

Star chromatic numbers and products of graphs*

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Abstract

The star-chromatic number of a graph, a concept introduced by Vince, is a natural generalization of the chromatic number of a graph. We point out an alternate definition of the star-chromatic number, which sheds new light on the relation of the star-chromatic number and the ordinary chromatic number. This new point of view allows us to answer several problems posed by Vince. We then study the star-chromatic number from the perspective of graph homomorphisms and of graph products.

1 Introduction

In what follows graphs are finite simple graphs unless otherwise specified. Let k and d be positive integers such that $k \geq 2d$. A (k, d) -coloring of a graph $G = (V, E)$ is a mapping $c : V \mapsto Z_k = \{0, 1, \dots, k - 1\}$ such that, for each edge $uv \in E$, $|c(u) - c(v)|_k \geq d$, where $|x|_k = \min\{|x|, k - |x|\}$. Thus a $(k, 1)$ -coloring of G is just a usual k -coloring of G . The *star-chromatic number* of G , $\chi^*(G)$, is defined as

$$\chi^*(G) = \inf\{k/d : G \text{ has a } (k, d)\text{-coloring}\}.$$

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If $|V(G)| = n$ and G has a (k, d) -coloring, then there exist integers k' and d' such that $k'/d' \leq k/d$ and $k' \leq n$ and G has a (k', d') -coloring, [2, 14]. Therefore to calculate $\chi^*(G)$, it is enough to consider those pairs k, d such that $2d \leq k \leq n = |V(G)|$. Thus

$$\chi^*(G) = \min\{k/d : G \text{ has a } (k, d)\text{-coloring}\}.$$

We introduce the circle-chromatic number of a graph and prove that for any graph G the star-chromatic number of G is equal to its circle-chromatic number. This alternate definition of the star-chromatic number allows us to determine the star-chromatic number of some graphs. In particular, we find an infinite family of edge critical 4-chromatic planar graphs which also have star-chromatic number 4, an infinite family of planar graphs which are different from odd cycles and have star-chromatic number strictly between 2 and 3. This answers some problems posed by Vince in [14]. The star-chromatic number of a graph can also be studied from the point of view of graph homomorphisms. In the study of the ordinary chromatic number, the complete graphs K_n play a special role, i.e., a graph G is n -colorable if and only if G is homomorphic to K_n . There are also similar special graphs in the study of the star-chromatic number. In fact, for each rational number $k/d \geq 2$ there is a special graph G_k^d such that a graph G is k/d -colorable if and only if G is homomorphic to G_k^d , [2, 14]. We study the similarities and differences of these special graphs G_k^d and the complete graphs K_n . We show that the graphs G_k^d are star-vertex-critical, but not star-edge-critical. Finally we study the star-chromatic number of various products of graphs. In particular, we prove that if G contains at least one edge then the star-chromatic number of the wreath product $G[H]$ is equal to the star-chromatic number of $G[K_n]$, where $n = \chi(H)$.

2 An alternate point of view

First we give an alternate definition of the concept of

Definition 1 *Let C be a circle in R^2 of length 1, and let $r \geq 1$ be any real number. Denote by $C^{(r)}$ the set of all open intervals of C of length $1/r$. An r -circle-coloring of a graph G is a mapping c from $V(G)$ to $C^{(r)}$ such*

that $c(x) \cap c(y) = \emptyset$ whenever $(x, y) \in E(G)$. If such an r -circle-coloring exists, we say that G is r -circle-colorable. The circle-chromatic number of G is $\chi^c(G) = \inf\{r : G \text{ is } r\text{-circle colorable}\}$.

Theorem 2.1 For any graph G we have $\chi^*(G) = \chi^c(G)$.

Proof. Suppose $c : V(G) \mapsto Z_k$ is a (k, d) -coloring of G . Put k points, p_0, p_1, \dots, p_{k-1} , evenly spaced on the unit circle C . For each vertex $x \in V(G)$, let $c'(x)$ be the interval of C of length d/k centered at $p_{c(x)}$. It is easy to verify that c' is a k/d -circle-coloring of G . Therefore $\chi^c(G) \leq \chi^*(G)$.

To prove that $\chi^*(G) \leq \chi^c(G)$, we first show that if $r = k/d$ is a rational number and G is r -circle-colorable then G is (k, d) -colorable. Suppose $c : V(G) \mapsto C^{(r)}$ is an r -circle-coloring of G . Choose any point of C and label it 0. A point x of C is called a *rational* point if the clockwise distance between 0 and x along the circle is a rational number. First we see that we can assume that each interval $c(g)$ is centered at a rational point. If $c(g)$ is not centered at a rational point, we can shift it to an interval centered at a rational point without creating new intersections between the intervals. If one of the end points of $c(g)$ is not an endpoint of any other interval $c(g')$, the shifting can be easily done since G is finite. If both end points of $c(g)$ are end points of other intervals, we consider the end points of their union and so on, obtaining a set of intervals which can then be shifted together. Since the distances between the centers are rational, if the center of $c(g)$ is shifted to a rational point then the centers of every interval involved in this shifting are also shifted to rational points.

Let $r(g)$ be the distance between 0 and the center of $c(g)$, which by assumption is a rational number. Let $p = mk$ be the common denominator of $1/k$ and all the $r(g), g \in V(G)$. Then the mapping $c' : V(G) \mapsto Z_p$ defined as $c'(g) = p \cdot r(g)$ is a (p, md) -coloring of G which then gives a (k, d) -coloring of G . Therefore $\chi^*(G) \leq k/d$.

Suppose $\chi^c(G) = r$ and $\{r_i : i = 1, 2, \dots\}$ is a sequence of rational numbers such that $r_i \geq r$ and $\lim_{i \rightarrow \infty} r_i = r$. Obviously G is r_i -circle colorable (given any r -circle coloring c of G , just shrink each interval $c(g)$ into an interval of length $1/r_i$ we get an r_i -circle coloring of G) and therefore $r_i \geq \chi^*(G)$ which implies that $r \geq \chi^*(G)$. The theorem is now proved. \square

If we replace the circle C in the definition of circle-coloring of a graph by an interval I of length 1, we define an interval-coloring of a graph and, by analogy, its interval-chromatic number.

Definition 2 *Let I be an interval of length 1 and $r \geq 1$ be any real number. Denote by $I^{(r)}$ the set of open intervals of I of length $1/r$. An r -interval-coloring of a graph G is a mapping c from $V(G)$ to $I^{(r)}$ such that $c(x) \cap c(y) = \emptyset$ whenever $(x, y) \in E(G)$. If such an r -interval-coloring exists, we say that G is r -interval-colorable. The interval-chromatic number of G is $\chi^I(G) = \inf \{ r : G \text{ is } r\text{-interval colorable} \}$.*

Theorem 2.2 *For any graph G we have $\chi^I(G) = \chi(G)$.*

The proof of this theorem is straightforward from the definition. This new point of view of looking at the chromatic number and the star-chromatic number of a graph gives us a clearer picture of the relation between them. It allows us to answer several questions posed by Vince in [14].

One of the problems posed by Vince [14] was to characterize those graphs G for which $\chi^*(G) = \chi(G)$. From the definitions of a circle-coloring and an interval-coloring, it is easy to see that the following condition is both necessary and sufficient for $\chi^*(G) = \chi(G)$: for any real number r , if G is r -circle-colorable then G has an r -circle-coloring c such that some point $x \in C$ is not covered by any of the intervals $c(g)$. This is because such an r -circle coloring induces an r -interval coloring (and any r -interval coloring induces such an r -circle coloring). We now use this characterization under which $\chi^*(G) = \chi(G)$.

We call a vertex v of a graph G *universal* if v is adjacent to every other vertex of G .

Theorem 2.3 *If G has a universal vertex then $\chi^*(G) = \chi(G)$.*

Proof. Suppose c is an r -circle-coloring of G , for some rational number r . If v is $c(v)$ is disjoint from $c(g)$ for all $g \in V(G)$, and hence the end points of $c(v)$ are not covered by any of the intervals $c(g)$. \square

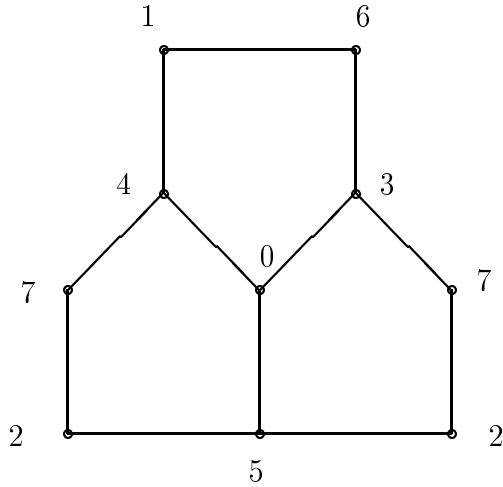
From the proof we see that the condition can be weakened - it suffices that v is adjacent to every other vertex of G but one. This is included in the following Corollary:

Corollary 1 *Suppose that G is a graph with $\chi(G) = n$ and that G has a vertex whose neighbours induce a subgraph of chromatic number $n - 1$. Then $\chi^*(G) = \chi(G)$. In particular if $\chi(G) = n$ and G contains a complete subgraph of order n then $\chi^*(G) = \chi(G)$.*

Theorem 2.3 allows us to determine the star-chromatic number of the odd wheel W_{2k+1} , that is a cycle of length $2k + 1$ to which a universal vertex has been added. Since $\chi(W_{2k+1}) = 4$ we know that $\chi^*(W_{2k+1}) = 4$. This provides a negative answer to a question asked by Vince in [14], namely whether or not all edge-critical 4-chromatic planar graphs have star-chromatic number less than 4. To see this, recall that the odd wheels are edge-critical, planar and 4-chromatic Vince [14] also asked for some infinite families of planar graphs with $3 < \chi^* < 4$. We construct an infinite family of such graphs as follows:

Suppose $W_{2k+1}, W_{2k'+1}$ are two odd wheels, ab, cd are edges of $W_{2k+1}, W_{2k'+1}$ which are not incident to the centers of $W_{2k+1}, W_{2k'+1}$ respectively. Let $G_{k,k'}$ be the graph obtained from the disjoint union of W_{2k+1} and $W_{2k'+1}$ by identifying a and c , deleting the edges ab and cd , adding an edge bd . i.e., $G_{k,k'}$ is a Hajos sum of W_{2k+1} and $W_{2k'+1}$, [1]. Then $3 < \chi^*(G_{k,k'}) \leq 7/2 < 4$. To see that $\chi^*(G_{k,k'}) \leq 7/2$, it is enough to verify that $\chi^*(G_{1,1}) \leq 7/2$ because for any other pair of (k, k') , $G_{k,k'}$ is homomorphic to $G_{1,1}$. As we will see in the next section, this implies that $\chi^*(G_{k,k'})$ is at most $\chi^*(G_{1,1})$. A $(7, 2)$ -coloring of $G_{1,1}$ is easy to find. To see that all $\chi^*(G_{k,k'}) > 3$, it is enough to observe that the Hajos sum of two 4-chromatic graphs still has chromatic number 4, [1].

Yet another question of Vince [14] we can answer is this: Are there planar graphs other than odd cycles which have star-chromatic number strictly between 2 and 3? The following example gives an infinite family of such



graphs. Fig. 1. Graph G and an $(8, 3)$ -colouring of G

Let G be the graph in Figure 1. It can be checked that $\chi^*(G) = 8/3$ and G is neither an odd cycle nor homomorphically equivalent to an odd cycle. If we subdivide the three thick edges into any odd number of edges, the resulting graph is still a planar graph with $2 < \chi^* < 3$. Such a graph has star-chromatic number greater than 2 because it has chromatic number at least number less than 3 because it is homomorphic to G and hence has star-chromatic number at most $\chi^*(G) = 8/3$. Also such a graph is neither an odd cycle nor homomorphically equivalent to an odd cycle.

This example can be easily modified to give some other planar graphs which have star-chromatic number strictly between 2 and 3. It seems that such graphs are abundant. However, we still do not know any characterization of planar graphs which has star-chromatic number strictly between 2 and 3.

3 Graph homomorphisms

When we write a rational number in the form k/d , we always assume that k and d are coprime integers. For a rational number $k/d \geq 2$, the graph G_k^d has vertex set $V(G) = \mathbb{Z}_k = \{0, 1, 2, \dots, k-1\}$ and edge set $E(G) =$

$\{(i, j) : \min\{|i - j|, k - |i - j|\} \geq d\}$. Suppose G and H are graphs. A homomorphism of G to H is a mapping f from $V(G)$ to $V(H)$ such that $(f(x), f(y)) \in E(H)$ whenever $(x, y) \in E(G)$. We say G is homomorphic to H , and write $G \rightarrow H$, if there is a homomorphism of G to H . It was proved in [14] that a graph G is (k, d) -colorable if and only if G is homomorphic to G_k^d . If G is homomorphic to H and H is homomorphic to G_k^d , then by composition of homomorphisms, we know that G is also homomorphic to G_k^d . Hence if G is homomorphic to H then the star-chromatic number of G is less than or equal to the star-chromatic number of H .

In the discussion of star-chromatic numbers, these graphs G_k^d take the role of the complete graphs as in the discussion of chromatic numbers. For instance we have G_k^d is homomorphic to $G_{k'}^{d'}$ if and only if $k/d \leq k'/d'$. This implies that a graph G is (k, d) -colorable if and only if $k/d \geq \chi^*(G)$, [2, 14]. Therefore we have $\chi(G) - 1 < \chi^*(G) \leq \chi(G)$ for any graph G , because $\chi^*(G) \leq \chi(G) - 1$ would imply that G is $\chi(G) - 1$ -colorable.

Despite the similarities of K_n and G_k^d , some trivial properties of the complete graphs K_n are not completely trivial and sometimes not even true for the graphs G_k^d .

Theorem 3.1 *Each graph G_k^d is star-vertex-critical, in the sense that for any vertex $v \in V(G)$, $G_k^d - v$ has star-chromatic number strictly less than k/d .*

Proof. We employ the idea used in [2]. Since G_k^d is vertex transitive we can assume $v = d$. To prove that $G_k^d - d$ has star-chromatic number strictly less than k/d , it suffices to give a (k', d') -coloring of $G_k^d - d$ for some k', d' such that $k'/d' < k/d$. Let α be the smallest positive integer such that $\alpha d = 1 \pmod{k}$. Let $c : V(G_k^d) \setminus \{d\} \mapsto Z_{k-\alpha}$ be the mapping defined as

$$c(i) = i - |\{0 < t \leq \alpha : td \leq i\}|$$

where the multiplication td is in the field Z_k and the order $td \leq i$ is the order of natural numbers. It is easy to verify that c gives a $(k - \alpha, d - \frac{\alpha d - 1}{k})$ -coloring of $G_k^d - d$ while

$$\frac{k - \alpha}{d - \frac{\alpha d - 1}{k}} = \frac{k(k - \alpha)}{d(k - \alpha) + 1} < \frac{k}{d}. \quad \square$$

Corollary 2 *Suppose G is a graph such that $\chi^*(G) = k/d$ and $c : V(G) \mapsto Z_k$ is a (k, d) -coloring of G . Then c is an onto mapping. In particular we have $|V(G)| \geq k$.*

Proof. A (k, d) -coloring of G is just a homomorphism of G to G_k^d . If c is not an onto mapping, then c is a homomorphism of G to $G_k^d - v$ for some vertex $v \in V(G_k^d)$. Therefore $\chi^*(G) \leq \chi^*(G_k^d - v) < k/d$, contradicting the assumption that $\chi^*(G) = k/d$. \square

This Corollary is useful when one tries to determine the star-chromatic number of a small graph. In general, it is more difficult to determine the star-chromatic number of a graph than number. (Once we know the star-chromatic number $\chi^*(G)$ of a graph G , then we know that $\chi(G) = \lceil \chi^*(G) \rceil$. In this sense, the star-chromatic number of a graph captures its structure more precisely than the ordinary chromatic number). By the above Corollary, if $\chi^*(G) = k/d$ then $k \leq |V(G)|$ and any (k, d) -coloring of G must use all the k colors. Furthermore if we know the chromatic number of G then we know that k/d is strictly greater than $\chi(G) - 1$. When G is small, this usually leaves very few rational numbers as candidates for $\chi^*(G)$.

Since the complete graphs K_n are also edge critical, one might expect that G_k^d are also edge critical. However this is not the case. Let us take the graph G_7^2 , delete the edge $(0, 3)$ from this graph, denote the resulting graph by H . It is easy to see H is not 3-colorable. Therefore $3 < \chi^*(H) \leq \chi^*(G_7^2) = 7/2$. Suppose $\chi^*(H) = p/q$. Then $q \geq 2$. Since $p \leq 7$ by the Corollary above, we must have $p = 7$ and $q = 2$, for otherwise p/q would be less than or equal to 3 which is impossible. Therefore G_7^2 is not edge critical. However the graphs G_k^d are edge saturated, i.e., adding any edge to G_k^d will increase the H is the graph obtained from G_k^d by adding an edge and $\chi^*(H) = k/d$. Let $c : V(H) \mapsto V(G_k^d)$ be a homomorphism. By the Corollary above, c must be onto and hence is one to one. This is impossible since H has one more edge than G_k^d .

The graphs G_k^d can also be used to construct some other interesting graphs, as shown in the proof of the next theorem and in the remark after the proof of Theorem 4.2. For any graph G and any vertex v , we know that $\chi(G - v) \geq \chi(G) - 1$. For the star-chromatic number of a graph we have a somewhat similar property.

Theorem 3.2 For any graph G and any vertex v we have $\chi^*(G - v) > \chi^*(G) - 2$. Moreover this is the best possible in the sense that for any $\delta > 0$ and any integer n , there exists a graph H and a vertex v such that $\chi^*(H) \geq n$ and $\chi^*(H - v) \leq \chi^*(H) - 2 + \delta$.

Proof. Since $\chi(G) \geq \chi^*(G) > \chi(G) - 1$, we have $\chi^*(G - v) > \chi(G - v) - 1 \geq \chi(G) - 2 \geq \chi^*(G) - 2$. To prove the other half of the theorem, let k be an integer such that $1/k \leq \delta$ and let H be the graph obtained by adding a universal vertex v to G_{nk+1}^k . By Theorem 2.3, $\chi^*(H) = \chi(H) = n + 2$ while $\chi^*(H - v) = \chi^*(G_{nk+1}^k) = n + 1/k$. Therefore $\chi^*(H - v) \leq \chi^*(H) - 2 + \delta$. \square

Although the deletion of some vertex of a graph could decrease its star-chromatic number graph has a vertex such that the deletion of this vertex decrease its star-chromatic number

Another interesting problem involving the deletion of a vertex is the following: which graphs have the property that the deletion of any vertex will decrease its star-chromatic number

Suppose G is a graph. An *extremely stable set* of G is a nonempty subset S of $V = V(G)$ such that for any $v \in V \setminus S$, v is either adjacent to every vertex of S or adjacent to no vertex of S . Based on a few examples which have the property that $\chi^*(G - v) = \chi^*(G) - 1$ for each vertex v , we pose the following question:

Is it true that if $|V(G)| > 2$ and $\chi^*(G - v) = \chi^*(G) - 1$ for each vertex v , then G has a non-trivial extremely stable set S , (i.e., $2 \leq |S| < |V(G)|$)?

4 Star chromatic number of graph products

This section studies the star-chromatic number of the wreath product, the categorical product and the Cartesian product of graphs.

Let G and H be graphs. The wreath product of G and H , denoted by $G[H]$ has vertex set $V(G[H]) = \{(g, h) : g \in V(G) \text{ and } h \in V(H)\}$ and edge set $E(G[H]) = \{((g, h), (g', h')) : \text{either } (g, g') \in E(G) \text{ or } g = g' \text{ and } (h, h') \in E(H)\}$. (The wreath product is also known as the lexicographic product, [7]). This section discusses the relation between $\chi^*(G)$, $\chi(H)$ and $\chi^*(G[H])$.

First we show that $\chi^*(G[H]) \leq \chi^*(G)\chi(H)$ for any graphs G and H . Suppose $\chi(H) = n$ and $\chi^*(G) = k/d$. Let $\phi : V(H) \mapsto \{0, 1, \dots, n-1\}$ be an n -coloring of H and $\psi : V(G) \mapsto Z_k$ be a (k, d) -coloring of G . Define $f : V(G[H]) \mapsto Z_{kn}$ as follows:

$$f((g, h)) = \psi(g) + \phi(h)k.$$

Then f is a (kn, d) -coloring of $G[H]$.

It is somewhat surprising that (if G contains at least one edge) the star-chromatic number of $G[H]$ only depends on the graph G and the chromatic number of H .

Theorem 4.1 *If G contains at least one edge and $\chi(H) = n$ then $\chi^*(G[H]) = \chi^*(G[K_n])$.*

Instead of proving this theorem directly, we prove a stronger result. It is obvious that Theorem 6 follows from Theorem 7 below.

Definition 3 *Given graphs G and H and a vertex g of G , we define $G(g, H)$ to be the graph obtained from G by replacing g with a copy of H . (In other words, we put a copy of H in place of g and connect each vertex of H to every neighbour of g .)*

Theorem 4.2 *Suppose G and H are graphs and $g \in V(G)$ is not an isolated vertex. If $\chi(H) = n$ then $\chi^*(G(g, H)) = \chi^*(G(g, K_n))$.*

Proof. There is an obvious homomorphism of $G(g, H)$ to $G(g, K_n)$. Therefore $\chi^*(G(g, H)) \leq \chi^*(G(g, K_n))$. Suppose $\chi^*(G(g, H)) = k/d$ and $c : V(G(g, H)) \mapsto Z_k$ is a (k, d) -coloring of $G(g, H)$. We will prove that $G(g, K_n)$ is also (k, d) -colorable, which will imply that $\chi^*(G(g, K_n)) = \chi^*(G(g, H))$. For convenience we assume that $V(K_n) = \{k_1, k_2, \dots, k_n\}$. To prove that $G(g, K_n)$ is (k, d) -colorable, it is enough to find a mapping $c' : V(K_n) \mapsto c(V(H))$ such that $|c'(k_i) - c'(k_j)|_k \geq d$ for any two distinct vertices k_i and k_j of K_n . We can extend this c' to a (k, d) -coloring of $G(g, K_n)$ by letting $c'(v) = c(v)$ for all $v \notin V(K_n)$.

Let $g' \in V(G)$ be a vertex such that $(g, g') \in E(G)$. Then g' is adjacent to every vertex of H in $G(g, H)$. Without loss of generality assume that $c(g') = 0$. Define $c'(k_1) = \min\{c(h) : h \in V(H)\}$. If $c'(k_i)$ has been defined, we define $c'(k_{i+1}) = \min\{c(h) : h \in V(H) \text{ and } c(h) \geq c'(k_i) + d\}$. To see that c' is well defined, it is enough to show that for $i \leq n - 1$, the sets $S = \{h \in V(H) : c(h) \geq c'(k_i) + d\}$ is not empty. Suppose $S = \emptyset$. We define a mapping $\Delta : V(H) \mapsto \{1, 2, \dots, i\}$ by $\Delta(h) = j$ if and only if $c'(k_j) \leq c(h) < c'(k_j) + d$. It is easy to see that Δ is an i -coloring of H , contradicting the assumption that $\chi(H) = n$.

To see that $|c'(k_i) - c'(k_j)|_k \geq d$, it is enough to check that $k - |c'(k_1) - c'(k_n)| \geq d$. This is obvious because $c(g') = 0$ and g' is adjacent to every vertex of H . Therefore $c'(k_1) \geq d$ and $c'(k_n) \leq k - d$. Thus Theorem 7 is proved. \square

Observe that Theorem 2.3 can now be proved as a corollary of Theorem 7. If v is a universal vertex of an n -chromatic graph G , then G is isomorphic to $K_2(k_1, G-v)$, where K_2 is a complete graph of order 2 with vertex set $\{k_1, k_2\}$. Therefore $\chi^*(G) = \chi^*(K_2(k_1, G-v)) = \chi^*(K_2(k_1, K_{n-1})) = \chi^*(K_n) = n = \chi(G)$. More generally, if we replace one vertex of K_2 by one graph H and replace the other vertex by another graph H' , the resulting graph G , i.e., those graphs G whose complement is disconnected, also has its star-chromatic number equal to its chromatic number. The latter result was first proved by Gao, Mendelsohn and Zhou, [6].

We have shown that $\chi^*(G[H]) \leq \chi^*(G)\chi(H)$. Using Theorem 6 we can show that equality holds for some graphs. First if G is a complete graph, then equality holds. In the following we show that equality holds if $G = C_5$.

Theorem 4.3 *Let G be an n -chromatic graph. Then*

$$\chi^*(C_5[G]) = \chi^*(C_5[K_n]) = n\chi^*(C_5) = 5n/2.$$

Proof. By Theorem 6, we only need to show that $\chi^*(C_5[K_n]) = 5n/2$. First suppose n is odd. Then it is easy to verify that $\chi(C_5[K_n]) = \frac{5n+1}{2}$. Therefore $\chi(C_5[K_n]) - 1 = \frac{5n-1}{2} < \chi^*(C_5[K_n]) \leq n\chi^*(C_5) = 5n/2$. Thus $\chi^*(C_5[K_n])$ is not an integer. Suppose $\chi^*(C_5[K_n]) = p/q$. Then $q \geq 2$. Since $p \leq 5n = |V(C_5[K_n])|$ and $p/q > \frac{5n-1}{2}$, we must have $p/q = 5n/2$.

Now suppose n is even. It is also easy to verify that $\chi(C_5[K_n]) = 5n/2$. Therefore $\chi(C_5[K_n]) - 1 = \frac{5n-2}{2} < \chi^*(C_5[K_n]) \leq 5n/2$. Suppose $\chi^*(C_5[K_n]) = p/q < \chi(C_5[K_n]) = 5n/2$. Then $q \geq 2$. Since $p \leq 5n$, we must have $p = 5n - 1$ and $q = 2$ in order to satisfy the above inequalities. Let $c : V(C_5[K_n]) \mapsto Z_{5n-1}$ be a $(5n - 1, 2)$ -coloring of $C_5[K_n]$. Since $|V(C_5[K_n])| = 5n$, there are two distinct vertices x, y of $C_5[K_n]$ such that $c(x) = c(y)$. Obviously $(x, y) \notin E(C_5[K_n])$.

Let G be the graph obtained from $C_5[K_n]$ by contracting x and y into a single vertex. Then c also gives a $(5n - 1, 2)$ -coloring of G . However the new vertex of G which is the contraction of x and y is adjacent to every other vertex of G . Hence by Theorem 4, $\chi^*(G) = \chi(G) \geq \chi(C_5[K_n]) = 5n/2$. This is a contradiction. \square

We note that $\chi^*(G[H])$ can be strictly less than $\chi^*(G)\chi(H)$. Let G be the graph obtained by adding an v to G_{nk+1}^k . We know that $\chi^*(G) = \chi(G) = n + 2$. So $\chi^*(G)\chi(K_k) = (n + 2)k$. However

$$\begin{aligned} \chi^*(G[K_k]) &\leq \chi(G[K_k]) \leq \chi(G_{nk+1}^k[K_k]) + k \\ &= \lceil \chi^*(G_{nk+1}^k[K_k]) \rceil + k \leq \lceil \chi^*(G_{nk+1}^k)k \rceil + k = (n + 1)k + 1. \end{aligned}$$

Therefore $\chi^*(G)\chi(K_k) - \chi^*(G[K_k]) \geq k - 1$.

The categorical product of graphs G and H , $G \otimes H$, has vertex set $V(G \otimes H) = \{(g, h) : g \in V(G), h \in V(H)\}$ and edge set $E(G \otimes H) = \{((g, h), (g', h')) : (g, g') \in E(G), (h, h') \in E(H)\}$. (The categorical product is also known as weak direct product, [7]). If $c : V(G) \mapsto Z_k$ is a (k, d) -coloring of G , then $c' : V(G \otimes H) \mapsto Z_k$ defined as $c'((g, h)) = c(g)$ is obviously a (k, d) -coloring of $G \otimes H$. Therefore $\chi^*(G \otimes H) \leq \min\{\chi^*(G), \chi^*(H)\}$.

If the star-chromatic number is replaced by inequality is still true, i.e., $\chi(G \otimes H) \leq \min\{\chi(G), \chi(H)\}$. It is a long standing open problem whether equality holds. Hedetniemi [9] conjectured that equality holds for any two graphs. Or equivalently, he conjectured that $\chi(G) > n$ and $\chi(H) > n$ imply

This conjecture has only been verified for $n = 2$ and 3 . While the case $n = 2$ is trivial, the case $n = 3$ is very difficult, [5]. Here we generalize this conjecture to the star-chromatic number.

Conjecture 1 *If $\chi^*(G) > r$ and $\chi^*(H) > r$ then $\chi^*(G \otimes H) > r$.*

Although this conjecture is much stronger than Hedetniemi's conjecture, it is true for infinitely many rational numbers. The following theorem is just a new interpretation of a result of Haggkvist et al., [7]

Theorem 4.4 *Suppose $k \geq 1$ is an integer, $\chi^*(G) > 2 + \frac{1}{k}$ and $\chi^*(H) > 2 + \frac{1}{k}$. Then $\chi^*(G \otimes H) > 2 + \frac{1}{k}$.*

Proof. We know that $\chi^*(G) > 2 + \frac{1}{k}$ if and only if G admits no homomorphism to C_{2k+1} . It was proved in [7] that if G admits no homomorphism to C_{2k+1} and H admits no homomorphism to C_{2k+1} then $G \otimes H$ admits no homomorphism to C_{2k+1} . This proves the theorem. \square

Although Hedetniemi's conjecture has only been verified for $n = 2, 3$, if extra conditions are put on the graphs in consideration the conjecture is true in many special cases [3, 4, 7, 8, 13, 15]. It would be interesting if some of them could be generalized to the stronger conjecture about star-chromatic number.

The definition of the infinite graphs, although in this case the star-chromatic number can be any real number greater than or equal to 2. For a given real number $r \geq 2$, let $\{r_i : i = 1, 2, \dots\}$ be a sequence of rational numbers such that $r_i \leq r$ and $\lim_{i \rightarrow \infty} r_i = r$. Let G_i be a finite graph of star-chromatic number r_i . Then the disjoint union of G_i has star-chromatic number r .

If infinite multiplication is allowed, Miller [12] observed that Hedetniemi's conjecture is not true. The graph $C_3 \otimes C_5 \otimes C_7 \dots$ has chromatic number 2 while each of the factors C_{2k+1} has chromatic number 3. This example does not work for the star-chromatic number because $\inf\{\chi^*(C_{2k+1}) : k = 1, 2, 3, \dots\} = 2$. Replacing the C_{2k+1} 's in the above example by a graph G_k of chromatic number at least 4 and girth at least k , the graph $G_1 \otimes G_2 \otimes G_3 \otimes \dots$ has star-chromatic number 2, because it contains no cycle. On the other hand, each of the factor graph has chromatic number at least 4, hence has star-chromatic number greater than 3. This shows that infinite multiplication can not be allowed even in our conjecture.

The following theorem is a generalization of a result of Hajnal [8] about chromatic number.

Theorem 4.5 *Suppose G and H are graphs (finite or infinite) such that $\chi^*(G) > r$ and $\chi(H)$ is infinite. Then $\chi^*(G \otimes H) > r$.*

Proof. Without loss of generality, we can assume that $r = k/d$ is a rational number, (for otherwise we can replace r by any rational number strictly between $\chi^*(G)$ and r). By the compactness theorem, G has a finite subgraph which has star chromatic number greater than r . So we can assume that G is a finite graph. If to the contrary $G \otimes H$ is (k, d) -colorable, let c be a homomorphism of $G \otimes H$ to G_k^d . We associate to each vertex $h \in V(H)$ a mapping $f_h : V(G) \mapsto V(G_k^d)$ by the formula $f_h(g) = c(g, h)$. Since $\chi(H) = \infty$ and there are only finitely many mappings from $V(G)$ to $V(G_k^d)$, there exist $h, h' \in V(H)$ such that $(h, h') \in E(H)$ and $f_h = f_{h'}$. Since c is a homomorphism of $G \otimes H$ to G_k^d , this implies that f_h is a homomorphism of G to G_k^d . This contradicts the assumption that G is not (k, d) -colorable. \square

The Cartesian product $G \times H$ of G and H has vertex set $V(G \times H) = \{(g, h) : g \in V(G), h \in V(H)\}$ and edge set $E(G \times H) = \{((g, h), (g', h')) : g = g' \text{ and } (h, h') \in E(H) \text{ or } (g, g') \in E(G) \text{ and } h = h'\}$. The relation between $\chi^*(G \times H)$ and $\chi^*(G), \chi^*(H)$ is simple, namely $\chi^*(G \times H) = \max\{\chi^*(G), \chi^*(H)\}$. This is because for any (k, d) -coloring f of G and any (k, d) -coloring g of H , the mapping $\Delta : V(G \times H) \mapsto Z_k$ defined by $\Delta((x, y)) = f(x) + g(y) \pmod{k}$ is a (k, d) -coloring of $G \times H$. On the other hand, $G \times H$ contains G and H as subgraphs. Therefore $\chi^*(G \times H) \geq \max\{\chi^*(G), \chi^*(H)\}$. This gives us an easy way to construct graphs with star-chromatic number equal to chromatic number. If $\chi^*(G) = \chi(G) \geq \chi(H)$, then we have $\chi^*(G \times H) = \chi(G \times H) = \chi(G)$.

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Note added in proof: G. Gao has recently given a negative answer to the question at the end of Section 3.

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