# Star chromatic numbers of graphs

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**Abstract**

We investigate the relation between the star-chromatic number $\chi^*(G)$ and the chromatic number $\chi(G)$ of a graph $G$. First we give a sufficient condition for graphs under which their star-chromatic numbers are equal to their ordinary chromatic numbers. As a corollary we show that for any two positive integers $k, g$, there exists a $k$-chromatic graph of girth at least $g$ whose star-chromatic number is also $k$. The special case of this corollary with $g = 4$ answers a question of Abbott and Zhou. We also present an infinite family of triangle-free planar graphs whose star-chromatic number equals their chromatic number. We then study the star-chromatic number of color-critical graphs. We prove that if an $(m+1)$-critical graph has large girth, then its star-chromatic number is close to $m$. We also consider $(m+1)$-critical graphs with high connectivity. An infinite family of graphs is constructed to show that for each $\epsilon > 0$ and each $m \geq 2$ there is an $m$-connected $(m+1)$-critical graph with star chromatic number at most $m + \epsilon$. This answers another question asked by Abbott and Zhou.

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# 1 Introduction

Let $k$ and $d$ be natural numbers such that $k \geq 2d$. A $(k,d)$-*coloring* of a graph $G=(V,E)$ is a mapping $\gamma : V \mapsto \mathcal{Z}_k$ such that, for each edge $(u,v) \in E$, $|\gamma(u) - \gamma(v)|_k \geq d$, where $|x|_k := min\{|x|, k-|x|\}$. Observe that a $(k,1)$-coloring of a graph $G$ is just an ordinary coloring of $G$. We say $G$ is $(k,d)$-*colorable* if there is a $(k,d)$-coloring of $G$. The *star chromatic number* $\chi^*(G)$ of $G$ is the minimum $\frac{k}{d}$ for which $G$ is $(k,d)$-colorable. (To be precise, the "minimum" in the definition of the star-chromatic number should be "infimum". However, it was proved in [10] that the infimum is attained.)

The notion of the star chromatic number of a graph was introduced by Vince [10]. His work relies on continuous methods. A pure combinatorial treatment of star-chromatic number was given by Bondy and Hell [3]. The following alternative definition of the star-chromatic number was given by the second author in [11], and we shall use both definitions in this paper.

Let $C$ be a circle in $R^2$ of length 1, and let $r \geq 1$ be any real number. Denote by $C^{(r)}$ the set of all open intervals of $C$ of length $1/r$. An $r$-*circular coloring* of a graph $G$ is a mapping $c$ from $V(G)$ to $C^{(r)}$ such that $c(x) \cap c(y) = \emptyset$ whenever $(x,y) \in E(G)$. If such an $r$-circular coloring exists, we say that $G$ is $r$-*circular colorable.*

It was proved in [11] that the star-chromatic number $\chi^*(G)$ of $G$ is equal to the minimum $r$ for which $G$ is $r$-circular colorable.

Thus we obtain an alternative definition of the star-chromatic number of a graph. This alternative definition is extended by Deuber and Zhu [5] to weighted graphs and some interesting problems are discussed there.

It follows from the definition that $\chi^*(G) \leq \chi(G)$ for any graph $G$. There is also a lower bound for $\chi^*(G)$ in terms of $\chi(G)$, namely $\chi(G) - 1 < \chi^*(G)$ (cf. [10]). Thus the parameter $\chi^*(G)$ can be regarded as a refinement of the parameter $\chi(G)$, and it contains more information about the graph. Note that if $\chi^*(G)$ is an integer then, necessarily, $\chi^*(G) = \chi(G)$. There are graphs $G$ for which $\chi(G) = \chi^*(G)$, and there are also graphs $G$ for which $\chi^*(G)$ is arbitrarily close to $\chi(G) - 1$. Vince [10] first asked the question as to what determines whether $\chi^*(G) = \chi(G)$. This question has been studied by many authors, and some sufficient conditions under which the star-chromatic number of a graph $G$ equals its ordinary chromatic number can be found in [1, 6, 9, 10, 11]. In general, it is intractable to decide whether a given graph $G$ satisfies $\chi^*(G) = \chi(G)$, (cf. [9]). In this paper we give a new sufficient condition for a graph $G$ to satisfy $\chi^*(G) = \chi(G)$. This condition is weaker (hence the result is stronger) than the previous known sufficient conditions, [1, 6, 9, 10, 11]. As a corollary, we show that uniquely colorable graphs $G$ satisfy $\chi^*(G) = \chi(G)$. Applying this corollary and an earlier result of Bollobás and Sauer [2], we prove that for any integers $k \geq 2$ and $g \geq 3$, there exists a graph $G$ with girth at least $g$ and $\chi^*(G) = \chi(G) = k$. The special case of this result with $g = 4$ gives a positive answer to a question asked by Abbott and Zhou in [1], namely, there are triangle-free graphs $G$ with $\chi^*(G) = \chi(G) = k$ for any $k$. In Section 3, we give an infinite family of triangle-free planar graphs whose star-chromatic number and their ordinary chromatic number are equal.

The "opposite" of the problem of characterizing graphs $G$ for which $\chi^*(G) = \chi(G)$ would be to find those graphs $G$ for which $\chi^*(G)$ is close to $\chi(G) - 1$. There has also been some work on this problem [1]. As mentioned in the previous paragraph, we shall prove that uniquely colorable graphs $G$ have $\chi(G) = \chi^*(G)$. In some sense, the "opposite" of uniquely colorable graphs are critical graphs, i.e., those graphs $G$ such that the deletion of any edge will decrease their chromatic numbers. Intuitively one may expect critical graphs $G$ to have star-chromatic number $\chi^*(G)$ close to $\chi(G) - 1$. This intuition is supported by the fact that 3-critical graphs are odd cycles $C_{2k+1}$ which have star-chromatic numbers $\chi^*(C_{2k+1}) = 2 + \frac{1}{k}$. Our result in Section 4 will, in some sense, confirm this intuition for $(m+1)$-critical graphs where $m$ is an arbitrary integer. We shall show that for any $\epsilon > 0$ and for any positive integer $m$, there is an integer $g = g(m, \epsilon)$ such that every $(m+1)$-critical graph $G$ with girth at least $g$ has star-chromatic number $\chi^*(G) \leq m + \epsilon$.

This result implies that there are many $(m+1)$-critical graphs $G$ for which $\chi^*(G)$ is close to $m$. However there is no easy construction of such graphs, and we have few examples of such graphs. The lack of concrete examples makes it difficult to investigate other properties of such graphs. For instance, what are their connectivities? The following question was asked by Abbott and Zhou [1]:

*Let $\epsilon > 0$ be a positive number. Do there exist critical graphs $G$ of high connectivity such that $\chi^*(G) \le \chi(G) - 1 + \epsilon$?*

Since critical graphs are always 2-connected, the lowest connectivity of critical graphs of interest is 3. An infinite family of 3-connected graphs $G$ with $\chi^*(G)$ arbitrarily close to $\chi(G) - 1$ was constructed in [1]. In Section 5, we answer the above question by constructing $(m+1)$-critical graphs $G$ of connectivity $m$ such that $\chi^*(G) \le m + \epsilon$ for any integer $m$ and any $\epsilon > 0$.

# 2 A sufficient condition

We present in this section a sufficient condition under which a graph $G$ satisfies $\chi^*(G) = \chi(G)$.

**Theorem 1** *Suppose $G$ is a $k$-chromatic graph. If there is a nonempty proper subset $A$ of $V(G)$ such that for any $k$-coloring $\Delta$ of $G$, and for any color class $S$ of $\Delta$, either $S \subset A$ or $S \cap A = \emptyset$, then $\chi^*(G) = \chi(G)$.*

**Proof.** We use the definition of circular coloring in the proof. Let $G$ be a $k$-chromatic graph with $\chi^*(G) = r < k$, and let $A$ be a nonempty proper subset of $V(G)$. We shall construct a $k$-coloring of $G$ which has a color class $S$ such that $S \cap A \neq \emptyset$ and $S \cap (V(G) - A) \neq \emptyset$.

Let $c : V(G) \to C^{(r)}$ be an $r$-circular coloring of $G$. Consider the two open subsets $\cup_{x \in A} c(x)$ and $\cup_{y \in V(G)-A} c(y)$ of $C$. If they are disjoint, then they cannot cover $C$, and hence there is a point $q_0 \in C$ such that $q_0 \notin c(x)$ for every $x \in V(G)$. Pick $k-1$ other points $q_1, q_2, \cdots, q_{k-1}$ on $C$ so that the clockwise distance from $q_i$ to $q_{i-1}$ $(1 \le i \le k-1)$ is equal to $1/r$. Since $r < k$, the clockwise distance from $q_0$ to $q_{k-1}$ is smaller than $1/r$. Note that (because of $k-1 < r$) $q_0$ lies "cyclically between" $q_{k-1}$ and $q_1$, i.e., the cyclic order of the points $q_i$ on $C$ is $\cdots, q_{k-2}, q_{k-1}, q_0, q_1, q_2, \cdots$. Now we can obtain a $(k-1)$-coloring $\Delta$ of $G$ as follows: $\Delta(x) = i$ if and only if either $q_i \in c(x)$ or $c(x)$ equals the open interval $(q_{i-1}, q_i)$ of $C$, for $i = 1, 2, \cdots, k-1$. We first note that every vertex $x$ of $G$ is colored. Indeed the points $q_0, q_1, \cdots, q_{k-1}$ partition the circle $C$ into $k$ intervals. All the intervals $(q_0, q_1), (q_1, q_2), \cdots, (q_{k-2}, q_{k-1})$ have length exactly $1/r$, and the interval $(q_{k-1}, q_0)$ has length strictly smaller than $1/r$. Since $c(x)$ is an open interval of length $1/r$ which does not contain $q_0$, either $q_i \in c(x)$ for some $i \in \{1, 2, \cdots, k-1\}$ or $c(x)$ is one of the open intervals $(q_0, q_1), (q_1, q_2), \cdots (q_{k-2}, q_{k-1})$. Thus $x$ is colored. It is easy to see that if $\Delta(x) = \Delta(y) = i$, then $c(x) \cap c(y) \neq \emptyset$, and hence $x$ is not adjacent to $y$. So $\Delta$ is indeed a $(k-1)$-coloring of $G$, contrary to our assumption that $\chi(G) = k$.

Therefore $(\cup_{x \in A} c(x)) \cap (\cup_{y \in V(G)-A} c(y)) \neq \emptyset$. So there is a point $q_0$ on $C$ such that $q_0 \in c(x) \cap c(y)$ for some vertices $x \in A$ and $y \in V(G) - A$. We now choose $k-1$ other points $q_1, q_2, \cdots, q_{k-1}$ on $C$ so that the clockwise distance from $q_i$ to $q_{i-1}$ is equal to $1/r$. Now we can obtain a $k$-coloring $\Delta$ of $G$ as follows: $\Delta(x) = i$ if and only if either $q_i \in c(x)$ or $c(x)$ is the open interval $(q_{i-1}, q_i)$ of $C$, for $i = 1, 2, \cdots, k-2$; $\Delta(x) = k-1$ if and only if either $q_{k-1} \in c(x)$ and $q_0 \notin c(x)$ or $c(x)$ is the open interval $(q_{k-2}, q_{k-1})$ of $C$; and $\Delta(x) = 0$ if and only if $q_0 \in c(x)$. As in the previous paragraph, one can verify that this is indeed a $k$-coloring of $G$. Moreover, the color class $\Delta^{-1}(0)$ contains both vertices from $A$ and vertices from $V(G) - A$. This completes the proof of Theorem 1. ∎

Suppose $G$ is a $k$-chromatic graph. If $G$ contains a $k$-chromatic subgraph $H$ with $\chi^*(H) = \chi(H) = k$, then we have $k = \chi(G) \geq \chi^*(G) \geq \chi^*(H) = k$, and hence $\chi^*(G) = \chi(G) = k$. Thus the above theorem actually applies to $k$-chromatic graphs which contain a $k$-chromatic subgraph satisfying the stated condition. However even this generalized version of Theorem 1 does not give a necessary condition. The Petersen graph does not satisfy this condition, and yet its star-chromatic number is equal to its chromatic number.

The following result was proved independently by Abbott and Zhou [1], and Gao, Mendelsohn and Zhou [6] (a short proof of this result can also be found in [11]):

**Corollary 1** *If the complement of $G$ is disconnected, then $\chi^*(G) = \chi(G)$.*

**Proof.** The graph $G$ is obtained from two disjoint graphs $X, Y$ by connecting every vertex of $X$ to every vertex of $Y$. Obviously for any coloring of $G$, no color class can contain both vertices from $V(X)$ and vertices from $V(Y)$. Thus $V(X)$, as well as $V(Y)$, satisfies the condition of Theorem 1. Therefore $\chi^*(G) = \chi(G)$. ∎

A graph $G$ is called a *uniquely $k$-colorable* graph if $G$ is $k$-colorable and every $k$-coloring of $G$ partitions $V(G)$ into the same $k$ color classes. The following corollary follows immediately from the definition and the comments after the proof of Theorem 1.

**Corollary 2** *If a $k$-chromatic graph $G$ contains a uniquely $k$-colorable subgraph, then $\chi^*(G) = \chi(G) = k$.* ∎

If a $k$-chromatic graph $G$ contains a clique of order $k$, then since $K_k$ is obviously uniquely $k$-colorable, we have $\chi^*(G) = \chi(G)$ by Corollary 2. This special case also follows easily from the definition, and was first noted in [10]. On the other hand, as observed in [1], a graph $G$ with $\chi^*(G) = \chi(G)$ need not have large clique number. Indeed, as Abbott and Zhou [1] noted, by using Corollary 1 one can construct $K_4$-free $k$-chromatic graphs $G$ with $\chi^*(G) = \chi(G)$ for any integer $k$. Abbott and Zhou then asked the following question: Given an integer $k$, does there exist a triangle-free graph $G$ with $\chi^*(G) = k$? For $k = 3, 4$, it was noted in [10] that the Petersen graph and the Grötzsch graph are such graphs. For $k \geq 5$, the answer was unknown. By applying Corollary 2 and a result of Bollobás and Sauer [2], we obtain a much stronger result:

**Corollary 3** *For any integers $k \geq 2$ and $g \geq 3$, there is a $k$-chromatic graph $G$ with girth at least $g$ and star-chromatic number $k$.*

**Proof.** Bollobás and Sauer proved in [2] that for any integers $k \geq 2$ and $g \geq 3$, there is a uniquely $k$-colorable graph $G$ with girth at least $g$. By Corollary 2, we have $\chi^*(G) = \chi(G) = k$ for such a graph. ∎

We note that the proof in [2] of the existence of $k$-uniquely colorable graphs of large girth is not constructive.

# 3 Planar graphs

Our next result concerns planar graphs. Vince [10] asked for families of planar graphs whose star-chromatic numbers are strictly between 2 and 3.

It seems that such graphs are abundant (see [1, 11]), while not many planar graphs are known to have star-chromatic number exactly 3. Obviously if a 3-chromatic planar graph $G$ contains a triangle, then $\chi^*(G) = 3$. The first triangle-free planar graph $G$ with $\chi^*(G) = 3$ was found by Gao (personal communication). Denote by $W_{2n+1}$ the graph obtained from the circuit $C_{2n+1}$ by adding a vertex $v$ and connecting $v$ to every vertex of the circuit $C_{2n+1}$. The graph $W_{2n+1}$ is called the $(2n+1)$-*wheel*, and the edges connecting $v$ to vertices of $C_{2n+1}$ are called the *spokes* of the wheel. Through an extensive check, Gao showed that the graph obtained from the 5-wheel by subdividing each of the five spokes by precisely one additional vertex has star-chromatic number 3. We prove that for all integers $n$, the graph obtained from $W_{2n+1}$ by subdividing its $2n+1$ spokes has star-chromatic number 3. This is the first (non-trivial) infinite family of triangle-free planar graphs with star-chromatic number 3.

**Theorem 2** *Let $G_{2n+1}$ be the graph obtained from $W_{2n+1}$ by subdividing each of its $2n+1$ spokes by precisely one additional vertex. Then $\chi^*(G) = 3$.*

**Proof.** We again use the definition of circular coloring in the proof. Let $V = \{v, x_0, x_1, \cdots, x_{2n}\}$ be the vertex set of $W_{2n+1}$, while $v$ is connected to all the $x_i$'s, and the set $\{x_0, x_1, \cdots, x_{2n}\}$ induces a circuit with edges $(x_i, x_{i+1})$. The graph $G_{2n+1}$ is obtained from $W_{2n+1}$ by subdividing each edge $(v, x_i)$ into two edges. Let $u_i$, $i = 0, 1, \cdots, 2n$, be the vertex which subdivides the edge $(v, x_i)$. It is easy to see that $\chi(G_{2n+1}) = 3$. Therefore $\chi^*(G_{2n+1}) \leq 3$. Suppose $\chi^*(G_{2n+1}) = r < 3$. Let $c : V(G_{2n+1}) \to C^{(r)}$ be an $r$-circular coloring of $G_{2n+1}$. As $v$ is adjacent to all the $u_i$'s, all the intervals $c(u_i)$ are disjoint from $c(v)$. Since $r < 3$ and $c(x_i)$ is disjoint from $c(u_i)$, we have $c(x_i) \cap c(v) \neq \emptyset$, for $i = 0, 1, \cdots, 2n$. For any $i \in \{0, 1, \cdots, 2n\}$, we cannot have $c(x_i) = c(v)$, for otherwise we would have $c(x_i) \cap c(x_{i+1}) \neq \emptyset$ while $x_i$ is adjacent to $x_{i+1}$. Therefore $c(x_i)$ contains one of the end points of $c(v)$. Let $p, q$ be the two end points of $c(v)$. Without loss of generality, we assume that $c(x_0)$ contains $p$. Since $c(x_1)$ is disjoint from $c(x_0)$, $c(x_1)$ must contain $q$. For the same reason, $c(x_2), c(x_3), \cdots, c(x_{2n})$ must alternately contain $p$ and $q$. Thus $p \in c(x_{2n})$ and hence $c(x_{2n}) \cap c(x_0) \neq \emptyset$, contrary to our assumption that $c$ is an $r$-circular coloring of $G_{2n+1}$. This completes the proof of Theorem 2. ∎

We pose two open problems:

(1): Is it true that every triangle-free planar graph $G$ with $\chi^*(G) = 3$ contains one of the graphs $G_{2n+1}$?

(2): It follows from Theorem 1, and is also noted in [11], that $W_{2n+1}$ has star-chromatic number 4. At present, no other minimal planar graphs with star-chromatic number 4 are known. Is it true that every planar graph of star-chromatic number 4 contains an odd wheel $W_{2n+1}$?

If answers to both the questions above are affirmative, and the chromatic number of a planar graph $G$ is given, then it can be decided in time polynomial in the size of $G$ whether or not $\chi^*(G) = \chi(G)$.

We note that the argument in [9] shows that even for planar graphs $G$, it is $NP$-hard to decide whether $\chi^*(G) = \chi(G)$ if the chromatic number of $G$ is not given. On the other hand, for general graphs $G$, if the chromatic number of $G$ is given, the complexity of deciding whether or not $\chi^*(G) = \chi(G)$ is still open, as far as we know.

# 4 Critical graphs with large girth

A graph $G$ is called $(m+1)$-*critical* if $\chi(G) = m+1$ and for any edge $e \in E(G)$, $\chi(G-e) = m$. We prove in this section that critical graphs $G$ with large girth have star-chromatic numbers $\chi^*(G)$ close to $\chi(G)-1$. Indeed we shall prove the following stronger statement:

**Theorem 3** *Let $m \geq 2$ and $t \geq 1$ be integers. Let $G$ be a graph. If $G$ has a vertex $x$ such that $G-x$ is $m$-colorable and any circuit of $G$ containing $x$ has length at least $m(t-1)+2$, then $\chi^*(G) \leq m + \frac{1}{t}$.*

To prove this theorem, we shall use a characterization of the star-chromatic number of a graph given by Goddyn, Tarsi and Zhang [8]. First we need a definition.

Let $G$ be a graph, and let $D$ be an orientation of $G$. For a cycle $C$ in $D$, denote by $C^+$ and $C^-$ the sets of edges of $C$ oriented 'forward' and 'backward' respectively, with respect to a sense of traversal of $C$ (which we may suppose chosen so that $|C^+| \geq |C^-|$). Set

$$f(C,D) = \frac{|C^+|}{|C^-|} + 1, f(D) = \max_C f(C,D).$$

It is shown in [8] that $\chi^*(G)$ is equal to the minimum of $f(D)$ over all orientations $D$ of $G$.

We now proceed to prove Theorem 3. Let $G$ be a graph and let $x$ be a vertex of $G$ such that $G-x$ is $m$-colorable, and any circuit of $G$ containing $x$ has length at least $m(t-1)+2$. To prove that $\chi^*(G) \leq m + \frac{1}{t}$, it suffices to find an orientation $D$ of $G$ such that $f(D) \leq m + \frac{1}{t}$.

Let $\Delta : V(G) - \{x\} \mapsto \{1, 2, \cdots, m\}$ be an $m$-coloring of $G - x$. Let $D$ be the orientation of $G$ in which $(u,v)$ is an arc of $D$ if and only if $(u,v)$ is an edge of $G$ and either $u = x$ or $\Delta(u) < \Delta(v)$. Note that $D - x$ contains no directed path of length more than $m-1$. Let $C$ be a cycle of $D$. If $x \notin C$ then $f(C,D) \leq m$ because $C$ contains no directed path of length more than $m-1$. If $x \in C$ then $C - x$ is a path $P$ of length at least $m(t-1)$, and $P$ contains no directed path of length more than $m-1$. Let $P^+$ and $P^-$ be the sets of forward and backward edges of $P$ respectively. Then

$$|P^+| + |P^-| \geq m(t-1), \ \ |P^+| \leq (m-1)(|P^-|+1).$$

Hence

$$f(C,D) = \frac{|P^+|+1}{|P^-|+1} + 1 \leq m + \frac{1}{|P^-|+1} \leq m + \frac{1}{t}.$$

Therefore $\chi^*(G) \leq f(D) \leq m + \frac{1}{t}$. This completes the proof of Theorem 3.

**Corollary 4** *Suppose $m \geq 2, t \geq 1$ are integers. Then all $(m+1)$-critical graphs $G$ of girth at least $m(t-1)+2$ have star-chromatic numbers $\chi^*(G) \leq m + \frac{1}{t}$.*

We note that there are many critical graphs $G$ for which $\chi(G) = \chi^*(G)$. Suppose $G_1, G_2$ are $m_1$-critical and $m_2$-critical, respectively. Let $G_1 + G_2$ be the graph obtained from the disjoint union of $G_1$ and $G_2$ by joining every vertex of $G_1$ to every vertex of $G_2$. It is easy to see that $G_1+G_2$ is $(m_1+m_2)$-critical. However $\chi^*(G_1+G_2)=\chi(G_1+G_2)=m_1+m_2$ by Theorem 1 (cf. also [1, 6, 11]). This shows that, in some sense, the condition on the girth of $G$ in the above Corollary is important. In case $m=2$, the bound $m(t-1)+2$ on the girth of $G$ is sharp for $G$ to have star-chromatic number $\chi^*(G)\leq 2+\frac{1}{t}$.

On the other hand, the girth condition alone does not imply that a graph $G$ has star-chromatic number close to $\chi(G)-1$, as shown by Corollary 3. Thus, in some sense, the condition of being critical is also important.

All critical graphs $G$ with $\chi(G)=\chi^*(G)$ presently known do contain triangles. Having failed in the search for triangle-free $(m+1)$-critical graphs $G$ with $\chi^*(G)=m+1$, we propose the following conjecture:

**Conjecture 1** *If $G$ is a triangle-free critical graph, then $\chi^*(G)<\chi(G)$.*

# 5 Critical graphs of high connectivity

In this section, we answer another question asked by Abbott and Zhou [1] by constructing $m$-connected $(m+1)$-critical graphs with star-chromatic number $\chi^*(G)$ arbitrarily close to $\chi(G)-1$.

**Theorem 4** *For all integers $m\geq 4$ and $k\geq 2$ there is an $m$-connected, $(m+1)$-critical graph $H_m^k$ with $\chi^*(H_m^k)\leq m+\frac{1}{k}$.*

**Proof.** Given $m\geq 4, k\geq 2$, we construct a graph $H_m^k$ as follows:

The vertex set $V$ of $H_m^k$ is $\mathcal{Z}_{mk+1}$.

Let

$V^*=\{nk|n=1,2,\cdots,m\}$

$V_1=\{0,1,\cdots,k-1\}$, and

$V_n=\{(n-1)k+1,(n-1)k+2,\cdots,nk-2,nk-1\}$ for $n=2,\cdots,m$

Note that $V^*, V_1, V_2, \cdots, V_m$ is a partition of $V$, $|V^*| = m, |V_1| = k$ and $|V_n| = k-1$ for $n = 2, \cdots, m$.

A pair of vertices $(i, j)$ is an edge of $H^k_m$ if and only if

i) either $|i-j|_{mk+1} = k$,

ii) or $i \in V^*$ and $|i-j|_{mk+1} > k$.

The mapping $\gamma(i) = i$ is an $(mk+1, k)$-coloring of $H^k_m$. Therefore $\chi^*(H^k_m) \leq m + \frac{1}{k}$. In the following we prove that the graph $H^k_m$ is $(m+1)$-critical and $m$-connected.

First we show that $H^k_m$ is not $m$-colorable. Suppose to the contrary that there is an $m$-coloring $\Delta$ of $H^k_m$. Since the subset $V^* = \{nk : n = 1, 2, \cdots, m\}$ of $V$ induces a complete graph of order $m$, we may assume that $\Delta(nk) = n$ for $n = 1, 2, \cdots, m$. Then the vertex $k-1$ is forced to be colored by color 1, as $k-1$ is adjacent to all the vertices $nk$ for $n = 2, 3, \cdots, m$. This in turn forces the vertex $2k-1$ to be colored by color 2, as $2k-1$ is adjacent to $k-1$ and to all the vertices $nk$ for $n = 3, 4, \cdots, m$. Following the sequence $(k-1, 2k-1, \cdots, mk-1, k-2, 2k-2, \cdots, mk-2, \cdots, 2, k+2, \cdots, (m-1)k+2, 1)$ of vertices of $H^k_m$, we see that the color of each vertex is forced by the coloring of the previous colored vertices, and at the end there is no color available for the vertex 1. Therefore $H^k_m$ is not $m$-colorable.

Next we show that $H^k_m - e$ is $m$-colorable for any edge $e \in E(H^k_m)$. In the following arguments, we shall frequently use the fact that any $k$ consecutive vertices $\{i, i+1, i+2, \cdots, i+(k-1)\}$ (addition modulo $mk+1$) form an independent set of $H^k_m$.

Let $e = (i, j)$ be an edge of $H^k_m$. First we consider the case that at least one of the two end vertices of $e$, say $i$, does not belong to $V^*$. Then $i$ has degree $m-1$ in $H^k_m - e$, because $i$ has degree $m$ in $H^k_m$. There is an obvious partition of the set $V - i$ into $m$ consecutive segments so that each of them is of order $k$ and independent. In other words, $H^k_m - i$ is $m$-colorable. Since $i$ has degree $m-1$ in $H^k_m - e$, this $m$-coloring of $H^k_m - i$ can be extended to an $m$-coloring of $H^k_m - e$.

We assume now that both $i, j \in V^*$. If $i = nk$ and $j = (n+1)k$, then let $W = \{nk, nk+1, \cdots, (n+1)k\}$ and partition the set $V - W$ into $m-1$

consecutive segments, each of order $k$. This gives an $m$-coloring of $H_m^k - e$.

Assume now that $i = n_1 k$, $j = n_2 k$ and $n_1 \leq n_2 - 2$. Since $m \geq 4$, there is an $n'$ such that $n_1 + 1 \leq n' \leq n_2$ and $V_{n'} \cup V_1$ is an independent set of $H_m^k$. There is a natural way to partition the set $V - (V_{n'} \cup V_1 \cup \{n_1 k, n_2 k\})$ into $m-2$ consecutive segments, each of them is of order $k$ and independent. Thus we can color these $m-2$ consecutive segments by $m-2$ colors, color $V_1 \cup V_{n'}$ by 1 color, and color $n_1 k, n_2 k$ by one color. Therefore $H_m^k - e$ is $m$-colorable.

To finish the proof of Theorem 4, it remains to show that the graphs $H_m^k$ are $m$-connected. By Menger's Theorem, we need only show that, for any two distinct vertices $i, j$ of $H_m^k$, there are $m$ internally disjoint paths connecting $i$ and $j$.

Any two vertices $i$ and $j$ $(i < j)$ of $H_m^k$ have either $m-4$ common neighbors in $V^*$, or $m-3$ common neighbors in $V^*$, or $m-2$ common neighbors in $V^*$.

Suppose $i$ and $j$ have exactly $m-4$ common neighbors in $V^*$. Then these common neighbors give $m-4$ pairwise disjoint paths connecting $i$ and $j$. Also it is easy to see that we must have $i, j \in V \setminus (V^* \cup \{0, k-1\})$ and $|i-j|_{mk+1} \geq 2k$. Let $i_1 k, i_2 k$ and $j_1 k, j_2 k$ be the vertices in $V^*$ not adjacent to $i$ and $j$, respectively. Without loss of generality we may assume $i_1 k < i < i_2 k$ and $j_1 k < j < j_2 k$, and hence $i - k < i_1 k < i < i_2 k < i + k$. If $j = i + 2k$ then four additional paths between $i$ and $j$ are $P_1 = [i, i+k, j]$; $P_2 = [i, i-k, i_2 k, j]$; $P_3 = [i, j_2 k, i_1 k, j]$ and $P_4 = [i, j_1 k, j+k, j]$. Otherwise the four paths are $P_1 = [i, i+k, i_1 k, j]$; $P_2 = [i, i-k, i_2 k, j]$; $P_3 = [j, j+k, j_1 k, i]$ and $P_4 = [j, j-k, j_2 k, i]$. The cases that $i$ and $j$ have exactly $m-3$ or $m-2$ common neighbors in $V^*$ are treated similarly, and we omit the details. ∎

We note that Theorem 4 remains true for $m = 2, 3$. For $m = 2$, the odd circuits are suitable critical graphs. For $m = 3$ and $k \geq 3$, the graphs $H_k^3 - e^*$ where $e^* = (k, 3k)$ are suitable graphs, while the verification is similar to the proof of Theorem 4.

Our result in the previous section says that critical graphs $G$ with large girth necessarily have star-chromatic number $\chi^*(G)$ close to $\chi(G) - 1$. The example in this section shows that a critical graph $G$ with large clique number may also have star-chromatic number $\chi^*(G)$ close to $\chi(G) - 1$.

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# References

[1] H. L. Abbott and B. Zhou, *The star chromatic number of a graph*, J. Graph Theory **17** (1993), 349-360

[2] B. Bollobás and N. Sauer, *Uniquely colorable graphs with large girth*, Can. J. Math. 28(1976) 1340-1344.

[3] J. A. Bondy and P. Hell, *A note on the star chromatic number*, J. Graph Theory **14** (1990), 479-482.

[4] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Macmillan, London, 1976

[5] W. Deuber and X. Zhu, *Circular coloring of weighted graphs*, manuscript, 1994.

[6] G. Gao, E. Mendelsohn and H. Zhou, *Computing star chromatic number from related graph invariants*, JCMCC, to appear.

[7] G. Gao and X. Zhu, *Star extremal graphs and the lexicographic product*, Disc. Math., to appear.

[8] L. A. Goddyn, M. Tarsi and C. Q. Zhang, *On $(k,d)$-colorings and fractional nowhere-zero flows*, J. Graph Theory, to appear.

[9] D. R. Guichard, *Acyclic graph coloring and the complexity of the star chromatic number*, J. Graph Theory 17(1993) 129-134.

[10] A. Vince, *Star chromatic number*, J. Graph Theory, 12(1988) 551-559.

[11] X. Zhu, *Star chromatic numbers and products of graphs*, J. Graph Theory **16** (1992), 557-569

[12] X. Zhu *Uniquely $H$-colorable graphs with large girth*, manuscript, 1994.