Fractional Chromatic Number and Circular Chromatic Number for Distance Graphs with Large Clique Size

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Abstract

Let M be a set of positive integers. The distance graph generated by M, denoted by $G(Z, M)$, has the set Z of all integers as the vertex

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set, and edges ij whenever $|i - j| \in M$. We investigate the fractional chromatic number and the circular chromatic number for distance graphs, and discuss their close connections with some number theory problems. In particular, we determine the fractional chromatic number and the circular chromatic number for all distance graphs $G(Z, M)$ with clique size at least $|M|$, except for one case of such graphs. For the exceptional case, a lower bound for the fractional chromatic number and an upper bound for the circular chromatic number are presented; these bounds are sharp enough to determine the chromatic number for such graphs. Our results confirm a conjecture of Rabinowitz and Proulx [21] on the density of integral sets with missing differences, and generalize some known results on the circular chromatic number of distance graphs and the parameter involved in the Wills' conjecture [25] (also known as the "lonely runner conjecture" [1]).

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1 Introduction

The notion of distance graphs arose from the search of the "plane chromatic number": What is the least number of colors needed to color the plane so that any two points of unit distance receive distinct colors ? This number is known to be bounded by 4 and 7 [15, 20], however, the exact value remains unknown.

Let M be a set of positive integers. The *distance graph* generated by M, denoted by $G(Z, M)$, has the set Z of all integers as the vertex set, and two vertices x and y are adjacent whenever $|x - y| \in M$. Initiated by Eggleton, Erdős and Skilton [10], the study of distance graphs has attracted considerable attention. Most of the efforts in the past are devoted to the chromatic number of such graphs [8, 10, 11, 12, 13, 17, 22, 23].

In this article, we investigate the fractional chromatic number and the circular chromatic number for distance graphs, and discuss their close connections with some number theory problems.

A *fractional coloring* of a graph G is a mapping f which assigns to each independent set I of G a non-negative weight $f(I)$ such that for each vertex x, $\sum_{x\in I} f(I) \geq 1$. The *fractional chromatic number* $\chi_f(G)$ of G is the least total weight of a fractional coloring of G.

The problem of determining the fractional chromatic number of a distance graph is equivalent to a number theory problem. For a set M of positive integers, a set S of non-negative integers is called an *M-set* if $a - b \notin M$ for any $a, b \in S$. Let $S(n)$ denote $|\{0, 1, \cdots, n\} \cap S|$. The upper density and the lower density of S are defined, respectively, by

$$\overline{\delta}(S) = \overline{\lim}_{n\to\infty} S(n)/n, \quad \underline{\delta}(S) = \underline{\lim}_{n\to\infty} S(n)/n.$$

We say S has density $\delta(S)$ if $\overline{\delta}(S) = \underline{\delta}(S) = \delta(S)$. The parameter of interest is the maximum density of an M-set, defined by

$$\mu(M) = \sup\{\delta(S) : S \text{ is an } M\text{-set}\}.$$

The problem of determining or estimating $\mu(M)$ was initially posed by Motzkin in an unpublished problem collection (cf. [2]). In 1973, Cantor and Gordon [2] proved the existence of $\mu(M)$ for any M. In addition to [2], the values of $\mu(M)$ for several special families of M have also been studied by Haralambis [16], Rabinowitz and Proulx [21], and Griggs and Liu [14].

The fractional chromatic number of distance graphs and the density of M-sets are indeed the *same* problem, due to the result of Chang, Liu and Zhu [4]:

Theorem 1.1 *For any finite set M, $\mu(M) = 1/\chi_f(G(Z, M))$.*

Throughout the article, we mainly use $\mu(M)$ instead of $\chi_f(G(Z, M))$, because some earlier results used in our proofs are formulated in terms of $\mu(M)$. In addition, we assume $\gcd(M) = 1$, since $\mu(M) = \mu(aM)$ for any integer a, where aM is the set obtained by multiplying every element in M by a.

If M is a singleton, then trivially $\mu(M) = 1/2$. If $M = \{a, b\}$ and a, b are both odd, then $\mu(M) = 1/2$ (all even numbers form an M-set). Cantor and Gordon [2] settled the case when a, b are of different parities.

Theorem 1.2 [2] *If $M = \{a, b\}$ and a and b are of different parities, then*

$$\mu(M) = \frac{\lfloor (a+b)/2 \rfloor}{a+b}.$$

For $|M| \geq 3$, the values of $\mu(M)$ are known only for some very special sets M. In [16, 21], the sets $M = \{1, a, b\}$, $M = \{1, 2, a, b\}$ and $M = \{a, b, a+b\}$, are considered, and $\mu(M)$ are determined for some special values of a and b.

A set M of positive integers is called *difference closed* if the difference of any pair of elements in M falls in M. A set M is *almost difference closed* if there exists $M' \subset M$ such that $|M'| = |M| - 1$ and $|a - b| \in M$ for any distinct $a, b \in M'$. Let $\omega(G)$ denote the maximum clique size of G. In the terminology of graph theory, M is almost difference closed if and only if $\omega(G(Z, M)) \geq |M|$, and M is difference closed if and only if $\omega(G(Z, M)) = |M| + 1$. Thus, the study of $\mu(M)$ for almost difference closed sets M is the same as the study of $\chi_f(G(Z, M))$ with $\omega(G(Z, M)) \geq |M|$. Many of the sets M for which the values of $\mu(M)$ have been investigated in the past are almost difference closed. For instance, if $|M| \leq 2$ or $M = \{a, b, a+b\}$, then M is almost difference closed.

In this paper, we determine the values of $\mu(M)$ for most of the almost difference closed sets M. There is only one exceptional case, $M = \{x, y, y - x, y + x\}$ where $y > x$, and both x and y are odd, for which we prove a lower bound and an upper bound of $\mu(M)$. These bounds are sharp enough to determine the chromatic number for such graphs.

Our results confirm a conjecture on the value of $\mu(M)$ for $M = \{a, b, a+b\}$, proposed by Rabinowitz and Proulx [21] in 1985, and generalize many previously known theorems. Moreover, results on $\mu(M)$ are used to investigate (or compare with) other parameters, including the chromatic number and the circular chromatic number of distance graphs generated by almost difference closed sets, and the parameter $\kappa(M)$ involved in the long standing open conjecture of Wills [25].

For a real number x, let $||x||$ denote the distance from x to the nearest integer, i.e., $||x|| = \min\{x - \lfloor x \rfloor, \ \lceil x \rceil - x\}$. For a real number t and a set of real numbers X, let $||tX|| = \inf\{||tx|| : x \in X\}$, and

$$\kappa(X) = \sup\{||tX|| : t \in R\}.$$

It was proved in [2] that for any set M, it is always true that

$$\kappa(M) \leq \mu(M). \tag{1.1}$$

We completely determine the values of $\kappa(M)$ for almost difference closed sets. The results imply the existence of an infinite family of four-element sets M with $\kappa(M) < \mu(M)$. It is known that $\kappa(M) = \mu(M)$ holds for all M with $|M| \leq 2$, and it is open whether there exists any three-element set M such that $\kappa(M) < \mu(M)$.

Let $k \geq d$ be positive integers. A (k, d)-coloring of a graph G is a mapping, $c : V(G) \to \{0, 1, \cdots, k-1\}$, such that $d \leq |c(u) - c(v)| \leq k - d$ for any $uv \in E(G)$. The *circular chromatic number* of G, $\chi_c(G)$, is the minimum ratio k/d such that G admits a (k, d)-coloring. It is known [26] that for any graph G,

$$\chi_f(G) \leq \chi_c(G) \leq \chi(G) = \lceil \chi_c(G) \rceil.$$

On the other hand, for any M, it is known [26] that $\chi_c(G(Z, M)) \leq 1/\kappa(M)$. In the last section, we determine the values of $\chi_c(G(Z, M))$ for all almost difference closed sets M, except for the case that $M = \{x, y, y - x, y + x\}$, where $y > x$, and x and y are odd.

A characterization of almost difference closed sets was given by Kemnitz and Marangio [18].

Theorem 1.3 [18] *Suppose M is a finite set of positive integers with $|M| = m$ and* $\gcd(M) = 1$*. Then M is almost difference closed if and only if M is one of the following:*

A.1) $M = \{a, 2a, 3a, \cdots, (m-1)a, b\}$.

A.2) $M = \{a, b, a + b\}$ *for some* $b \neq 2a$.

A.3) $M = \{x, y, y - x, y + x\}$ *for some* $y \neq 2x$.

Hence, almost difference closed sets are partitioned into three types, A.1, A.2 and A.3. Treatments for these types are different. In the next three sections, we investigate the values of $\mu(M)$ for Types A.1, A.2 and A.3, respectively.

By (1.1), a method of getting lower bounds for $\mu(M)$ is to determine or estimate $\kappa(M)$. Moreover, the inequalities (1.2) below are also useful for bounding $\mu(M)$.

$$1/(|M|+1) \leq \mu(M) \leq 1/|M|. \quad (1.2)$$

Note that (1.2) follows from the fact that $\omega(G) \leq \chi_f(G) \leq \chi(G)$ for any graph G, and the result $\chi(G(Z,M)) \leq |M|+1$ for any M [24, 8]. Another useful tool for establishing upper bounds for $\mu(M)$ is the following lemma proved in [16]:

Lemma 1.4 [16] *Let M be a set of positive integers, α a real number in the interval $(0,1]$. If $\mu(M) \geq \alpha$ then there is an M-set S such that for any $n \geq 0$, $S(n) \geq \alpha(n+1)$. In particular, $S(0) \geq \alpha$ implies that $0 \in S$.*

2 Almost difference closed sets of Type A.1

This type turns out to be the easiest one. We completely determine the values of $\mu(M)$ and $\kappa(M)$, which are always equal for this type.

For a set A and an integer a, denote $A + a = \{x + a : x \in A\}$.

Theorem 2.1 *Suppose $M = \{a, 2a, \cdots, (m-1)a, b\}$, where $\gcd(a,b) = 1$. If $a = 1$, then*

$$\mu(M) = \kappa(M) = \begin{cases} \frac{1}{m}, & \text{if } b \text{ is not a multiple of } m; \\ \frac{k}{km+1}, & \text{if } b = km \text{ for some } k. \end{cases}$$

If $a \geq 2$, then $\mu(M) = \kappa(M) = 1/m$.

Proof. Assume $a = 1$. If b is not a multiple of m, let $t = 1/m$. Then $||tM|| = 1/m$, so $\mu(M) \geq \kappa(M) \geq 1/m$. By (1.1) and (1.2), the equalities hold.

Now assume $M = \{1, 2, 3, \cdots, m-1, mk\}$ for some integer $k \geq 1$. Let $t = k/(mk+1)$. Then $||Mt|| = k/(mk+1)$, so $\mu(M) \geq \kappa(M) \geq k/(mk+1)$. To prove that $\mu(M) \leq k/(mk+1)$, by Lemma 1.4, it suffices to show that for any M-set S with $0 \in S$, we have $S(mk) \leq k$. As $0 \in S$, one has $b = mk \notin S$. Partition the set $\{0, 1, 2, \cdots, mk-1\}$ into

$$X_i = \{0, 1, \cdots, m-1\} + im, \text{ for } i = 0, 1, \cdots, k-1.$$

Obviously $|S \cap X_i| \leq 1$, so $S(mk) \leq k$. Therefore $\mu(M) \leq k/(mk+1)$.

Assume $a \geq 2$. By (1.1) and (1.2), it suffices to prove that $\kappa(M) \geq 1/m$, i.e., to show that there exists t such that $||Mt|| \geq 1/m$. Let $b = q(am) + r$, $0 \leq r < am$. Then $r > 0$, since $\gcd(a, b) = 1$. If $a \leq r \leq (m-1)a$, let $t = 1/(am)$. It is easy to see that $||Mt|| = 1/m$.

Assume $1 \leq r \leq a-1$. Let l be the smallest integer with $rl \geq a$. Then $2 \leq l \leq a$ and $a \leq rl \leq 2a-1$. Moreover, there exists some $k \in \{0, 1, 2, \cdots, m-3\}$ such that $l + k \equiv \pm 1 \pmod{m}$, so $a(l+k) \equiv \pm a \pmod{am}$. Take $t = (l+k)/(am)$. Then $||bt|| = ||r(l+k)/(am)|| \geq 1/m$, since

$$\begin{aligned} a \leq (l+k)r \leq (l+m-3)r &= (l-1)r + (m-2)r \\ &< a + (m-2)a = (m-1)a. \end{aligned}$$

Because $a(l+k) \equiv \pm a \pmod{am}$, one has, for any $j = 1, 2, \cdots, m-1$, $(l+k)(ja) = \pm ja \pmod{am}$, so $||t(ja)|| \geq 1/m$. Hence, $\kappa(M) \geq ||tM|| \geq 1/m$.

The proof for the case that $(m-1)a + 1 \leq r \leq am - 1$ is similar, we omit the details. ∎

3 Almost difference closed sets of Type A.2

Let T be a set of non-negative integers with $0 \in T$. A *T-coloring* of a graph G, with span k, is a function $f : V(G) \to \{0, 1, 2, 3 \cdots, k\}$ such that $|f(u) - f(v)| \notin T$ if $u \sim v$. For a given set T, denote $\delta_n(T)$ by the minimum span of a T-coloring for K_n. The *asymptotic T-coloring ratio* is defined as

$$rt(T) = \lim_{n \to \infty} \delta_n(T)/n.$$

The parameter $rt(T)$ has been studied by Rabinowitz and Proulx [21] and by Griggs and Liu [14]. It was noted in [14] that $\mu(M) = rt(T)$, if $T = M \cup \{0\}$.

Some known results used in this section are originally given in terms of $rt(T)$. Due to the equivalence of the two parameters, we shall consistently use $\mu(M)$.

Suppose $M = \{a, b, a+b\}$. If none of a, b or $a+b$ is a multiple of 3, then it is easy to prove that $\kappa(M) = \mu(M) = 1/3$ ([21]) (all multiples of 3 form an M-set). If $a = 1$, i.e. $M = \{1, b, b+1\}$, then the value of $\mu(M)$ was determined in [16].

For other sets of Type A.2, a lower bound for $\mu(M)$ was proved in [21] and the authors conjectured that the lower bound is sharp:

Conjecture 1 [21] *Suppose* $M = \{a, b, a+b\}$*,* $\gcd(a, b) = 1$*, and one of* $a, b, a+b$ *is a multiple of* 3*. Then*

$$\mu(M) = \max\{\frac{\lfloor (2b+a)/3 \rfloor}{2b+a}, \frac{\lfloor (2a+b)/3 \rfloor}{2a+b}\}.$$

We prove this conjecture, and hence determine the values of $\mu(M)$ for all Type A.2 sets. To confirm Conjecture 1, by the lower bound shown in [21], it suffices to prove

$$\mu(M) \leq \max\{\frac{\lfloor (2b+a)/3 \rfloor}{2b+a}, \frac{\lfloor (2a+b)/3 \rfloor}{2a+b}\}.$$

The inequality above is established by the following theorem.

Theorem 3.1 *Suppose* $M = \{a, b, a+b\}$*, where* $0 < a < b$ *and* $\gcd(a, b) = 1$*. Then*

$$\mu(M) \leq \begin{cases} 1/3, & \text{if } b-a = 3k; \\ (a+k)/(3a+3k+1), & \text{if } b-a = 3k+1; \\ (a+2k+1)/(3a+6k+4), & \text{if } b-a = 3k+2. \end{cases}$$

Proof. Let $c = a + b$. The result for the case $b - a = 3k$ was proved in [21].

Case 1: $b - a = 3k + 1$

Suppose S is an M-set, $0 \in S$. By Lemma 1.4, it suffices to show that there exists an integer n satisfying the inequality

$$S(n) \leq \frac{a+k}{3a+3k+1}(n+1). \tag{3.1}$$

Consider $S(c+a-1)$. Note that $a, b, c \notin S$, since $0 \in S$. For $i = 1, 2, \cdots, a-1$, define the *Triangle-i* by

$$T_i = \{i, i+b, i+c\}.$$

Since the difference of any two elements of T_i lies in M, one has $|S \cap T_i| \leq 1$. Set

$$I = \{1 \leq i \leq a-1 : S \cap T_i = \emptyset\},$$
$$T = S \cap \{a+1, a+2, \cdots, b-1\}, \quad \text{and} \quad t = |T|.$$

Hence $S(c+a-1) = t+a-|I|$. If $t \leq k$, then $S(c+a-1) \leq a+k$, implying (3.1), as $n = c+a-1$.

Assume $t \geq k+1$. We prove that (3.1) holds when $n = b+c-1$. Note that $b+c-1 = 3a+6k+1$. Since $t \geq k+1$, we have

$$\frac{a+k}{3a+3k+1}(3a+6k+2) \geq a+2k \geq a+3k+1-t.$$

Hence, it suffices to show the following

$$S(b+c-1) \leq a+3k+1-t. \tag{3.2}$$

Observe that $S(b+c-1) = S(c+a-1) + |S \cap \{c+a, c+a+1, \cdots, c+b-1\}|$. Let $U = \{c+a, c+a+1, \cdots, c+b-1\} - S$. Then

$$S(b+c-1) = a+t-|I|+(b-a)-|U|.$$

Note that $a+3k+1-t = a+t+(b-a)-2t$. Hence, to prove (3.2), it amounts to show that $|I|+|U| \geq 2t$. That is, to prove that, for each $i \in T$, there exist two distinct elements, i_1, $i_2 \in I \cup U$, such that $\{i_1, i_2\}$ and $\{j_1, j_2\}$ are disjoint whenever $i \neq j$.

For any $i \in T$, let $i_1 = i+c$, then $i_1 \in U$.

Let $m \geq 0$ be the smallest integer such that $S \cap T_{i-a+(m-1)(b-a)} = \emptyset$ or $i+b+m(b-a) \geq a+c$. If $S \cap T_{i-a+(m-1)(b-a)} = \emptyset$, let $i_2 = i-a+(m-1)(b-a)$.

Otherwise, let $i_2 = i + b + m(b-a)$. Note that, if it is the former case, then $m \geq 1$ and $i_2 \in I$.

We now show that if it is the latter case, then $i_2 \in U$. Assume $i_2 = i + b + m(b-a)$. Then $i + b + (m-1)(b-a) < a + c \leq i + b + m(b-a)$, which implies

$$a + c \leq i + b + m(b-a) = i_2 < b + c. \tag{$*$}$$

We claim that for $0 \leq m' \leq m$,

$$i + m'(b-a) \in S. \tag{$**$}$$

If $m' = 0$, then (**) follows from our assumption that $i \in T$. Assume $0 \leq m' \leq m-1$ and $i + m'(b-a) \in S$. Then $i - a + m'(b-a) \notin S$ and $i + m'(b-a) + b = i - a + m'(b-a) + c \notin S$. Consider $T_{i-a+m'(b-a)} = \{i - a + m'(b-a), i - a + m'(b-a) + b, i - a + m'(b-a) + c\}$. Note, $T_{i-a+m'(b-a)}$ is well-defined since, by definition, $i + b + m'(b-a) < a + c$, so $i - a + m'(b-a) < a$. By the definition of i_2, we have $T_{i-a+m'(b-a)} \cap S \neq \emptyset$. Hence, it must be that $i - a + m'(b-a) + b = i + (m'+1)(b-a) \in S$, so (**) holds. In particular, we have $i + m(b-a) \in S$, which implies that $i_2 = i + b + m(b-a) \notin S$. Combining this with (*), we have $i_2 \in U$.

It remains to show that for any $i \in T$, $i_1 \neq i_2$, and for any $i \neq j \in T$, $\{i_1, i_2\} \cap \{j_1, j_2\} = \emptyset$. Assume to the contrary that $i_p = j_q$, where $p, q \in \{1, 2\}$ and $(i, p) \neq (j, q)$. If $i_p = j_q \in I$, then by definition, $p = q = 2$, and $i_p = i - a + m(b-a) = j_q = j - a + m'(b-a)$ for some $m, m' \geq 0$. As $(i, p) \neq (j, q)$ and $p = q = 2$, we have $i \neq j$ and hence $m \neq m'$. Assume $m' > m$. Then $i = j + (m' - m)(b-a) \geq a + 1 + b - a = b + 1$ (as $j \geq a+1$), contradicting the assumption that $i \in T \subseteq \{a+1, a+2, \cdots, b-1\}$.

Assume $i_p = j_q \in U$. Then $i_p \in \{i+c, i+b+m(b-a)\}$ and $j_q \in \{j+c, j+b+m'(b-a)\}$ for some $m, m' \geq 0$. By the same argument in the above, we cannot have both $i_p = i+b+m(b-a)$ and $j_q = j+b+m'(b-a)$. Moreover, as $(i, p) \neq (j, q)$, it is impossible that $i_p = i+c = j_q = j+c$. This leaves the only possibility (by symmetry) that $i_p = i + c = i + b + a = j_q = j + b + m'(b-a)$, which implies that $i + a = j + m'(b-a)$. This is again impossible, as by (**), $j + m'(b-a) \in S$, and by the assumption that $i \in S$.

Case 2: $b - a = 3k + 2$

The proof of this case is similar to the one of Case 1. Suppose S is an M-set with $0 \in S$. By Lemma 1.4, we shall prove that there exists an integer n satisfying the inequality

$$S(n) \leq \frac{a+2k+1}{3a+6k+4}(n+1). \tag{3.3}$$

Consider $S(c+a-1)$. Similar to Case 1, set

$$T_i = \{i, i+b, i+c\}, \text{ for } i = 1, 2, \cdots, a-1,$$

$$I = \{1 \leq i \leq a-1 : S \cap T_i = \emptyset\},$$

$$T = S \cap \{a+1, a+2, \cdots, b-1\}, \quad \text{and} \quad t = |T|.$$

Hence $S(c+a-1) = t + a - |I|$. If $t \leq k$, then $S(c+a-1) \leq a+k$. This implies that (3.3) holds when $n = c+a-1 = 3k+3a+1$, because

$$a+k \leq \frac{a+2k+1}{3a+6k+4}(3a+3k+2).$$

Assume $t \geq k+1$. By the same argument used in Case 1 (the one about assigning i_1 and i_2 to each $i \in T$), one can show that

$$S(b+c-1) = S(3a+6k+3) \leq a+t+(b-a)-2t \leq a+2k+1.$$

Hence (3.3) is satisfied when $n = b+c-1 = 3a+6k+3$. ∎

4 Almost difference closed sets of Type A.3

This type turns out to be the most complicated family of almost difference closed sets, as far as the values of $\mu(M)$ are concerned. The case that x and y are of distinct parity is easy. It was proved by Kemnitz and Kolberg [17] that for this case, $\chi(G(Z,M)) = 4$. Hence, $\chi(G(Z,M)) = \omega(G(Z,M)) = 4$, so we have

Theorem 4.1 *If $M = \{x, y, y-x, x+y\}$, $y > x$, and x, y are of distinct parity, then $\mu(M) = 1/4$.*

Because $\gcd(x, y) = 1$, for Type A.3 sets, it remains to consider the case that x, y are both odd. Note that for this case, it was proved by Kemnitz and Marangio [19] that $\chi(G(Z, M)) = 5$, however, this result does not provide further information to the value of $\mu(M)$ other than the bound $\mu(M) \geq \frac{1}{\chi(G)} = 1/5$, which can also be obtained directly from (1.2). Below, we prove a better lower bound for $\mu(M)$.

Lemma 4.2 *Suppose* $M = \{x, y, y - x, y + x\}$, *where* $y > x$, $x = 2k + 1$, $y = 2m + 1$ *and* $\gcd(x, y) = 1$. *Then* $\mu(M) \geq \frac{(k+1)m}{4(k+1)m+1}$.

Proof. Let $n = xy + y - x = 4(k + 1)m + 1$. Set

$$I = \{0, 2x, 4x, \cdots, (m - 1)2x\}; \quad Y = \cup_{i=0}^{k}\{I + 2iy \pmod{n}\};$$

$$S = \cup_{k=0}^{\infty}(Y + kn).$$

We shall prove that S is an M-set with density $\frac{(k+1)m}{n}$. Note that S is "periodic" with period $n = xy + y - x$. Thus, to show that S has density $\frac{(k+1)m}{n}$, it suffices to verify that $|S \cap \{0, 1, \cdots, n - 1\}| = (k + 1)m$.

To prove this, it suffices to show that for $i \neq i'$, one has $(I + 2iy) \cap (I + 2i'y) = \emptyset$. Suppose to the contrary, there exist $i \neq i'$ and $j \neq j'$, $0 \leq i \leq i' \leq k$, $0 \leq j, j' \leq m - 1$, such that one of the following holds:

$$2jx + 2iy = 2j'x + 2i'y, \quad \text{or} \quad 2jx + 2iy = 2j'x + 2i'y + xy + y - x.$$

This implies that $(j - j')x = (i' - i)y$ or $(2(j - j') + 1)x = 2(i' - i - 1)y + xy$. Neither one is possible, since $2|j - j'| < y - 1$ and $\gcd(x, y) = 1$.

We now show that S is an M-set. Let

$$u = 2tx + 2iy, \ v = 2t'x + 2i'y, \ \text{for some } 0 \leq t, t' \leq m - 1, \ 0 \leq i, i' \leq k.$$

Then

$$\begin{aligned} |u - v| &= |2(t - t')x + 2(i - i')y| \\ &\leq (y - 3)x + (x - 1)y \\ &= 2xy - 3x - y < 2n - (x + y). \end{aligned}$$

Therefore, it suffices to show that

$$|u - v| \notin \{x, y, (y - x), (y + x), n \pm x, n \pm y, n \pm (y - x), n \pm (y + x)\}.$$

Note that $|u-v|$ is even, so it remains to show that

$$|u-v| \notin \{y-x, y+x, n \pm x, n \pm y\}.$$

By definition of u and v, we have

$$|2(t-t')| \leq 2(m-1) = y-3, \tag{4.1}$$

$$|2(i-i')| \leq 2k = x-1. \tag{4.2}$$

Without loss of generality, we assume that $u-v = 2(t-t')x + 2(i-i')y > 0$.

If $2(t-t')x+2(i-i')y = y \pm x$, then $(2t-2t' \mp 1)x + (2i-2i'-1)y = 0$, which is impossible, because $0 < |2t-2t' \pm 1| \leq y-2$ (by (4.1)), and $\gcd(x,y) = 1$.

If $2(t-t')x+2(i-i')y = n+x = xy+y$, then $(2t-2t')x = (2i'-2i+x+1)y$. This is impossible, because $|2i'-2i+x+1| > 0$ (by (4.2)), $|2t-2t'| < y$ (by (4.1)), and $\gcd(x,y) = 1$.

If $2(t-t')x + 2(i-i')y = n-x = xy+y-2x$, then $2(t-t'+1)x = (2i'-2i+x+1)y$. Again this is impossible.

Similarly, one can show that $2(t-t')x + 2(i-i')y \neq n \pm y$. ∎

We conjecture that the equality in Lemma 4.2 always holds.

Conjecture 2 *If* $M = \{x, y, y-x, y+x\}$, *where* $y > x$, $x = 2k+1$, $y = 2m+1$ *and* $\gcd(x,y) = 1$, *then* $\mu(M) = \frac{(k+1)m}{4(k+1)m+1}$.

The following result confirms Conjecture 2, for the case that $x = 1$.

Theorem 4.3 *If* $M = \{1, 2m, 2m+1, 2m+2\}$ *for some* $m > 1$, *then* $\mu(M) = \kappa(M) = m/(4m+1)$.

Proof. By Lemma 4.2, it suffices to show that $\mu(M) \leq m/(4m+1)$. Let S be an M-set. By Lemma 1.4, it suffices to show that if $0 \in S$, then $S(4m) \leq m$. Partition the set of integers $\{0, 1, \cdots, 4m\} - \{2m\}$ into

$$R_i = \{i, i+1, i+2m+1, i+2m+2\}, i = 0, 2, \cdots, 2m-2.$$

Then $|S \cap R_i| \leq 1$. Furthermore, since $0 \in S$, we have $2m \notin S$, implying that $S(4m) \leq m$. ∎

Let $\beta = \frac{1}{(k+1)m}$. Lemma 4.2 asserts that $\mu(M) \leq \frac{1}{4+\beta}$. In the following, we establish an upper bound for $\mu(M)$, in a similar format

Lemma 4.4 *Suppose $M = \{x, y, y-x, y+x\}$, where $y > x$, $x = 2k+1$, $y = 2m+1$ and $\gcd(x,y) = 1$. Let $\delta = \frac{1}{k^2+2km+3k+m+1}$. Then $\mu(M) \leq \frac{1}{4+\delta}$.*

Proof. Let S be an M-set of maximum density. By some result in [14] and [2], we may assume that S is periodic, i.e., there is an integer n such that for any $k \geq 0$, $S \cap \{kn, kn+1, \cdots, (k+1)n-1\} = B + kn$, for some subset B of $\{0, 1, \cdots, n-1\}$.

In the remaining of the proof, we regard B as a subset of Z_n. All calculations are carried out in the group Z_n (i.e., modulo n). For instance, for each $i \in Z_n$, $B + i = \{b + i : b \in B\} \subset Z_n$ ($b + i$ is carried out in Z_n). Let $|B| = q$. Then $|B+i| = q$ for any i. The density of S is q/n. Let $z = n - 4q$. It suffices to show that $z \geq \delta q = \frac{q}{k^2+2km+3k+m+1}$.

Because S is an M-set, for any $i, j \in B$, one has $i - j \pmod{n} \notin M$. Hence, $(B+i) \cap (B+j) = \emptyset$, whenever $|i-j| \in M$.

Claim 1 $$|(B \cup (B+y)) \cap (B + (xy+x))| \geq q - (m+1)z. \tag{4.3}$$

Proof of Claim 1: Let

$$A = B \cup (B+y) \cup (B+x) \cup (B+(x+y)) \cup (B+2x) \cup (B+(2x+y)),$$

$$C = (B \cup (B+y)) - ((B+2x) \cup (B+(2x+y))).$$

We first show that $|C| \leq z$. Note that $B, B+y, B+x, B+(x+y)$ are pairwise disjoint, and $B+x, B+2x, B+(x+y), B+(2x+y)$ are pairwise disjoint. Hence

$$|C| = |A - ((B+x) \cup (B+(x+y)) \cup (B+2x) \cup (B+(2x+y)))| \leq |A| - 4q \leq z.$$

Next, we show that $|(B \cup (B+y)) - ((B+(xy+x)) \cup (B+(xy+x+y))| \leq (m+1)z$. Observe that $2(m+1)x = xy + x$. So

$$|(B \cup (B+y)) - ((B+(xy+x)) \cup (B+(xy+x+y))|$$

$$
\begin{aligned}
&= |(B \cup (B+y)) - ((B+2(m+1)x) \cup (B+2(m+1)x+y))| \\
&\le \sum_{j=0}^{m} |((B+2jx) \cup (B+2jx+y)) - ((B+2(j+1)x) \cup (B+2(j+1)x+y))| \\
&= \sum_{j=0}^{m} |C+2jx| \le (m+1)z.
\end{aligned}
$$

As $|(B \cup (B+y))| = |((B+(xy+x)) \cup (B+(xy+x+y))| = 2q$, we conclude

$$|(B \cup (B+y)) \cap ((B+(xy+x)) \cup (B+(xy+x+y))| \ge 2q-(m+1)z. \quad (4.4)$$

This implies (4.3), since $|B+(xy+x+y)| = q$.

Claim 2 $\quad |(B+(x+y)) \cap (B+(xy+x))| \ge q-k(k+2m+3)z. \quad (4.5)$

Proof of Claim 2: Let

$$A' = B \cup (B+x) \cup (B+y) \cup (B+(x+y)) \cup (B+2y) \cup (B+(x+2y)),$$

$$C' = (B \cup (B+x)) - ((B+2y) \cup (B+(x+2y))).$$

Similarly as in the proof of Claim 1, we have $|C'| = |A'| - 4q \le z$, and the following: (Note that $2(k+1)y = xy+y$.)

$$|(B \cup (B+x)) - ((B+(xy+y)) \cup (B+(xy+x+y))| \le (k+1)z.$$

This implies that

$$|(B \cup (B+x)) \cap ((B+(xy+y)) \cup (B+(xy+x+y))| \ge 2q-(k+1)z.$$

Hence

$$|(B \cup (B+x)) \cap (B+(xy+x+y))| \ge q-(k+1)z. \quad (4.6)$$

It follows from (4.4) that

$$|(B \cup (B+y)) \cap (B+(xy+x+y))| \ge q-(m+1)z. \quad (4.7)$$

As $B, (B+x), (B+y)$ are pairwise disjoint, we have

$$(B+(xy+x+y)) \cap (B+x) \subseteq (B+(xy+x+y)) - (B \cup (B+y)).$$

Hence, by (4.7), $|(B+(xy+x+y)) \cap (B+x)| \leq (m+1)z$. Combining this with (4.6), we conclude that

$$|B \cap (B+(xy+x+y))| \geq q-(m+k+2)z. \tag{4.8}$$

By adding y to the involved sets of (4.4), we have

$$|((B+y)\cup(B+2y))\cap((B+(xy+x+y))\cup(B+(xy+x+2y))| \geq 2q-(m+1)z.$$

This implies that $|(B+(xy+x+y))-((B+y)\cup(B+2y))| \leq (m+1)z$, and hence

$$|((B+(xy+x+y))\cap B)-((B+y)\cup(B+2y))| \leq (m+1)z.$$

Because $B \cap (B+y) = \emptyset$, we have

$$|((B+(xy+x+y))\cap B)-(B+2y)| \leq (m+1)z. \tag{4.9}$$

Combining (4.8) and (4.9), one has

$$|(B+(xy+x+y))\cap B \cap (B+2y)| \geq q-(k+2m+3)z.$$

Hence, $|B \cap (B+2y)| \geq q-(k+2m+3)z$. This implies that

$$|B-(B+2y)| \leq (k+2m+3)z.$$

Note that $B+2ky = B+(xy-y)$. So

$$\begin{aligned} |B-(B+xy-y)| &= |B-(B+2ky)| \\ &\leq \sum_{j=0}^{k-1} |(B+2jy)-(B+2(j+1)y)| \\ &= \sum_{j=0}^{k-1} |(B-(B+2y))| \leq k(k+2m+3)z. \end{aligned}$$

This implies that

$$|B \cap (B+(xy-y))| \geq q-k(k+2m+3)z. \tag{4.10}$$

Hence, (4.5) is established by adding $x + y$ to the sets in (4.10). This completes the proof of Claim 2. $\square$

Because $B, B + y, B + (x + y)$ are pairwise disjoint, we conclude that

$$\begin{aligned} q &= |B + xy + x| \\ &\geq |(B \cup (B + y)) \cap (B + (xy + x))| + |(B + (x + y)) \cap (B + (xy + x))| \\ &\geq 2q - (k(k + 2m + 3) + (m + 1))z. \end{aligned}$$

Hence $z \geq \frac{q}{k^2+2km+3k+m+1}$. This completes the proof. ∎

5 Consequences and related problems

In this section, we determine the values of $\chi_c(G(Z, M))$ for all almost difference closed sets M, except for the sub-case of Type A.3, $M = \{x, y, y-x, y+x\}$, where x and y are odd. For the exceptional case, the bounds for $\mu(M)$ proved in the previous section are sharp enough to determine the chromatic number. In addition, we completely determine the values of $\kappa(M)$ for all almost difference closed sets.

We begin with the Type A.1 sets, $M = \{a, 2a, \cdots, (m - 1)a, b\}$ with $\gcd(a, b) = 1$. The following is a corollary of Theorem 2.1, where the $a = 1$ case was first proved in [3] and the $a \geq 2$ case was first proved in [18].

Throughout the section, we denote $G(Z, M)$ by G.

Corollary 5.1 *Suppose* $M = \{a, 2a, \cdots, (m - 1)a, b\}$, $\gcd(a, b) = 1$. *Then*

$$\chi_f(G) = \chi_c(G) = \frac{1}{\kappa(M)} = \begin{cases} m, & \text{if } a = 1 \text{ and } m \not| \, b, \text{ or if } a \geq 2; \\ m + \frac{1}{k}, & \text{if } a = 1 \text{ and } b = km \text{ for some } k. \end{cases}$$

For Type A.2 sets, the values of $\kappa(M)$ were determined by Y. Chen [6]; and the chromatic numbers were determined by J. Chen et al. [8] and by Voigt [22]. Partial results on the circular chromatic number and the fractional chromatic number were obtained by Zhu [27].

Using Theorem 3.1, we are able to completely determine $\chi_f(G)$ and $\chi_c(G)$ for Type A.2 sets. Combining Theorem 3.1 with the values of $\kappa(M)$ obtained in [6], we conclude:

Theorem 5.2 *Suppose* $M = \{a, b, a+b\}$*,* $0 < a < b$*,* $\gcd(a,b) = 1$*. Then*

$$\chi_f(G) = \chi_c(G) = 1/\kappa(M) = \begin{cases} 3, & \text{if } b-a = 3k; \\ 3 + \frac{1}{a+k}, & \text{if } b-a = 3k+1; \\ 3 + \frac{1}{b-k-1}, & \text{if } b-a = 3k+2. \end{cases}$$

The rest of this section deals with Type A.3 sets. Note that by using Theorem 4.1, Lemma 4.4, and (1.2), we can determine the chromatic number for this type of sets M, which was obtained in [17] and [18] by different approaches.

Corollary 5.3 *Suppose* $M = \{x, y, y-x, x+y\}$*, where* $\gcd(x,y) = 1$*. If* x, y *are of distinct parity, then* $\chi_f(G) = \chi_c(G) = \chi(G) = 4$*. If* x *and* y *are both odd, then* $\chi(G) = 5$*.*

As remarked earlier, $\mu(M) \geq \kappa(M)$ for any M. The question whether the equality always holds was first raised in [2], and then discussed in [16]. An infinite family of sets M for which $\mu(M) > \kappa(M)$ was given in [16].

Our results indicate that $\mu(M) = \kappa(M)$ for Types A.1 and A.2 sets. For Type A.3 sets, however, as shown in the next few results, the equality does not always hold.

If $M = \{x, y, y-x, x+y\}$, where x, y are of distinct parity and none of x, y is a multiple of 4, then $||\frac{1}{4}M|| = 1/4$, so $\kappa(M) = 1/4$. Hence by Theorem 4.1, $\mu(M) = \kappa(M)$. In what follows, we give the $\kappa(M)$ values for the remaining Type A.3 sets.

It is known and not hard to show (cf. [16]) that $\kappa(M)$ is a fraction whose denominator always divides the sum of some pair of elements in M. Indeed, suppose $\kappa(M) = ||tM|| = p/q$, then there exist $a, b \in M$ such that $at = k_1 + p/q$ and $bt = k_2 - p/q$, for some integers k_1 and k_2 (otherwise, one may increase or decrease t by a small amount so that $||tM||$ increases). This implies that $t = (k_1 + k_2)/(a+b)$, and hence $q|(a+b)$.

Lemma 5.4 *Suppose* $M = \{x, y, y-x, y+x\}$*, where* $\gcd(x,y) = 1$*. If one of* x *and* y *is a multiple of* 4 *and the other is odd, or if both* x *and* y *are odd, then* $\kappa(M) < 1/4 = \mu(M)$*.*

Proof. By Theorem 4.1, it suffices to show that $\kappa(M) \neq 1/4$. Assume $\kappa(M) = ||tM|| = 1/4$. Then $t = k/(a+b)$ for some $a, b \in M$ and $a+b$ is a multiple of 4 (by the explanation above). For the sets M considered in this theorem, there are only two cases that M contains such a, b: 1) $\{a, b\} = \{x, y-x\}$ where y is a multiple of 4 and x is odd; 2) $\{a, b\} = \{x, y\}$ where one of x, y is $\equiv 1 \pmod 4$ and the other is $\equiv 3 \pmod 4$. Neither one of these is possible, since 1) implies $||ty|| = 0$; and 2) implies $||t(x+y)|| = 0$. ∎

Let $\phi_4(n)$ denote $\lfloor \frac{n}{4} \rfloor / n$.

Lemma 5.5 *Suppose* $M = \{x, y, y-x, y+x\}$*, where* $\gcd(x, y) = 1$*. Then* $\kappa(M) \geq \phi_4(a+b)$*, where* $\{a, b\}$ *is any one of the following 2-element subsets of* M*:* $\{x+y, y\}, \{x+y, x\}, \{y-x, y\}$*.*

Proof. Because $\gcd(x, y) = 1$, so $\gcd(a, b) = \gcd(a, a+b) = 1$. Hence, there exists an integer k such that $ak \equiv (a+b)\phi_4(a+b) \pmod{(a+b)}$. Let $t = \frac{k}{a+b}$. Then $||ta|| = \phi_4(a+b)$, $||tb|| = ||-ta|| = \phi_4(a+b)$. The other two elements of M belong to the set $X = \{\pm(a-b), \pm(2a-b)\} \pmod{(a+b)}$. It is straightforward to verify that $||tx|| \geq \phi_4(a+b)$ holds for each $x \in X$. Therefore, $\kappa(M) \geq ||tM|| = \phi_4(a+b)$. ∎

Corollary 5.6 *Suppose* $M = \{x, y, y-x, y+x\}$*, where* $\gcd(x, y) = 1$*. Then*

$$\kappa(M) = \begin{cases} \phi_4(2y+x), & \text{if } x \equiv 0 \pmod 4 \text{ and } y \equiv 3 \pmod 4, \text{ or} \\ & x \equiv 1 \pmod 4 \text{ and } y \equiv 0 \pmod 4, \text{ or} \\ & x \equiv 3 \pmod 4 \text{ and } y \equiv 1, 3 \pmod 4; \\ \phi_4(2x+y), & \text{if } x \equiv 0 \pmod 4 \text{ and } y \equiv 1 \pmod 4, \text{ or} \\ & x \equiv 1 \pmod 4, \ y \equiv 3 \pmod 4, \text{ and } y < 3x; \\ \phi_4(2y-x), & \text{if } x \equiv 3 \pmod 4 \text{ and } y \equiv 0 \pmod 4, \text{ or} \\ & x \equiv y \equiv 1 \pmod 4, \text{ or} \\ & x \equiv 1 \pmod 4, \ y \equiv 3 \pmod 4, \text{ and } y \geq 3x. \end{cases}$$

Proof. Denote by $\beta(M)$ the corresponding value on the right-hand-side of the above equality. It is straightforward to verify that

$$\beta(M) = \max\{p/q : p/q < 1/4, \text{ and } q \text{ divides the sum of two elements of } M\}.$$

By Lemma 5.4, and the discussion preceding it, we have $\kappa(M) \leq \beta(M)$. By Lemma 5.5, $\kappa(M) \geq \beta(M)$. So the equality holds. ∎

By Theorem 4.1, Lemma 4.2 and Corollary 5.6, we conclude that

Theorem 5.7 *Let $M = \{x, y, y-x, y+x\}$, $\gcd(M) = 1$. Then $\mu(M) = \kappa(M)$, when $x = 1$ and y is odd, or when x and y are of different parities and none of them is a multiple of 4; otherwise, $\kappa(M) < \mu(M)$.*

Besides almost difference closed sets, the values of $\mu(M)$ and $\kappa(M)$ are known only for very few sets M. In particular, the following remains open:

Question Is it true that $\mu(M) = \kappa(M)$ for all M with $|M| = 3$?

A long standing open question concerning $\kappa(M)$ is the following conjecture due to Wills [25].

Conjecture 3 *Suppose M is a finite set of positive integers with $|M| = m$. Then $\kappa(M) \geq 1/(m+1)$.*

Conjecture 3 is also known as the "lonely runner conjecture" by Bienia et al. [1] due to the interpretation: Suppose m runners run laps on a circular track of unit length. Each runner maintains a distinct constant speed. A runner is called lonely if the distance (on the circular track) between him (or her) and every other runner is at least $1/m$. The conjecture is equivalent to assert that for each runner, there is a time when (s)he is lonely.

Although Conjecture 3 has attracted considerable attention, it remains open for $m \geq 5$. For $m \leq 4$, the conjecture is confirmed [1, 5, 7, 9].

If Conjecture 3 is true, then the bound $1/(m+1)$ is sharp for difference closed sets. Only very few other sets M attaining this bound are known. (This is another motivation we consider almost difference closed sets). As an analogy to Conjecture 3, we have $\mu(M) \geq 1/(m+1)$ (cf. (1.2)). This bound is attained by difference closed sets. However, we do not know whether there is any other set M attaining this bound. It seems natural to conjecture the following:

Conjecture 4 *Suppose M is a finite set of positive integers with $|M| = m$. If M is not almost difference closed, then $\chi_f(G(Z, M)) \leq m$, or equivalently, $\mu(M) \geq 1/m$.*

The conjecture above is weaker than the following conjecture of [27]:

Conjecture 5 *Suppose M is a finite set of positive integers with $|M| = m$. If M is not almost difference closed, then $\chi(G(Z, M)) \leq m$.*

We note here that for $m \leq 3$, both Conjectures 4 and 5 are true [5, 7, 27].

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