Circular chromatic number of subgraphs

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Abstract

This paper proves that every $(n+1)$-chromatic graph contains a subgraph H with $\chi_c(H) = n$. This provides easy methods for constructing sparse graphs G with $\chi_c(G) = \chi(G) = n$. It is also proved that for any $\epsilon > 0$, for any fraction $k/d > 2$, there exists an integer g such that if G has girth at least g and $\chi_c(G) = k/d$ then for every vertex v of G, $\chi_c(G - v) > k/d - \epsilon$. This implies that G has an induced subgraph H with $\chi_c(G) - \epsilon < \chi_c(H) < \chi_c(G)$.

Keywords: Circular chromatic number, critical graphs, subgraphs.

1 Introduction

The circular chromatic number $\chi_c(G)$ of a graph G (also known as the star chromatic number and denoted by $\chi^*(G)$) is a natural generalization of the chromatic number. For positive integers $k \geq 2d$, a (k,d)-coloring of G is a mapping $c : V \to \{0, 1, \cdots, k-$

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$1\}$ such that for any edge xy of G, $d \le |f(x) - f(y)| \le k - d$. The *circular chromatic number* $\chi_c(G)$ of G is defined by

$$\chi_c(G) = \inf\{k/d : \text{ there exists a } (k,d)\text{-coloring of } G\}.$$

Note that a $(k,1)$-coloring is simply a k-coloring. So $\chi_c(G) \le \chi(G)$. It was proved in [19] that $\chi(G) - 1 < \chi_c(G)$. Therefore $\chi(G) = \lceil \chi_c(G) \rceil$.

It is trivial that if $\chi(G) = n$ then for any $1 \le k \le n-1$, G contains a subgraph H with $\chi(H) = k$. However, the possible values of $\chi_c(H)$ for subgraphs H of G is an open problem. The following question was asked in [25]:

Question 1 *Are there two rational numbers $2 < r < r'$ such that any graph of circular r' contains a subgraph of circular chromatic number r ?*

In this paper, we shall answer this question in affirmative. We prove that if $1 \le n \le k/d$ is an integer, then for any graph G with $\chi_c(G) = k/d$, there exists a subgraph H of G with $\chi_c(H) = n$. This result provides a simple way of constructing graphs G of large girth and with $\chi_c(G) = \chi(G) = n$ for arbitrary integer n. In particular, the Mycielski's construction can be used to construct triangle free graphs G with $\chi_c(G) = n$ for arbitrary integer $n \ge 2$. Namely, for $t \ge 1$, denote by $M^t(K_2)$ the t-th iterated Mycielskian of K_2 (here $M(H)$ is the Mycielskian of H and $M^t(H) = M(M^{t-1}(H))$), and let $G = M^t(K_2) - e$ for an arbitrary edge e of $M^t(K_2)$. Then it follows that $\chi_c(G) = t+1$. Compared to previous known methods, the graphs G constructed this way is much smaller. Indeed, with this construction, we can give some meaningful upper bound on the minimum number of vertices of K_k-free graphs G with $\chi_c(G) = n$.

Then we shall discuss the circular chromatic number of subgraphs of sparse graphs. We shall prove the result stated in the abstract. Intuitively speaking, the result says that if G has large girth, then by deleting a vertex, the circular chromatic number can decrease only a very small amount.

2 Deleting one edge

Delete an edge or a vertex of a graph G, how much can its circular chromatic number decrease ? The problem was discussed in [22], where it was shown that by deleting a vertex, the decrease of the circular chromatic number is always less than 2, but could be larger than $2 - \epsilon$ for any $\epsilon > 0$. On the other hand, by deleting an edge, the decrease of the circular chromatic number is at most 1. In this section, we prove Theorem 1 below, which is a strengthening of the above result. For a vertex v of a graph G, we denote by $N_G(v)$ the neighbourhood of v, i.e., $N_G(v) = \{x : xv \in E(G)\}$. A *clique* X of G is a subset of $V(G)$ that induces a complete subgraph.

Theorem 1 *Suppose G is a graph, v is a vertex of G and $X \subseteq N_G(v)$ is a clique of G contained in the neighbourhood of v. Let G' be obtained from G by deleting all the edges $\{vx : x \in X\}$. Then*

$$\chi_c(G') \geq \chi(G) - 1.$$

Proof. Assume that $\chi_c(G') = k/d$ (where $\gcd(k,d) = 1$), and that f is a (k,d)-coloring of G'. If $d = 1$ (i.e., $k/d = k$ is an integer), then by assigning a new color to v, we obtain a $(k+1)$-coloring of G. So $\chi(G) \leq \chi_c(G') + 1$.

Assume that $d \geq 2$. Without loss of generality, assume that $f(v) = 0$. If $|X| = 1$, then let $X = \{x\}$. Without loss of generality, we may assume that $0 \leq f(x) \leq k/2$. Let $n = \lfloor k/d \rfloor$. Define a coloring c as follows: $c(u) = \lfloor f(u)/d \rfloor$ if $u \neq v$ and $c(v) = n$. We prove that c is a proper $(n+1)$-coloring of G. If ab is an edge of G and v *not* $\in \{a,b\}$ then $|f(a) - f(b)| \geq d$, and hence $|c(a) - c(b)| \geq 1$. If $ab = vx$, then $c(x) \leq k/2d < n$ and $c(v) = n$, so $c(x) \neq c(v)$. If $ab = vu$ for some $u \neq x$, then $f(u) \leq k - d$ and hence $c(u) \leq n - 1$. Therefore $\chi(G) \leq n + 1 \leq \chi_c(G') + 1$.

Assume $|X| \geq 2$. Let $m = \min\{f(x) : x \in X\}$ and $M = \max\{f(x) : x \in X\}$. Since X is a clique, we conclude that $d \leq M - m \leq k - d$. If $m = 0$, then it is easy to verify that the coloring c defined in the previous paragraph is a proper $(n+1)$-coloring of G. Assume $m > 0$. If $m < d$, then define a coloring c as follows: $c(u) = 0$ if $f(u) < m$ or $f(u) > k - d + m$; $c(u) = \lfloor (f(u) - m)/d \rfloor + 1$ otherwise. It is easy to verify that c is a proper $(n+1)$-coloring of G. If $m \geq d$, then define a coloring c by $c(u) = \lfloor f(u) \rfloor$ for all u. Then it is easy to verify that c is a proper $(n+1)$-coloring of G. In any case, $\chi(G) \leq n + 1 \leq \chi_c(G') + 1$. ∎

An important special case of Theorem 1 is the case $|X| = 1$.

Corollary 2 *Suppose G is a graph e is an edge of G. Then*

$$\chi_c(G - e) \geq \chi(G) - 1.$$

A graph G is called n-critical if $\chi(G) = n$ and $\chi(G - e) = n - 1$ for every edge e of G.

Corollary 3 *If G is n-critical, then for any edge e of G, $\chi_c(G-e) = n-1 = \chi(G-e)$.*

Proof. By Theorem 1, $\chi_c(G-e) \geq n-1$. On the other hand, $\chi_c(G-e) \leq \chi(G-e) = n - 1$. ∎

It is obvious that every n-chromatic graph contains an n-critical subgraph. So we have the following corollary.

Corollary 4 *If G is an n-chromatic graph, then G has a subgraph H with $\chi_c(H) = n - 1$.*

The essence of Theorem 1 and its corollaries is that the deletion of an edge can never move the circular chromatic number "across an integer", that is, there is never an integer m such that $\chi_c(G - e) < m < \chi_c(G)$. Thus, given a graph G of chromatic number n, a sequence of edge deletions must eventually results in a graph G' with $\chi_c(G') = n - 1$.

The problem of constructing graphs G of large girth with $\chi_c(G) = k/d$ for any fraction $k/d \geq 2$ was discussed in [1, 16, 17, 20, 24, 21, 23]. By using Corollary 4 above, the question of constructing large girth graphs G with $\chi_c(G) = n$ becomes very easy. One simply construct a large girth graph G with $\chi(G) = n + 1$, then by deleting some edges, one obtains a large girth graph H with $\chi_c(H) = n$. In particular, to construct triangle free graphs G with $\chi_c(G) = n$ (which is a problem first raised by Abbott and Zhou in [1]), one may simply use the Mycielski construction. There are some previously known methods for constructing triangle free graphs G with $\chi_c(G) = n$. However, the graphs constructed by Mycielski method are much smaller than those constructed by other known methods [20, 24].

As another interesting consequence of Theorem 1, we derive a sufficient condition for a graph G to have $\chi_c(G) = \chi(G)$. The problem of deciding whether a graph G satisfies $\chi_c(G) = \chi(G)$ has been studied in [7, 11, 19, 22, 21, 25, 24, 23]. There are quite a few sufficient conditions under which the equality $\chi_c(G) = \chi(G)$ holds. The following is a new sufficient condition.

Corollary 5 *Suppose G is an n-chromatic graph, v is a vertex of G and X is a clique of G, no vertex of which is adjacent to v. If for every n-coloring f of G one vertex of X is colored by the same color as v, then $\chi_c(G) = \chi(G)$.*

Proof. Let H be obtained from G by adding the edges $\{vx : x \in X\}$. Then H is not n-colorable, and hence $\chi(H) = n + 1$. By Theorem 1, $\chi_c(G) \geq n$. On the other hand, $\chi_c(G) \leq \chi(G) = n$. The result follows. ∎

The following is a special case of Corollary 5.

Corollary 6 *Suppose G is an n-chromatic graph. If G has two distinct vertices x, y such that every n-coloring of G must color x, y by the same color, then $\chi_c(G) = \chi(G)$.*

Corollary 7 *Suppose G is an n-chromatic graph, v is a vertex. If for every n-coloring f of G, the neighbours of v use all of the remaining $n - 1$ colors, i.e., $|f(N_G(v))| = n - 1$, then $\chi_c(G) = \chi(G)$.*

Proof. Let H be obtained from G by duplicate v, i.e., add a vertex v' and join v' to every neighbour of v by an edge. It is obvious that $\chi_c(G) = \chi_c(H)$. Let f be an n-coloring of H. Since $N_H(v) = N_H(v')$ use $n-1$ colors, we conclude that $f(v) = f(v')$. By Corollary 6, $\chi_c(H) = \chi(H)$. Hence $\chi_c(G) = \chi(G)$. ∎

The following result of [17] is a special case of Corollary 6.

Corollary 8 *Suppose G is a uniquely n-colorable graph, then $\chi_c(G) = \chi(G)$.*

Theorem 1 says that by deleting edges joining a clique and a vertex, the decrease of the circular chromatic number is bounded above by $\chi_c(G) + 1 - \chi(G)$. Another point of view of looking at this result is that by adding edges joining a vertex to a clique, the increase of the circular chromatic number is at most $\lfloor \chi_c(G) \rfloor + 1 - \chi_c(G)$.

Corollary 9 *Suppose G is a graph, v is a vertex and X is a clique of G. Let H be obtained from G by adding the edges $\{vx : x \in X\}$. Then $\chi_c(H) \leq \lfloor \chi_c(G) \rfloor + 1$.*

Corollary 9 is just another way of stating Theorem 1. However, Corollary 9 is sharp in the following sense.

For every $k/d > 2$ there exists a graph G and two nonadjacent vertices x, y of G such that $\chi_c(G) = k/d$ and $\chi_c(G + xy) = \lfloor \chi_c(G) \rfloor + 1$.

Indeed, for this purpose, let G be obtained from G^k_d by duplicating the vertex 0, i.e., add a new vertex $0'$ and add edges connecting $0'$ to all the neighbours of 0. Let $\{x, y\} = \{0, 0'\}$. Then $\chi_c(G) = k/d$ and $\chi_c(G + xy) = \lfloor \chi_c(G) \rfloor + 1$. Recall that G^k_d is the graph with vertex set $\{0, 1, \cdots, k-1\}$ in which $i \sim j$ if and only if $d \leq |i - j| \leq k - d$.

On the other hand, Corollary 2 is not sharp in the above sense, i.e., there are fractions k/d such that for any graph G with $\chi_c(G) = k/d$, for any edge e of G, $\chi_c(G - e) > \chi(G) - 1$. As an example, take $k/d = 8/3$. A core graph G with $\chi_c(G) = 8/3$ is not an odd cycle. But it is easy to show that if G is a core graph which is nonbipartite and not an odd cycle, then for any edge e of G, $G - e$ is not bipartite. Hence $\chi_c(G - e) > 2$. It is likely that for most fractions k/d, Theorem 1 is also not sharp in the above sense.

3 Small graphs with $\chi_c(G) = n$

Given integers k, n, let $\mu(n, k)$ be the number of vertices of a smallest graph G with $\chi(G) = n$ and $\omega(G) = k$. The value $\mu(n, k)$ has been studied in a few papers [12, 9]. Let $\mu_c(n, k)$ be the number of vertices of a smallest graph G with $\chi_c(G) = n$ and $\omega(G) = k$. It follows from the definition that $\mu_c(n, k) \geq \mu(n, k)$. By Theorem 1, any $(n+1)$-chromatic graph G contains a graph H with $\chi_c(H) = n$. If $\omega(H) < k$, then consider a n-colouring of H and add edges between differently coloured vertices until we obtain a clique of size k, this cannot increase the circular chromatic number of new graph as $\chi_c(H)$ is already equal to $\chi(G)$. Therefore we have the following result.

Theorem 10 *For any positive integers k, n, we have $\mu_c(n, k) \leq \mu(n + 1, k)$.*

For arbitrary integers k, n, it is very difficult to determine the value of $\mu(n, k)$ or $\mu_c(n, k)$. When $k = 2$, determining $\mu(n, 2)$ means to find a smallest triangle free graph G with $\chi(G) = n$, a problem first asked P. Erdős [5]. The Mycielski's construction gives an upper bound for $\mu(n, 2)$: $\mu(n, 2) \leq 2\mu(n-1, 2)+1$. As $\mu(2, 2) = 2$, we have $\mu(n, 2) \leq 3 \cdot 2^{n-2} - 1$. This bound is sharp for $n \leq 4$. The problem of determining $\mu(5, 2)$ was raised by Toft [18], and solved by Jensen and Royle [12] through a computer search. They proved that $\mu(5, 2) = 22$. For large n, it follows from a result of Erdős [6] that $\mu(n, 2) \leq c \cdot (n \cdot \log n)^2$ for some constant c. It follows from a result of Ajtai, Komlós, and Szemerédi [2] (see [13]) that $\mu(n, 2) \geq c' \cdot n^2 \cdot \log n$ for some constant c'. As $\mu_c(n, 2) \leq \mu(n + 1, 2)$, we conclude that

$$c' \cdot n^2 \cdot \log n \leq \mu_c(n, 2) \leq c'' \cdot (n \cdot \log n)^2.$$

For $n = 3, 4$, the exact values of $\mu_c(n, 2)$ are known. Let H be obtained from Petersen graph by deleting one vertex. Then H is triangle free, $\chi_c(H) = 3$ and H has 9 vertices. So $\mu_c(3, 2) \leq 9$. On the other hand, it follows from a result of Brandt [3] that for any triangle free graph G, $\chi_c(G) \geq 3$ if and only if every maximal triangle free supergraph of G contains H as a subgraph. So $\mu_c(3, 2) = 9$. Let G be the Grötzch graph. It was proved in [4] that $\chi_c(G) = 4$. Hence $\mu_c(4, 2) \leq 11$. As $\mu(4, 2) = 11$ and $\mu_c(4, 2) \geq \mu(4, 2)$, we have $\mu_c(4, 2) = 11$.

In the following, we consider the case that k is not much smaller than n.

Theorem 11 *If $n \geq 4$ then $\mu_c(n, n - 1) = \mu(n, n - 1) = n + 2$. If $n \geq 6$ then $\mu_c(n, n - 2) = \mu(n, n - 2) = n + 4$.*

Proof. For $n \geq 4$, let G be obtained from K_{n+2} by deleting a C_5. It is easy to verify that $\chi_c(G) = n$ and $\omega(G) = n - 1$. Therefore $\mu(n, n - 1) \leq \mu_c(n, n - 1) \leq n + 2$. For $n \geq 6$, let G be obtained from K_{n+4} by deleting two vertex disjoint copies of C_5. The resulting graph shows that $\mu(n, n - 2) \leq \mu_c(n, n - 2) \leq n + 4$.

To prove that $\mu(n, n - 1) \geq n + 2$ and $\mu(n, n - 2) \geq n + 4$, we first prove the following claim. For convenience, if X is a subset of $V(G)$, we also use X to denote the subgraph of G induced by X.

Claim 1 *Suppose X is a clique of G of size $\omega(G) = m$ and $Y \subset V(G)$ such that $X \cap Y = \emptyset$. If each connected component of Y is a complete graph. Then $X \cup Y$ has chromatic number m.*

Proof of Claim 1. For each $y \in Y$, let $S_y = \{x \in X : x \not\sim y\}$. Suppose Z is a component of Y (which is a complete graph by assumption). If there is a subset Z' of Z such that $|\cup_{y\in Z'} S_y| < |Z'|$, then $(X - \cup_{y\in Z'}S_y) \cup Z'$ would be a clique of G larger than X, contrary to our assumption. Therefore $|\cup_{y\in Z'} S_y| \geq |Z'|$ for each subset Z' of Z. By the marriage theorem, there is a mapping $\phi : Y \to \cup_{y\in Y}S_y$ such that $\phi(y) \in S_y$ and $\phi(y) \neq \phi(y')$ whenever $y \sim y'$. Thus we may color each vertex of X by a distinct color, and color each vertex $y \in Y$ by the color of $\phi(y)$.

Assume to the contrary that $\mu(n, n-1) < n+2$ for some $n \geq 4$. Then there is a graph G on $n+1$ vertices for which $\omega(G) = n-1$ and $\chi(G) = n$. Let X be a clique of G of size $n-1$. Let $Y = V(G) - X$. Since $|Y| = 2$, each component of Y is a clique. By Claim 1, $\chi(G) = n-1$, contrary to our assumption. This proves that for $n \geq 4$, $\mu_c(n.n-1) = \mu(n.n-1) = n+2$.

For the second half, assume to the contrary that $\mu(n, n-2) < n+4$ for some $n \geq 6$. Then there is a graph G on $n+3$ vertices for which $\omega(G) = n-2$ and $\chi(G) = n$. Let X be a $(n-2)$-clique of G. Let $Y = V(G) - X$. Then $|Y| = 5$. It is easy to verify that except for the case that Y induces a copy of W_4 (i.e., a C_4 plus a universal vertex) then Y has an independent Z such that $Y - Z$ is the union of cliques. Then by Claim 1, $X \cup (Y - Z)$ can be colored by $n-2$ colors, and Z can be colored by an extra color, contrary to the assumption that $\chi(G) = n$.

Assume that $Y = W_4$, where $\{y_1, y_2, y_3, y_4\} \subset Y$ induces a C_4 and y_5 is a universal vertex of Y. Note that $\{y_1, y_3\}$ is an independent set because otherwise $Y - \{y_4\}$ is the union of cliques, contrary to the above assumption. If $X \cup \{y_2, y_4, y_5\}$ can be colored by $n-2$ colors, then we can color y_1, y_3 by a new color to obtain an $(n-1)$-coloring, contrary to our assumption. Assume that $X \cup \{y_2, y_4, y_5\}$ cannot be colored by $n-2$ colors. Then by the argument as the proof of Claim 1, it is easy to show that we must have $S_{y_5} = \{a, b\}$, $S_{y_2} = \{a\}$ and $S_{y_4} = \{b\}$ for some $a, b \in X$. (Here S_{y_i} are as defined in the proof of Claim 1). Using the same argument, we also must have $\{y_2, y_4\}$ is an independent set, and $S_{y_1} \cup S_{y_3} = \{a, b\}$. But this implies that $(X - \{a, b\}) \cup \{y_1, y_2, y_5\}$ is a clique of G larger than X, contrary to our assumption.
∎

For $\frac{2n}{3} \leq k \leq n$ and $n \geq 4$, we have $\mu(n, k) \leq 3n - 2k$ because the graph G obtained from K_{3n-2k} by deleting $n-k$ vertex disjoint copies of C_5 has chromatic number n and clique size k. Moreover, the complement of G is disconnected. By a result in [1, 22], G has its chromatic number equal its circular chromatic number. So $\chi_c(G) = n$ and hence $\mu_c(n,k) \leq 3n-2k$. Theorem 11 shows that this bound is sharp if $k \geq n-2$. However, for large n, and for large $n-k$, the bound is not sharp. Indeed, it was known [15] that $R(k,3) > ck^2/\log k$. Therefore for large m, there exists a graph G on m vertices whose clique size is less than $m^{\frac{1}{2}+\epsilon}$, whose complement is K_3-free. So $\chi(G) \geq m/2$. Let $n = m/2$ and $k = m^{\frac{1}{2}+\epsilon}$. Then we have $\mu(n,k) \leq 2n < 3n-2k$. By adding universal vertices to the graph G, we conclude that for any $t \geq 1$, $\mu_c(n+t, k+t) \leq \mu(n,k) + t < 3(n+t) - 2(k+t)$.

4 Subgraphs of a sparse graph

A sparse graph here means a graph of large girth. In this section, we prove that for sparse graphs, by deleting a vertex, the decrease of the circular chromatic number is very small.

Suppose G is a graph and $\vec{G}$ is an orientation of G. For a cycle C of $\vec{G}$ with a given traversal, we denote by C^+ the set of forward edges of C (i.e., edges whose direction agree with the direction of the traversal), and C^- the set of backward edges of C. The following lemma was proved in [8].

Lemma 12 *A graph G has $\chi_c(G) \leq k/d$ if and only if there exists an orientation $\vec{G}$ of G such that for any cycle C of $\vec{G}$,*

$$|C^+|/|C^-| \leq (k-d)/d.$$

It follows from Lemma 12 that $\chi_c(G) \leq r$ if and only if G has an orientation $\vec{G}$ such that for each cycle C, $|C^+|/|C^-| \leq r-1$.

Recall the graph G^k_d has vertex set $\{0, 1, \cdots, k-1\}$ and $i \sim j$ if $d \leq |i-j| \leq k-d$. We assign an orientation to G^k_d as follows: For an edge ij of G^k_d, orient the edge from i to j if $i < j$. Denote this oriented graph by $\vec{G^k_d}$.

Theorem 13 *Suppose G is a graph with $\chi_c(G) = k/d$ and G has girth at least g. Then for any vertex v of G, $\chi_c(G-v) > k/d(1-\frac{2}{g-2})$.*

Proof. Assume $\chi_c(G-v) = k'/d' \le k/d$. Let f be a (k',d')-coloring of $G-v$. Let $\vec{G}-v$ be the orientation of $G-v$ in which an edge xy of G is orient from x to y if and only if $f(x) < f(y)$. Extend this orientation to G by orient all the edges incident to v toward v (so that v becomes a sink). Denote the resulting oriented graph by $\vec{G}$. We shall prove that in $\vec{G}$, every cycle C satisfies

$$|C^+|/|C^-| < \frac{k'}{d'} - 1 + \frac{2}{g-2}\frac{k}{d}.$$

By Lemma 12, this implies that

$$\frac{k}{d} = \chi_c(G) < \frac{k'}{d'} + \frac{2}{g-2}\frac{k}{d}.$$

Hence

$$\frac{k'}{d'} > \frac{k}{d}(1 - \frac{2}{g-2}).$$

Let C be an oriented cycle of $\vec{G}$. If C does not contain v, then C is an oriented cycle of $\vec{G}-v$. Hence $|C^+|/|C^-| \le \frac{k'}{d'} - 1$ (see [8, 25]).

Assume now that C contains v. Then $P = C - v$ is an oriented path of $G-v$. Moreover, $|C^+| = |P^+| + 1$ and $|C^-| = |P^-| + 1$.

Let $x_1, x_2, \cdots, x_s$ be the vertices of P and for each $j = 1, 2, \cdots, s-1$, either $e_j = x_j x_{j+1}$ is a forward edge of P, or $e_j = x_{j+1}x_j$ is a backward edge of P. For each j, let $\ell(e_j) = |f(x_j) - f(x_{j+1})|$. Then $d' \le \ell(e_j) \le k' - d'$.

Now

$$\begin{aligned} f(x_s) - f(x_1) &= (f(x_2) - f(x_1)) + (f(x_3) - f(x_2)) + \cdots + (f(x_s) - f(x_{s-1}) \\ &= \Sigma_{e_j \in P^+}\ell(e_j) - \Sigma_{e_j \in P^-}\ell(e_j). \end{aligned}$$

Since $d' \le \ell(e_j) \le k' - d'$ for each j, we have $\Sigma_{e_j \in P^+}\ell(e_j) \ge d'|P^+|$ and $\Sigma_{e_j \in P^-}\ell(e_j) \le (k'-d')|P^-|$. Therefore

$$f(x_s) - f(x_1) \ge d'|P^+| - (k'-d')|P^-|.$$

As $f(x_s) - f(x_1) \le k' - 1$, we have $d'|P^+| - (k'-d')|P^-| \le k' - 1$, and hence $|P^+| \le (k'-d')/d'|P^-| + (k'-1)/d'$. This implies that

$$|C^+| \le (k'-d')/d'|C^-| + (2d'-1)/d' < (k'-d')/d'|C^-| + 2.$$

As G has girth at least g, so $|C^+| + |C^-| \geq g$. Together with the inequality above, we conclude that $|C^-| > \frac{d'}{k'}(g-2)$. Therefore

$$|C^+|/|C^-| < \frac{k'-d'}{d'} + \frac{2k'}{d'(g-2)} \leq \frac{k'}{d'} - 1 + \frac{2}{g-2}\frac{k}{d}.$$

This completes the proof. ∎

Theorem 13 can also be viewed as a bound on the increase of the circular chromatic number by adding a vertex under the assumption that the resulting graph has large girth. Actually a similar proof as that of Theorem 13 establishes the following result.

Theorem 14 *Suppose $\chi_c(G) \leq k/d$ and g is a positive integer. Let G' be obtained from G by adding a vertex v. If every cycle of G' containing v has length at least g, then*

$$\chi_c(G') \leq \frac{k}{d} + \frac{(2d-1)}{d\tau},$$

where $\tau = \lceil \frac{dg-2d+1}{k} \rceil$.

When $k/d = m$ is an integer, then the formula above simplifies to

$$\chi_c(G') \leq m + \frac{1}{\lceil \frac{g-1}{m} \rceil}.$$

So we have the following consequence which was proved in [17].

Corollary 15 *Let $m \geq 2$ and $t \geq 1$ be integers. If a graph G has a vertex v such that $G-v$ is m-colorable and any cycle of G containing v has length at least $g = m(t-1)+2$, then $\chi_c(G) \leq m + \frac{1}{t}$.*

We have shown that if n is an integer less than k/d, then any graph G with $\chi_c(G) = k/d$ contains a subgraph with $\chi_c(G) = n$. However, the following questions remain open:

Question 2 *Let $r > 2$ be a rational number. For which rational numbers r', every graph G with $\chi_c(G) = r$ contains a subgraph H with $\chi_c(H) = r'$?*

Question 3 *Let $r > 2$ be a rational number. For which rational numbers r', every graph G with $\chi_c(G) = r$ contains an induced subgraph H with $\chi_c(H) = r'$?*

Given a fraction p/q with $\gcd(p, q) = 1$. Let a, b be the unique integers $0 < a < p$ and $0 < b < q$ such that $pb - aq = 1$. In some sense, a/b is the fraction preceding p/q (see properties of Farey sequence). It is unknown whether or not every graph G with $\chi_c(G) = p/q$ contains a subgraph (or even an induced subgraph) H with $\chi_c(H) = a/b$.

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