (recall that $d = \gcd(m', k) = \gcd(m + 1, k)$), otherwise we say (u, v) is an *irregular pair*. Note that if (u, v) and (v, w) are both regular and that u is non-adjacent to w , then (u, w) is also regular. We say an irregular pair (u, v) is *strongly irregular* if $|u - v| \leq m'$.

Suppose $I = \{x_1, x_2, \dots, x_t\}$ is an independent set of G , and that $x_1 < x_2 < \dots < x_t$. We call the sequence $(x_2 - x_1, x_3 - x_2, \dots, x_t - x_{t-1})$ the *gap sequence* of I , and each entry of the gap sequence is called a *gap* in I . We shall refer $x_2 - x_1$ as the first gap of I , $x_3 - x_2$ as the second gap of I , etc. Each gap in I is either equal to jk for some integer $1 \leq j \leq s$, or is greater than or equal to $m + 1$. If I contains an irregular pair, then at least one of the gap is greater than or equal to $m + 2$.

The following are some easy observations about the size $y_{u,j}$.

Observation 1 For any $0 \leq u \leq 3k - 1$,

1. $y_{u,j} \leq s + 2$.
2. if $y_{u,j} = s + 2$ then $u, u + m' \in Y_j$.
3. $q(m' + 1) = \sum_{j=0}^{p-1} y_{u,j} > p(s + 1)$.
4. $|\{j : y_{u,j} = s + 2\}| \geq |\{j : y_{u,j} \leq s\}| + 1$. In particular, there is at least 1 index j such that $y_{u,j} = s + 2$.
5. If Y_j contains an irregular pair then Y_j cannot contain both u and $u + m'$ and hence $y_{u,j} \leq s + 1$.
6. If $u \leq (3 - t)k - 1$, $Y_j[u + tk, u + tk + m']$ contains an irregular pair (v, u) with $v < u$ and $u + tk + m' \notin Y_j$, then $y_{u+tk,j} \leq s$.
7. $|\{j : y_{u,j} = s + 2\}| \leq q$.

(1) and (2) follow from the observation that $G[u, u + m' - 1]$ has independence number $s + 1$ and $G[u, u + m']$ has independence number $s + 2$. (3) follows from the fact that $\sum_{j=0}^{p-1} y_{u,j}$ count each of the vertices of $G[u, u + m']$ q times and the assumption that $p/q < (m' + 1)/(s + 1)$. By (3), the average size of $y_{u,j} > s + 1$, so we have (4).

Assume to the contrary of (5) that Y_j contains an irregular pair (x, y) ($x < y$) and Y_j contains both $u, u + m'$. Then one of x, y , say x is contained in $[u, u + m']$. This implies that one of the gap $x - u, u + m' - x$ is less than $m + 2$. So (x, u) or $(x, u + m')$ is regular, which implies that both (x, u) and $(x, u + m')$ are regular (as $(u, u + m')$ is regular). Now $y - (u + m') < m + 2$, so $(u + m', y)$ is regular, which is a contradiction. Thus Y_j cannot contain both $u, u + m'$, and the rest of (5) follows from (2).

If $Y_j[u + tk, u + tk + m']$ contains an irregular pair (v, u) (with $v < u$) and $u + tk + m' \notin Y_j$, then as u is adjacent to each vertex of $[u + (t-1)k + 1, u + tk - 1]$, we conclude that $Y_j[u + tk, u + tk + m'] = Y_j[u + tk, u + (t-1)k + m']$. So the sum of the gaps of $Y_j[u + tk - m', u + tk]$ is at most $m' - k$. On the other hand, $Y_j[u + tk - m', u + tk]$ contains both u and v , which is an irregular pair. Hence $Y_j[u + tk - m', u + tk]$ has a gap $\geq m + 2$. Therefore $Y_j[u + tk - m', u + tk]$ has at most $s - 1$ gaps, and $|Y_j[u + tk, u + tk + m']| \leq s$. So we have (6).

If $u \in X_i$, then only for $j = i - q + 1, i - q + 2, \dots, i$, Y_j contains u . So (7) follows from (2).

Claim 1 For any $0 \leq u \leq 3k - 1$, there exists an index i such that $Y_i[u, u + m' - 1]$ contains a strongly irregular pair.

Proof. Assume to the contrary that for some $0 \leq u \leq 3k - 1$, none of the sets $Y_j[u, u + m' - 1]$ contains an irregular pair. (Note that any irregular pair contained in $[u, u + m' - 1]$ is strongly irregular.)

For $i = 0, 1, \dots, d - 1$, let

$$S_i = \{i, i + d, i + 2d, \dots\} \cap [u, u + m' - 1].$$

These sets are disjoint and form a partition of $[u, u + m' - 1]$. As none of the sets $Y_j[u, u + m' - 1]$ contains an irregular pair, we conclude that each of $Y_j[u, u + m' - 1]$ is contained in one of the S_i 's. Since the sets $Y_j[u, u + m' - 1]$ form a partition of the set $\{u, u + 1, \dots, u + m' - 1\}$, it follows that there is an index j_0 such that $Y_{j_0}[u, u + m' - 1]$ and $Y_{j_0+1}[u, u + m' - 1]$ are disjoint (the addition in the indices are modulo p). Therefore

$$X_{j_0+1}[u, u + m' - 1], X_{j_0+2}[u, u + m' - 1], \dots, X_{j_0+q-1}[u, u + m' - 1]$$

are all empty sets. Thus we obtain an $\lfloor p/q \rfloor$ -colouring of $G[u, u + m' - 1]$ as follows:

$$f(w) = j \text{ if } w \in Y_{j_0+jq} \text{ for } j = 1, 2, \dots, \lfloor p/q \rfloor.$$

However, $G[u, u + m' - 1]$ has independence number $s + 1$, and hence

$$\chi(G[u, u + m' - 1]) \geq m'/(s + 1) > \lfloor p/q \rfloor.$$

(Note that by assumption, $m'/(s + 1)$ is not an integer. Since $m'/(s + 1) \leq p/q < (m' + 1)/(s + 1)$, we have $m'/(s + 1) > \lfloor m'/(s + 1) \rfloor = \lfloor p/q \rfloor$.) \blacksquare

Claim 2 $p/q > m'/(s + 1)$.

Proof. Assume to the contrary that $p/q = m'/(s + 1)$. For $0 \leq i \leq k - 1$, the subgraph $G[i, i + m' - 1]$ has independence number $s + 1$. If for some j , $|Y_j[i, i + m' - 1]| \leq s$, then

$$\sum_{j=0}^{p-1} |Y_j[i, i + m' - 1]| \leq p(s + 1) - 1.$$

However, in the sum $\sum_{j=0}^{p-1} |Y_j[i, i + m' - 1]|$, each vertex of $G[i, i + m' - 1]$ is counted q times. Hence $qm' < p(s + 1)$, contrary to the assumption that $p/q = m'/(s + 1)$. Therefore for any $0 \leq i \leq k - 1$ and for any $0 \leq j \leq p - 1$, $|Y_j[i, i + m' - 1]| = s + 1$.

By Claim 1, there is an index j^* , such that $Y_{j^*}[0, m' - 1]$ contains an irregular pair. Thus $Y_{j^*}[0, m' - 1]$ has a gap $\geq m + 2$. Since $|Y_{j^*}[0, m' - 1]| = s + 1$, the sum of the gaps is at least $(s - 1)k + (m + 2)$. This implies that $Y_{j^*}[0, k - 2] \neq \emptyset$.

Assume $w \in Y_{j^*}[0, k - 2]$. If the first gap of $Y_{j^*}[0, m' - 1]$ is $\geq m + 2$, then $Y_{j^*}[w + 1, m' + w] \subset Y_{j^*}[w + m + 2, m' + w]$. This implies that the sum of

the gaps of $Y_{j^*}[w+1, m'+w]$ is at most $sk-1$. Since each gap is at least k , we conclude that $Y_{j^*}[w+1, m'+w]$ has at most $s-1$ gaps, and hence $|Y_{j^*}[w+1, m'+w]| \leq s$. If the first gap of $Y_{j^*}[0, m'-1]$ is $\leq m+1$, then $Y_{j^*}[w+1, m'+w] \subset Y_{j^*}[w+k, m'+w]$ and contains an irregular pair, and hence has a gap $\geq m+2$. As the sum of all the gaps of $Y_{j^*}[w+k, m'+w]$ is at most $m'-k = m+(s-1)k+1$, it follows that there are at most $s-1$ gaps and hence $|Y_{j^*}[w+k, m'+w]| \leq s$. So, in any case, $|Y_{j^*}[w+1, m'+w]| \leq s$. This is in contrary to our previous conclusion that $|Y_{j^*}[i, i+m'-1]| = s+1$. ■

Corollary 2 *Either $q = s+2$ or $q \leq s$.*

Proof. Since $m'/(s+1) < p/q < (m'+1)/(s+1)$, we know that $q \neq s+1$. Assume to the contrary that $q \geq s+3$. As c is an onto colouring, we know that $p \leq |V(G[0, m'+3k-1])| = m'+3k$. On the other hand, $m'/(s+1) < p/q \leq p/(s+3)$, which implies that $p > m'+2m'/(s+1)$. As $m \geq (s+1)k$ and $m' = m+sk+1$, it follows that $p > m'+3k$, which is a contradiction. ■

Remark. The arguments in the proofs of Claims 1 and 2 actually prove that $\chi_c(G[0, m'+k-1]) > m'/(s+1)$ (under the assumption that $d > 1$ and $(s+1) \nmid m'$). In case $s = 1$, we can conclude from this that $\chi_c(G[0, m'+k-1]) = (m'+1)/2$. Indeed, if this is not true, then $\chi_c(G[0, m'+k-1]) = p'/q' < (m'+1)/2$, where $(p', q') = 1$. However $m'/2 < p'/q' < (m'+1)/2$ implies that $q' \geq 3$, and hence $p' > 3m'/2 = m' + m'/2 > m' + m/2 \geq m' + k$. (Recall that $m' = m+sk+1 = m+k+1$, and $m \geq (s+1)k = 2k$.) This is in contrary to the well-known fact that $p' \leq |V(G[0, m'+k-1])| = m'+k$. This shows that Theorem 1 is true when $s = 1$, which is a result proven in [3].

For the remaining part of the proof, we assume that $s \geq 2$.

We note that for any u , the graph $G_u = G[u, u+m']$ has independence number $s+2$, and any independent set I of G_u of size $s+2$ is of the form

$$\{u, u+k, \dots, u+\alpha k, u+\alpha k+m+1, u+(\alpha+1)k+m+1, \dots, u+sk+m+1\},$$

for some $0 \leq \alpha \leq s$. In other words, all the gaps of I are equal to k , except one gap which is equal to $m+1$. In particular, I does not contain an irregular pair, and contains both vertices $u, u+m'$.

Claim 3 *For $a = 0, 1, \dots, p-1$, the set X_a does not contain an irregular pair.*

Proof. Assume to the contrary that X_a contains an irregular pair, say (u, v) . Assume that $v < u$.

If $v < 3k$, then none of the Y_i 's contains both v and $v+m'$. By (5) of Observation 1, $|Y_i[v, v+m']| \leq s+1$ for all i , contrary to Observation 1. If $u \geq m'$ then we consider the subgraph $G_{u-m'}$. The same contradiction would be derived.

Assume now that $v \geq 3k$ and $u \leq m'-1$. Since $u-v \geq m+2$, we know that $u \geq m+3k+2$. Let t be the largest integer such that $u+tk \leq m'+k-1$. Then $G_{u+tk-m'}$ contains both vertices v and u .

Assume that $u+tk \in X_h$. If $q \leq |h-a| \leq p-q$, then by (6) of Observation 1 for $j \in \{a-q+1, a-q+2, \dots, a\}$, $|Y_j[u+tk-m', u+tk]| \leq s$. for $j = a, a-1, \dots, a-q+1$. Thus, at least q of the sets $Y_j[u+tk-m', u+tk]$ has cardinality at most s . On the other hand, any independent set of $G_{u+tk-m'}$ of size $s+2$ must contain $u+tk$, so at most q of the sets $Y_j[u+tk-m', u+tk]$, namely for $j = h-q+1, h-q+2, \dots, h$, has cardinality $s+2$. This contradicts (4) of Observation 1.

If either $|h - a| < q$ or $|h - a| \geq p - q$, then a similar calculation shows that for at least $|h - a|$ index j , $y_{u+tk-m',j} \leq s$, and for at most $|h - a|$ index j , $y_{u+tk-m',j} = s + 2$, again contrary to (4) of Observation 1. \blacksquare

Assume that the set Y_j contains a strongly irregular pair (u, v) (such an index j exists by Claim 1). Assume that $u \in X_a, v \in X_b$, and without loss of generality, we may assume that $v < u$ and $b < a$ (by Claim 3, we know that $b \neq a$). We choose the strongly irregular pair (u, v) so that $a - b$ is minimum.

Claim 4 For a, b defined above, we have $a - b = 1$.

Proof. It is easy to see that there is an index i such that there is an irregular pair (u, v) with $v \in X_{i-1}$ and $u \in X_i$. For otherwise, every element of X_j form regular pair with every element of X_{j+1} . Then by the ‘‘transitivity’’ of the regular relation, we conclude that Y_ℓ contains no irregular pair for every ℓ , contrary to Claim 1

Without loss of generality, we assume that $v < u$. If (u, v) is strongly irregular, then we are done. Assume that (u, v) is not strongly irregular. Then $u \geq m'$. Consider the subgraph $G_{u-m'}$. We may assume that u is a minimum element of X_i , because for any element $u' \neq u$ of X_i , the pair (u, u') is regular (by Claim 3), and hence (v, u') is irregular.

Now for $j \notin \{i - q + 1, i - q + 2, \dots, i\}$, the set Y_j does not contain u . For $j \in \{i - q + 1, \dots, i - 1\}$, the set Y_j contains both u and v and hence does not contain $u - m'$ (by (5) of Observation 1). Therefore, if $j \neq i$, then $y_{u-m',j} \leq s + 1$.

It follows from (3) of Observation 1 that $y_{u-m',i} = s + 2$ and $y_{u-m',j} = s + 1$, for $j \neq i$. Note that $y_{u-m',j} = \sum_{t=j}^{j+q-1} x_{u-m',t}$. Hence $y_{u-m',j} - y_{u-m',j+1} = x_{u-m',j} - x_{u-m',j+q}$. Therefore we have

$$\begin{aligned} x_{u-m',i} &= x_{u-m',i+q} + 1, \\ x_{u-m',i-1} &= x_{u-m',i+q-1} - 1, \\ x_{u-m',j} &= x_{u-m',j+q} \quad \text{for } j \neq i, i - 1. \end{aligned}$$

This system of equations can be easily solved (together with the equation $\sum_{j=0}^{p-1} x_{u-m',j} = m' + 1$). Indeed, let i_0 be the integer such that $0 \leq i_0 \leq p - 1$ and $i_0 q \equiv 1 \pmod{p}$, then $x_{u-m',i+jq-1} = \lceil (m' + 1)/p \rceil$ for $j = 1, 2, \dots, i_0$, and $x_{u-m',i+jq-1} = \lfloor (m' + 1)/p \rfloor$ for $j = i_0 + 1, i_0 + 2, \dots, p - 1, 0$. (Such solutions to a system of simple linear equations will be used a few more times in the latter discussion, and we shall omit the details hereafter.)

A particular feature of the solution is that for each j , either $x_{u-m',j} = \lceil (m' + 1)/p \rceil$ or $x_{u-m',j} = \lfloor (m' + 1)/p \rfloor$, and that $x_{u-m',i} = \lceil (m' + 1)/p \rceil$. However, u is the minimum element of X_i , hence $x_{u-m',i} = 1$. Therefore for each j , $x_{u-m',j}$ is either 0 or 1. This implies that $p > m + sk + 2$. By Corollary 2, we have $q = s + 2$. We now list the solution for $x_{u-m',j}$ as a 0-1 sequence, starting from $x_{u-m',i}$. As the first $q = s + 2$ of the items in the sequence sum up to $s + 2$, and every other $s + 2$ (cyclically) consecutive items in the sequence sum up to $s + 1$, the sequence is

$$(11 \dots 1011 \dots 1011 \dots 1 \dots 11 \dots 10),$$

which is $s + 2$ 1's followed by a 0, and then every $s + 1$ 1's follows by a 0, and ending with a 0.

Therefore $p = \alpha(s + 2) + 1$ for some integer α . As $m'/(s + 1)$ is not an integer, and $m'/(s + 1) < p/q < (m' + 1)/(s + 1)$, we conclude that $\alpha = \lfloor m'/(s + 1) \rfloor =$

$\lfloor p/q \rfloor$. However, this is an obvious contradiction, as $m'/(s+1) - \alpha \geq 1/(s+1) > 1/(s+2) = p/q - \alpha$. ■

Without loss of generality, we may assume that $b = 0$ and $a = 1$, i.e., $v \in X_0$ and $u \in X_1$, $v < u$ and (u, v) is a strongly irregular pair. Assume that v is the largest element of X_0 which is less than u . The next claim gives the possible locations of v and u .

Claim 5 *For the strongly irregular pair (u, v) given above, $u < m'$ and $v > 3k - 1$.*

Proof. First we shall prove that either $u < m'$ or $v > 3k - 1$.

Assume to the contrary that there are vertices $v \in X_0, u \in X_1$ such that (u, v) is a strongly irregular pair, $v \leq 3k - 1$ and $u \geq m'$.

For $j \neq p - q + 1$, either Y_j does not contain v , or Y_j contains the irregular pair, and hence by (2) and (4) of Observation 1, $y_{v,j} \leq s + 1$. Then it follows from (3) of Observation 1 that $y_{v,p-q+1} = s + 2$ and $y_{v,j} = s + 1$ for $j \neq p - q + 1$.

Similar to the corresponding part of the proof of Claim 4, we conclude that

$$\begin{aligned} x_{v,0} &= x_{v,p-q} + 1, \\ x_{v,1} &= x_{v,p+1-q} - 1, \\ x_{v,j} &= x_{v,j-q}, \quad \text{for } j \neq 0, 1. \end{aligned}$$

Together with equation $\sum_{j=0}^{p-1} x_{v,j} = m' + 1$, the variables $x_{v,j}$ can be easily determined. A particular feature of the solution is that $x_{v,0} = x_{v,1} + 1$.

Now by considering the graph $G_{u-m'}$, and by using the same argument as above, we may conclude that $x_{u-m',1} = x_{u-m',0} + 1$. As (u, v) is strongly irregular, we know that $u \leq v + m'$. Hence $X_1[u - m', v] \neq \emptyset$ and $X_0[u, v + m'] \neq \emptyset$. However, $u' \in X_1[u - m', v]$ implies that (u, u') is an irregular pair (as $v - u' < m + 2$ and hence (u', v) is regular), contrary to Claim 3.

This proves that either $u < m'$ or $v > 3k - 1$. By symmetry, we assume that $u < m'$. It remains to show that $v > 3k - 1$.

Assume to the contrary that $v \leq 3k - 1$. Consider the subgraph G_v . By the same argument as above, we can conclude that $x_{v,0} = \lceil (m' + 1)/p \rceil$ and $x_{v,1} = \lfloor (m' + 1)/p \rfloor = x_{v,0} - 1 \geq 1$. Thus $x_{v,0} \geq 2$. Let $v' \in X_0[v + 1, v + m']$. Then $v' > v$ and (u, v') is strongly irregular (for otherwise we would have (v, v') being irregular, contrary to Claim 3). Thus $|u - v'| \geq m + 2$. Because $u \geq v + m + 2$ and $v' \leq v + m'$, we know that $v' < u$. This is in contrary to the choice of v . ■

Since $u - v \geq m + 2$, we have $m' > u \geq m + 3k + 2$. Now let t_0 be the least integer such that $u + t_0 k \geq m'$. Then $u + (t_0 + 2)k \leq m' + 3k - 1$. For $t = t_0, t_0 + 1, t_0 + 2$, assume that $u + tk \in X_{h(t)}$. A careful calculation (which is presented in the appendix) gives us the size of the set $X_1[u + tk - m', u + tk]$ depending on $h(t)$, for $t = t_0, t_0 + 1, t_0 + 2$.

Claim 6 *For $t = t_0, t_0 + 1, t_0 + 2$, if $h(t) = 1$, then $x_{u+tk-m',1} = \lceil (m' + 1)/p \rceil$. If $h(t) \neq 1$, then $x_{u+tk-m',1} = \lfloor m'/p \rfloor$.*

Proof. By Claim 3, we know that $h(t) \neq 0$. If $h(t) = 1$, then among the independent sets Y_j , only Y_1 contains $u + tk$ and contains no irregular pair. By Observation 1, $y_{u+tk-m',1} = s + 2$, and $y_{u+tk-m',j} = s + 1$ for $j \neq 1$. This implies that $x_{u+tk-m',1} = \lceil (m' + 1)/p \rceil$, (cf. proof of Claim 4).

Assume now that $h(t) \neq 1$. Suppose first that $q \leq h(t) - 1 \leq p - q$. If $j \in \{0, p-1, \dots, p-q+2\}$, then $Y_j[u+tk-m', u+tk]$ contains both u and v but not $u+tk$ (as $h(t) \neq j$). By Observation 1, $y_{u+tk-m',j} \leq s$ for $j \in \{0, p-1, \dots, p-q+2\}$. Therefore at least $q-1$ of the sets $Y_j[u+tk-m', u+tk]$ has cardinality at most s . By (7) and (4) of Observation 1, we conclude that

$$\begin{aligned} y_{u+tk-m',j} &= s, & \text{for } j = 0, p-1, \dots, p-q+2, \\ y_{u+tk-m',j} &= s+2, & \text{for } j = h(t), h(t)-1, \dots, h(t)-q+1, \\ y_{u+tk-m',j} &= s+1, & \text{for all other } j. \end{aligned}$$

Similar to the proof of Claim 4, this induces a system of equations for $x_{u+tk-m',j}$ which can be easily solved. A particular feature of the solution is that $x_{u+tk-m',1} = \lfloor m'/p \rfloor$.

If $0 < h(t)-1 < q$, then the same argument as above shows that $y_{u+tk-m',j} \leq s$, for $j = p-q+2, p-q+3, \dots, p-q+h(t)$. On the other hand, because any independent set of $G[u+tk-m', u+tk]$ of size $s+2$ must contain $u+tk$ and must not contain an irregular pair, we conclude that

$$|Y_j[u+tk-m', u+tk]| \leq s+1,$$

for $j \neq 1, 2, 3, \dots, h(t)$. Thus there are at most $h(t)$ of the indices j with $y_{u+tk-m',j} = s+2$. Then by (4) of Observation 1, we have

$$\begin{aligned} y_{u+tk-m',j} &= s, & \text{for } j = p-q+2, p-q+3, \dots, p-q+h(t), \\ y_{u+tk-m',j} &= s+2, & \text{for } j = 1, 2, \dots, h(t), \\ y_{u+tk-m',j} &= s+1, & \text{for all other } j. \end{aligned}$$

This also induces a system of equations for $x_{u+tk-m',j}$, and by solving the equations, we have $x_{u+tk-m',1} = \lfloor m'/p \rfloor$. (Note that the equations for the case $h(t) < q$ and the case $h(t) = q$ are a little bit different, but $x_{u+tk-m',1}$ has the same value in both systems of equations).

The case that $h(t) > p-q$ can be treated similarly, and we omit the details. \blacksquare

Claim 7 For $t \in \{t_0, t_0+1, t_0+2\}$, at most one of the $h(t)$'s is equal to 1.

Proof. Assume to the contrary that $t < t'$ are two indices of $\{t_0, t_0+1, t_0+2\}$ such that $h(t) = h(t') = 1$. By Claim 6, $x_{u+tk-m',1} = x_{u+t'k-m',1} = \lceil (m'+1)/p \rceil$. Since

$$u+t'k \in X_1[u+t'k-m', u+t'k] - X_1[u+tk-m', u+tk],$$

we conclude that

$$\begin{aligned} X_1[u+tk-m', u+tk] - X_1[u+t'k-m', u+t'k] \\ = X_1[u+tk-m', u+t'k-m'] \neq \emptyset. \end{aligned}$$

Suppose $u' \in X_1[u+tk-m', u+t'k-m']$. Since $v \geq 3k$, we have $u' < v$, hence (u', v) is regular, which implies that (u', u) is irregular, contrary to Claim 3. \blacksquare

Claim 8 Among the three indices t_0, t_0+1, t_0+2 , there are no two consecutive indices t, t' such that $h(t), h(t') \neq 1$.

Proof. Suppose to the contrary that $t' = t+1$ and that $h(t), h(t') \neq 1$. Consider the subgraph $G[u + t'k - m', u + t'k]$. If $q \leq h(t') - 1 \leq p - q$ then by the proof of Claim 6, we have

$$\begin{aligned} y_{u+t'k-m',j} &= s, & \text{for } j = 0, p-1, \dots, p-q+2, \\ y_{u+t'k-m',j} &= s+2, & \text{for } j = h(t'), h(t')-1, \dots, h(t')-q+1, \end{aligned}$$

and $y_{u+t'k-m',j} = s+1$ for all other j .

Now we consider the value of $h(t)$. By assumption, either $2 \leq h(t) \leq p-q+1$, or $h(t) \geq p-q+2$. (Note that $h(t) \neq 0$.) Assume first that $2 \leq h(t) \leq p-q+1$. Then $Y_{p-q+2}[u + t'k - m', u + t'k]$ contains both u and v but contains neither of $u + tk, u + t'k$. The same argument as in the proof of (6) of Observation 1 shows that $y_{u+t'k-m',p-q+2} \leq s-1$, contrary to the previous paragraph.

If $h(t) \geq p-q+2$, then the same argument shows that $y_{u+t'k-m',0} \leq s-1$, again in contrary to the first paragraph of this proof.

Assume now that $h(t') - 1 < q$. Then by the same argument as in the proof of Claim 6 shows that

$$\begin{aligned} y_{u+t'k-m',j} &= s, & \text{for } j = p-q+2, p-q+3, \dots, p-q+h(t'), \\ y_{u+t'k-m',j} &= s+2, & \text{for } j = 1, 2, \dots, h(t'), \end{aligned}$$

and $y_{u+t'k-m',j} = s+1$ for all other j .

The same reason as above, we know that if $2 \leq h(t) \leq p-q+1$, then $y_{u+t'k-m',p-q+2} \leq s-1$. Assume $h(t) \geq p-q+2$. Then $Y_1[u + t'k - m', u + t'k]$ contains both u and $u + t'k$, but does not contain $u + tk = u + (t-1)k$. Since $u \geq m + 3k + 2$, it follows that $t' < s$ and hence $t'k < m$. Therefore the last gap of $Y_1[u + t'k - m', u + t'k]$ is neither k nor $m+1$. This implies that $y_{u+t'k-m',1} \leq s+1$, because any independent set of $G[u + t'k - m', u + t'k]$ of size $s+2$ has all the gaps equal to k , except one which is equal to $m+1$. This is in contrary to the previous paragraph.

The case $h(t') - 1 > p-q$ can be treated similarly, and we omit the details. \blacksquare

Combining Claims 7 and 8, we conclude that $h(t_0) \neq 1, h(t_0 + 1) = 1$ and $h(t_0 + 2) \neq 1$.

Let $t = t_0$ and $t' = t_0 + 2$. By Claim 6,

$$x_{u+tk-m',1} = x_{u+t'k-m',1}.$$

As $h(t_0 + 1) = 1$, i.e.,

$$u + (t_0 + 1)k \in X_1[u + t'k - m', u + t'k] - X_1[u + tk - m', u + tk],$$

we conclude that

$$X_1[u + tk - m', u + t'k - m'] \neq \emptyset.$$

Let $u' \in X_1[u + tk - m', u + t'k - m']$. Then (u, u') is irregular (cf. proof of Claim 7), contrary to Claim 3. This completes the proof of Proposition 7.

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