Graphs of large girth with prescribed partial circular colourings

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Abstract

This paper completes the constructive proof of the following result: Suppose $p/q \geq 2$ is a rational number, A is a finite set and $f_1, f_2, \cdots, f_n$ are mappings from A to $\{0, 1, \cdots, p-1\}$. Then for any integer g, there is a graph $G = (V, E)$ of girth at least g with $A \subset V$, such that G has exactly n (p, q)-colourings (up to equivalence) $g_1, g_2, \cdots, g_n$, and each g_i is an extension of f_i. A probabilistic proof of this result was given in [8]. A constructive proof of the case $p/q \geq 3$ was given in [7].

Keywords: circular chromatic number, girth, uniquely colourable

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1 Introduction

Suppose $G = (V, E)$ is a graph and $r \geq 2$ is a real number. An r-colouring of G is a mapping $f : V \to [0, r)$ such that for every edge xy of G, $1 \leq |f(x) - f(y)| \leq r - 1$.

The set $[0, r)$ of "colours" is viewed to form a circle C^r which is obtained from the interval $[0, r]$ by identifying 0 and r into a single point. For $a, b \in [0, r)$, let $[a, b]_r$

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denote the interval of C^r from a to b along the "increasing" direction. To be precise, if $a < b$, then $[a,b]_r = \{t : a \leq t \leq b\}$. If $a > b$, then $[a,b]_r = \{t : a \leq t < r$, or $0 \leq t \leq b\}$. For two points $a, b \in C^r$, $|a-b|_r$ is the circular distance between a and b, and is defined as $|a-b|_r = \min\{|a-b|, r-|a-b|\}$. So an r-colouring of G is a mapping $f : V \to [0,r)$ such that for every edge xy of G, $|f(x)-f(y)|_r \geq 1$.

We say G is *r-colourable* if G has an r-colouring. The *circular chromatic number* $\chi_c(G)$ of G is defined as

$$\chi_c(G) = \inf\{r : G \text{ is } r\text{-colourable}\}.$$

If $r = k$ is an integer, then an ordinary k-colouring of G is an r-colouring as defined above, and conversely, for an r-colouring f defined as above, $g(x) = \lfloor f(x) \rfloor$ is an ordinary k-colouring of G. Therefore $\chi_c(G) \leq \chi(G)$ for any graph G. On the other hand, it is known [2, 10, 13] that for any graph G, $\chi_c(G) > \chi(G) - 1$, and hence $\chi(G) = \lceil \chi_c(G) \rceil$. So $\chi_c(G)$ is a refinement of $\chi(G)$.

An equivalent definition of the circular chromatic number is as follows: For a rational number $r = p/q \geq 2$, let $K_{p/q}$ be the graph with vertex set $\{0, 1, \cdots, p-1\}$, in which two vertices i, j are adjacent if $q \leq |i-j| \leq p-q$. A graph G is called *(p,q)-colourable* if there is a homomorphism from G to $K_{p/q}$ (i.e., a mapping $f : V(G) \to \{0, 1, \cdots, p-1\}$ such that $q \leq |f(x)-f(y)| \leq p-q$ for any edge xy of G). Then

$$\chi_c(G) = \inf\{p/q : G \text{ is } (p,q)\text{-colourable}\}.$$

Two (p,q)-colourings f, g of a graph G is *equivalent* if there is an automorphism σ of $K_{p/q}$ such that $f = \sigma \circ g$. A graph G is called *uniquely (p,q)-colourable*, if there is an onto homomorphism f from G to $K_{p/q}$, and all (p,q)-colourings of G are equivalent. It is known [11] that if G is uniquely (p,q)-colourable, then $\chi_c(G) = r$.

The existence of graphs of large girth with given circular chromatic number has been studied in a few papers. It was first asked by Abbott and Zhou [1] whether there exists, for any integer $n \geq 3$, a triangle free graph G with $\chi_c(G) = n$. Zhu [11] answered this question in affirmative, by showing that for any integer g and for any rational $r \geq 2$, there is a graph G of girth at least g which is uniquely r-colourable. Later on, Nešetřil and Zhu [8] proved the following much stronger result:

Theorem 1.1 *Suppose $p/q \geq 2$ is a rational number, A is a finite set and $f_1, f_2, \cdots, f_n$ are mappings from A to $\{0, 1, \cdots, p-1\}$. Then for any integer g, there is a graph $G = (V, E)$ of girth at least g with $A \subset V$, such that G has exactly n non-equivalent (p,q)-colourings $g_1, g_2, \cdots, g_n$, and each g_i is an extension of f_i.*

Theorem 1.1 is a generalization of a result of V. Müller [6] who proved this for k-colourings. The proofs in [11] and in [8] use the probabilistic method. A constructive proof of Theorem 1.1 for $p/q \geq 3$ is given in [7]. For the case $2 < p/q < 3$, a constructive proof of Theorem 1.1 was missing. In this paper, we give a constructive proof of Theorem 1.1, for all $p/q \geq 2$.

2 Preliminaries

An undirected graph is viewed as a symmetric directed graph. Each edge $e = xy$ of G corresponds to a pair of opposite arcs $a = \vec{xy}$ and $a^{-1} = \vec{yx}$. We shall denote by $A(G)$ the arc set of G. For each arc a, a^{-1} is the opposite arc of a which corresponds to the same edge as a. All graphs considered in this paper are undirected graphs (and hence are symmetric directed graphs). A *rooted graph* is a pair (G, a) such that G is a graph and a is an arc of G. The arc a is called the root arc of G. The edge corresponding to a is called the *root edge* of G. A *walk* W of G is a sequence $(x_0, x_1, \cdots, x_k)$ of vertices of G such that for each i, $\overrightarrow{x_i x_{i+1}} \in A(G)$. For convenience, we also use W to denote the set of arcs in W, i.e., $W = \{\overrightarrow{x_i x_{i+1}} : i = 1, 2, \cdots, k-1\}$. If $x_k = x_0$, then W is called a *closed walk*.

A *tension* of a graph G is a mapping $f : A(G) \to [0, r)$ such that for each closed walk W of G, $\sum_{a \in W} f(a) = 0$. In particular, for each arc a, $f(a^{-1}) = -f(a)$. Suppose $r > 2$ is a real number. An *r-tension* of G is a tension f of G such that for each arc a, $1 \leq |f(a)| \leq r - 1$. An r-tension of a graph G corresponds to an r-colouring of G. Indeed, if f is an r-colouring of G, then $\delta f : A(G) \to [0, r)$ defined as $\delta f(\vec{xy}) = f(y) - f(x)$ is an r-tension. Conversely, if g is an r-tension of G, then let x_0 be a fixed vertex of G, and for each vertex x of G, let W_x be a walk from x_0 to x. Then $f(x) = \sum_{a \in W_x} g(a) \pmod r$ is an r-colouring of G. So

$$\chi_c(G) = \inf\{r : G \text{ has an } r\text{-tension}\}.$$

If (G, a) is a rooted graph with root edge e, then a *rooted r-tension* of G is a tension of G such that for each arc $b \neq a, a^{-1}$, $1 \leq |f(b)| \leq r - 1$. The r-label set of (G, a) is defined as

$$L_T^r(G, a) = \{f(a) \pmod r : f \text{ is a rooted } r\text{-tension of } G\}.$$

Similar to the correspondence between r-colouring and r-tension, it is easy to see that the following is an equivalent definition of the label set:

$$L_T^r(G, a) = \{f(x) - f(y) \pmod r : f \text{ is an } r\text{-colouring of } G - e\}.$$

In case the root arc a is clear from the context, or is irrelevant, we write $L_T^r(G)$ for $L_T^r(G, a)$.

As an example, it is easy to verify that for any $r \geq 2$, $L_T^r(C_2) = [1, r-1]_r$, where C_2 is the 2-cycle, i.e., the graph with two vertices and two parallel edges.

Definition 2.1 *A rooted graph (H, a) is an r-superedge if $L^r_T(H) = [1, r-1]_r$.*

The concept of superedge is the main tool in [7] for the construction of the required large girth graphs. The reason that the proof in [7] only works for $r \geq 3$ is that r-superedges are constructed for $r \geq 3$ only. The main result of this paper is the following:

Theorem 2.1 *For any integer g and for any rational number $r \geq 2$, there is an r-superedge (H, a) of girth at least g.*

Theorem 1.1 follows from Theorem 2.1 and the following result of [7]:

Lemma 2.1 *[7] Suppose $r = p/q \geq 2$ is a rational, where $(p, q) = 1$. If there is an r-superedge (H, a) of girth at least g, then for any finite set A, for any mappings $f_1, f_2, \cdots, f_n$ from A to $\{0, 1, \cdots, p-1\}$, there is a graph $G = (V, E)$ of girth at least g with $A \subset V$, such that G has exactly n non-equivalent (p, q)-colourings $g_1, g_2, \cdots, g_n$, and each g_i is an extension of f_i.*

The proof of Lemma 2.1 is a little bit technical, and the readers are referred to [7] for the details. Here we only give a sketch of how an superedge is used to construct such a graph.

Suppose (H, a) is an r-superedge, G is a graph and $e' = x'y'$ is an edge of G. To *replace e' with superedge (H, a)* means to delete the edge e', add a (vertex disjoint) copy of $H - e$, where $e = xy$ is the root edge of H, and identify x with x', y with y'.

To construct the graph G as required in Lemma 2.1, we take the categorical product $(K_{p/q})^n$ of n copies of $K_{p/q}$. The graph $(K_{p/q})^n$ has exactly n (p, q)-colourings, namely, the n projections π_i $(i = 1, 2, \cdots, n)$. Duplicate each vertex of $(K_{p/q})^n$ m times, where m is a sufficiently large integer. Denote the resulting graph by Q. There are exactly n (p, q)-colourings, say π'_i, of Q, where π'_i is an extension of π_i for $i = 1, 2, \cdots, n$. When m is large enough, we can embed A into $V(Q)$ so that f_i is a restriction of π'_i for $i = 1, 2, \cdots, n$. Now replace each edge of Q by a copy of the superedge (H, a). Denote the resulting graph by Q', which is a graph of large girth. The vertices of Q' are partitioned into two parts: the "old" vertices of G, and the "new" vertices that are added together with the copies of the superedge. The set $\mathcal{C}$ of all (p, q)-colourings of Q' is partitioned into n subsets $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_n$, and for each i, all the colourings in $\mathcal{C}_i$ are extensions of π'_i. In other words, if we only look at the colours of the old vertices, there are exactly n (p, q)-colourings of Q'. However, there are probably many ways to colour the new vertices.

Next we build a graph R which has many layers $V_1, V_2, \cdots, V_t$, where t is sufficiently large and $V_i = V(Q) \times \{i\}$. Edges are added between the vertices of two consecutive layers in such a way that: (1) the resulting graph has large girth, and (2) if the ith layer is (p,q)-coloured as $f(x,i)=\pi'_j(x)$ for all $x \in V(Q)$, then it is uniquely extended to the $(i+1)$th layer as $f(x,i+1)=\pi'_j(x)$ for all $x \in V(Q)$. Now identify the first layer of R with the set of old vertices of Q', we obtain a graph R'. The graph R' has large girth, and if we only look at the colours of the vertex set of R, there are precisely n (p,q)-colourings of R'. The next goal is to fix the colours of the other vertices of R'. Fix n colourings $g_1, g_2, \cdots, g_n$, of R', where g_i is an extension of π'_i. Then by adding edges between new vertices of Q' and vertices of R, we can make sure that the colours of the new vertices of Q' are also fixed as in the n colourings $g_1, g_2, \cdots, g_n$. Since R' has t layers, where t is sufficiently large, and vertices of one layer is only adjacent to vertices of the two adjacent layers, the edges can be added in such a way that no short cycles will be produced. So the resulting graph G satisfies the requirements of Lemma 2.1.

The remainder of this paper is devoted to the construction of r-superedges of large girth.

3 r-indicators

Definition 3.1 *Suppose $r \geq 2$ is a real number. A rooted graph H with root edge e is called an* r-indicator *if $L^r_T(H) = \{0\}$.*

In this section, we show that if there is an r-indicator H, then one can easily construct an r-superedge.

Given two rooted graphs (G,a) and (G',a') with root arcs $a=\vec{xy}$ and $a'=\vec{x'y'}$, respectively, the *series join* of (G,a) and (G',a') is the rooted graph (G'',a'') obtained from the vertex disjoint union of G and G' by deleting the root edges $e=xy$ and $e'=x'y'$, then identify y' and x into a single vertex z, and add an root edge $e''=xy'$ with root arc $\overrightarrow{xy'}$. The following lemma is easy and is proved in [9].

Lemma 3.1 *If (G'',a'') is the series join of (G,a) and (G',a'), then for any $r \geq 2$,*

$$L^r_T(G'') = L^r_T(G) + L^r_T(G') = \{t+t' \pmod r : t \in L^r_T(G), t' \in L^r_T(G')\}.$$

Theorem 3.1 *Suppose (H,a) is an r-indicator. Let (H',a') be the series join of H and C_2. Then (H',a') is an r-superedge. Moreover, if H has girth g, then H' has girth at least g.*

Proof. If (H,a) is an r-indicator, then it follows from Lemma 3.1 that $L^r_T(H') = L^r_T(H) + L^r_T(C_2) = [1, r-1]_r$. The moreover part of the theorem is trivial. ∎

So to prove the existence of an r-superedge of girth at least g, it suffices to prove the existence of an r-indicator of girth at least g.

The following lemma shows that every r-indicator corresponds to a graph of circular chromatic number r.

Lemma 3.2 *If a rooted graph H with root edge $e = uv$ is an r-indicator, then $\chi_c(H - e) = r$.*

Proof. Since $L^r_T(H) \neq \emptyset$, it follows that $H - e$ is r-colourable. Assume to the contrary that $\chi_c(H - e) = r' < r$. Then $L^{r'}_T(H) \neq \emptyset$. Assume $t \in L^{r'}_T(H) \neq \emptyset$ and f is a rooted r'-tension of H with $f(a) = t$, where $a = \vec{uv}$ is the rooted arc. Let $D = D^+_f(H)$ be the orientation of G defined as $\vec{xy} \in D$ if and only if $f(\vec{xy}) > 0$. As f is a rooted tension, it is easy to see that D is acyclic. For a subset X of $V(G)$, denote by $[X, \overline{X}]$ the cut which consists of all the arcs $\vec{xy}$ with $x \in X$ and $y \notin X$. Then there is a cut $[X, \overline{X}]$ of H which contains the arc a such that for any arc $\vec{xy} \in [X, \overline{X}]$, $f(\vec{xy}) \geq 0$. For any $0 \leq \epsilon \leq r - r'$, let f' be defined as

$$f'(\vec{xy}) = \begin{cases} f(\vec{xy}), & \text{if } \vec{xy}, \vec{yx} \notin [X, \overline{X}], \\ f(\vec{xy}) + \epsilon, & \text{if } \vec{xy} \in [X, \overline{X}], \\ f(\vec{xy}) - \epsilon, & \text{if } \vec{yx} \in [X, \overline{X}]. \end{cases}$$

Then f' is a rooted r-tension of H with $f'(a) = f(a) + \epsilon$. Thus $L^r_T(H)$ contains an interval of positive length, in contrary to the assumption that $L^r_T(H) = \{0\}$. ∎

4 Construction of r-indicators of large girth

In this section, we present a method that construct an r-indicator of large girth, for each rational $r \geq 2$.

Definition 4.1 *Suppose $r \geq 2$ is a rational number. A rooted graph (H, a) is called an* r-semi-indicator *if $L^r_T(H) \neq \emptyset$ and $L^r_T(H) \cap [1, r-1]_r = \emptyset$.*

Lemma 4.1 *For any integer $g \geq 3$, for any $r \geq 2$, there is an r-semi-indicator (H, a) of girth at least g.*

Proof. It is well-known [3, 5] that one can construct a graph G of girth at least g such that $\chi(G) > r + 1$. As $\chi_c(G) > \chi(G) - 1$, we have $\chi_c(G) > r$. By deleting edges from G, if necessary, we obtain a graph H and an edge e of H such that (i) $\chi_c(H) > r$, and (ii) $\chi_c(H - e) \leq r$. Let a be an arc corresponding to e. It follows from definition that $L^r_T(H) \neq \emptyset$ and $L^r_T(H) \cap [1, r-1]_r = \emptyset$. ∎

Suppose (H, a) is an r-semi-indicator, where $a = \vec{xy}$ is the root arc. Let $\tau(H) = \max\{|f(a)|_r : f$ is a rooted r-tension of $H\}$. It follows from the definition of an r-semi-indicator that $0 \leq \tau(H) < 1$.

Lemma 4.2 *If $r = p/q$ is a rational number and $(p, q) = 1$, then for any r-semi-indicator (H, a), $\tau(H) = \frac{i}{q}$ for some non-negative integer $i \leq q$.*

Proof. Assume $\tau(H) = t$ and $a = \vec{xy}$. Let $e = xy$. By definition, there is an r-colouring f of $H - e$ such that $f(x) = 0$ and $f(y) = t$. Let $g : V(G) \to \{0, 1, \cdots, p-1\}$ be defined as $g(v) = \lceil f(v)q \rceil$. It is straightforward to verify that g is a (p, q)-colouring of $H - e$. Then $f'(v) = g(v)/q$ is an r-colouring with $f'(x) = 0$ and $f'(y) = g(y)/q \geq f(y)$. By definition of $\tau(H)$, we conclude that $f'(y) = f(y)$. ∎

It follows from the definition that an r-indicator is an r-semi-indicator (H, a) with $\tau(H) = 0$. Suppose f is a rooted r-tension of H with $f(a) \geq 0$. By definition of $\tau(H)$, we have $|f(a)|_r \leq \tau(H)$. Let

$$\xi(f) = \begin{cases} 0, & \text{if } f(a) = 0; \\ 1, & \text{if } 0 < f(a) \leq \tau(H); \\ -1, & \text{if } r - \tau(H) \leq f(a) < r. \end{cases}$$

Note that $f(a) = \xi(f)|f(a)|_r \pmod{r}$.

Suppose A is a subset of $[0, r)$ and $s \geq 2$ is an integer. Let $sA = \{t_1 + t_2 + \cdots + t_s \pmod{r} : t_i \in A\}$.

Given an r-semi-indicator (H, a) of girth at least g for which $\tau(H) > 0$. Let s be the least integer such that $s\tau(H) \geq 1$. Then $s \geq 2$. Let $\delta = 1 - (s-1)\tau(H) > 0$. Let k be the largest integer such that $k(r-2) < \delta$.

Let (Q, b) be the series join of s copies of (H, a). Denote by (H_i, a_i) the ith copy of (H, a) in (Q, b). Assume that e is the root edge of Q, and e_i is the root edge of H_i for $i = 1, 2, \cdots, s$. Figure 1 below is an illustration of (Q, b), where the thick edge indicates the root edge of (Q, b), and the dotted lines indicate the (deleted) root edges of the copies of H.

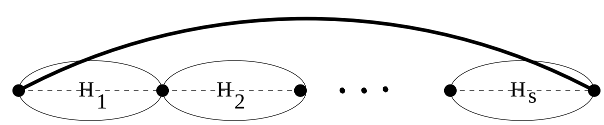

Figure 1: The rooted graph (Q, b)

It follows from Lemma 3.1 that $L_T^r(Q) = sL_T^r(H)$. Let (Q', b') be the parallel join of (Q, b) and (C_{2k+2}, d), where d is an arbitrary arc of C_{2k+2}. As C_{2k+2} is the series join of $2k+1$ copies of C_2, and $L_T^r(C_2) = [1, r-1]_r$, it follows from Lemma

3.1 that $L_T^r(C_{2k+2}) = [1-k(r-2), r-1+k(r-2)]_r$, because $k(r-2) < \delta < 1$. Thus $L_T^r(Q') \neq \emptyset$ if and only if $sL_T^r(H) \cap [1-k(r-2), r-1+k(r-2)]_r \neq \emptyset$. Figure 2 below is an illustration of (Q', b') for the case $k = 2$. Note that in the case as depicted in Figure 2, Q' can be viewed as obtained from Q by adding a path of length 5 connecting the two end vertices of the root edge. In the general case, we add a path of length $2k+1$ joining the two end vertices of Q. In particular, if $k = 0$, we simply add an edge joining the two end vertices of the root edge.

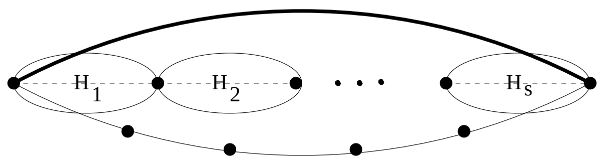

Figure 2: The rooted graph (Q', b')

Lemma 4.3 *Suppose f is a rooted r-tension of (Q', b'). Let f_i be the restriction of f to H_i. Then either for all i, $\xi(f_i) = 1$, or for all i, $\xi(f_i) = -1$.*

Proof. By Lemma 3.1,

$$f(b') = \sum_{i=1}^{s} f_i(a_i) \pmod r = \sum_{i=1}^{s} \xi(f_i)|f(a_i)|_r \pmod r.$$

If there are indices i, j such that $\xi(f_i) = 1$ and $\xi(f_j) = -1$, or there is an index with $\xi(f_i) = 0$, then

$$|\sum_{i=1}^{s} \xi(f_i)|f(a_i)|_r| \leq (s-1)\tau(H) < 1 - k(r-2),$$

contrary to the fact that $f(b') \in L_T^r(C_{2k+2}) = [1-k(r-2), r-1+k(r-2)]_r$. ∎

Let Q'' be the graph constructed as follows: Take g copies of $Q' - e'$, say $Q'_1 - e'_1, Q'_2 - e'_2, \cdots, Q'_g - e'_g$, where e'_i is the root edge of Q'_i. Recall that each copy of $Q' - e'$ consists of s copies of H. Let $H_{i,j}$ denotes the jth copy of H in $Q'_i - e'_i$, and $a_{i,j} = \overrightarrow{x_{i,j}y_{i,j}}$ be the root arc of $H_{i,j}$ and let $e_{i,j}$ denote the root edge of $H_{i,j}$. For $i = 1, \cdots, g-1$, identify $H_{i,s}$ with $H_{i+1,1}$. Figure 3 is an illustration of the construction of Q''.

Lemma 4.4 *The graph Q'' is r-colourable if and only if $Q' - e'$ is r-colourable. Moreover, if f is an r-tension of Q'', then either for all i, j, $\xi(f_{i,j}) = 1$, or for all i, j, $\xi(f_{i,j}) = -1$, where $f_{i,j}$ is the restriction of f to $H_{i,j}$.*

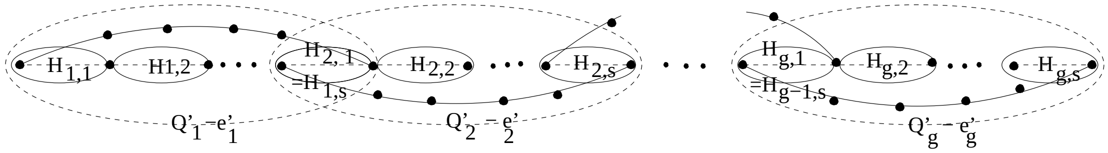

Figure 3: The graph Q''

Proof. As $Q' - e'$ is a subgraph of Q'', so if Q'' is r-colourable, then $Q' - e'$ is r-colourable. Conversely, if Q' is r-colourable, then $sL_T^r(H) \cap [1-k(r-2), r-1+k(r-2)]_r \neq \emptyset$, and hence there exist $t_1, t_2, \cdots, t_s \in L_T^r(H)$ such that

$$t_1 + t_2 + \cdots + t_s \in [1-k(r-2), r-1+k(r-2)]_r.$$

Let f_j be a rooted r-tension of H with $f_j(a) = t_j$. Let $g_{i,j}$ be a rooted r-tension on $H_{i,j}$ defined as

$$g_{i,j} = \begin{cases} f_j, & \text{if } i \text{ is odd;} \\ f_{s+1-j}, & \text{if } i \text{ is even.} \end{cases}$$

It is straightforward to verify that the union of $g_{i,j}$ $(i = 1, 2, \cdots, g, j = 1, 2, \cdots, s)$ is an r-tension of Q'', i.e., Q'' is r-colourable.

By applying Lemma 4.3 to each copy of Q' in Q'', we obtain the moreover part. ∎

Lemma 4.5 *If (H, a) is an r-semi-indicator of girth at least g and with $\tau(H) > 0$, then there is an r-semi-indicator $(H'a')$ of girth at least g for which $\tau(H') < \tau(H)$.*

Proof. Assume (H, a) is an r-semi-indicator of girth at least g for which $\tau(H) > 0$. Let s be the least integer such that $s\tau(H) \geq 1$. Let $\delta = 1 - (s-1)\tau(H) > 0$. Let k be the largest integer such that $k(r-2) < \delta$. We consider two cases.

Case 1. $sL_T^r(H) \cap [1-k(r-2), r-1+k(r-2)]_r \neq \emptyset$.

Let Q'' be the graph constructed as above. Let (H', a') be the rooted graph constructed as follows: Take two copies of Q'', say Q''_1 and Q''_2. The copies of H in Q''_1 and Q''_2 are denoted by $H^1_{i,j}$ and $H^2_{i,j}$, respectively. The root arc of $H^l_{i,j}$ is $a^l_{i,j} = \overrightarrow{x^l_{i,j}y^l_{i,j}}$. Identify $H^1_{1,1}$ with $H^2_{1,1}$ and identify $y^1_{g,s}$ and $y^2_{g,s}$. Add an edge e' joining $x^1_{g,s}$ and $x^2_{g,s}$. The root arc is $a' = \overrightarrow{x^1_{g,s}x^2_{g,s}}$. Figure 4 is an illustration of the construction of H'.

By Lemma 4.4, Q'' is r-colourable. Hence $H' - e'$ is r-colourable, implying that $L_T^r(H', a') \neq \emptyset$. Let f be an arbitrary rooted r-tension of H'. Let $f^l_{i,j}$ be the restriction of f to $H^l_{i,j}$. By the moreover part of Lemma 4.4, either for all l, i, j, $\xi(f^l_{i,j}) = 1$, or for all l, i, j, $\xi(f^l_{i,j}) = -1$. This implies that

$$|f(a')| = ||f(a^2_{g,s})|_r - |f(a^1_{g,s})|_r|.$$

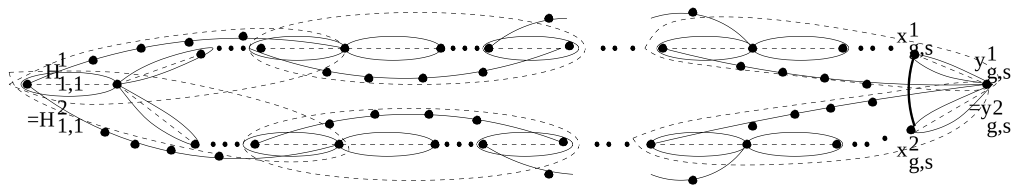

Figure 4: The rooted graph (H', a') for Case 1

As

$$0 < ||f(a^2_{g,s})|_r - |f(a^1_{g,s})|_r| < \tau(H),$$

we conclude that $|f(a')|_r < \tau(H)$. So $\tau(H') < \tau(H)$.

Case 2. $sL^r_T(H) \cap [1 - k(r-2), r - 1 + k(r-2)]_r = \emptyset$.

Let $\delta' = 1 - k(r-2) - (s-1)\tau(H) = \delta - k(r-2)$ (note that it is possible that $k = 0$, in which case $\delta' = \delta$). By the maximality of k, $\delta' \leq r - 2$. Moreover,

$$L^r_T(H) \cap [\delta', r - 2 + \delta'] = \emptyset,$$

for otherwise, there exists $t \in L^r_T(H) \cap [\delta', (2k+1)(r-2) + \delta']$, which implies that

$$(s-1)\tau(H) + t \in sL^r_T(H) \cap [1 - k(r-2), r - 1 + k(r-2)]_r,$$

contrary to the assumption that $sL^r_T(H) \cap [1 - k(r-2), r - 1 + k(r-2)]_r = \emptyset$.

Furthermore, $sL^r_T(H) \cap [1 - k(r-2), r - 1 + k(r-2)]_r = \emptyset$ implies that $(2k+1)(r-2) + \delta' < \tau(H)$. In particular, $2\delta' \leq (r-2) + \delta' < \tau(H)$.

Let (H', a') be constructed as follows: Take g copies of $H - e$, say $H_1 - e_1, H_2 - e_2, \cdots, H_g - e_g$, where $e_i = x_i y_i$. Identify $x_1, x_2, \cdots, x_g$ into a single vertex x. For $i = 1, 2, \cdots, g-1$, connect y_i to y_{i+1} by a path of length 2. Finally, add an edge e' connecting y_1 and y_g, and the root arc is $a' = \overrightarrow{y_1 y_g}$. Figure 5 is an illustration of the construction of H'.

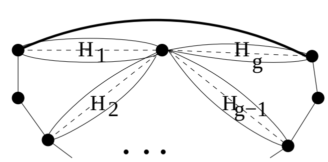

Figure 5: The rooted graph (H', a') for Case 2

It is obvious that H' has girth at least g and $H' - e'$ is r-colourable. Let f be an arbitrary r-colouring of $H' - e'$. We need to prove that $|f(y_1) - f(y_g)|_r < \tau(H)$. Without loss of generality, we may assume that $f(x) = 0$. Then $f(y_i) \in L^r_T(H)$ for

$i = 1, 2, \cdots, g$. Note that $L_T^r(H) \cap [\delta', r-2+\delta] = \emptyset$. Let

$$\begin{aligned} B_1 &= L_T^r(H) \cap (r-\delta', \delta')_r, \\ B_2 &= L_T^r(H) \cap ((2k+1)(r-2)+\delta', \tau(H)]_r, \\ B_3 &= L_T^r(H) \cap [r-\tau(H), r-(2k+1)(r-2)-\delta')_r. \end{aligned}$$

Then for each $i = 1, 2, \cdots, g$, there is an index $\sigma(i) \in \{1, 2, 3\}$ such that $f(y_i) \in B_{\sigma(i)}$. Since y_i and y_{i+1} is connected by a path of length 2, it is easy to verify that $|f(y_i) - f(y_{i+1})| \in [2, r-2]_r$. It follows that for any $j \in \{1, 2, 3\}$, if $f(y_i) \in B_j$, then $f(y_{i+1}) \in B_j$, i.e., $\sigma(i) = \sigma(i+1)$. Thus if $f(y_1) \in B_j$ then $f(y_g) \in B_j$. If $j = 1$, then

$$|f(y_1) - f(y_g)|_r < 2\delta' < \tau(H).$$

If $j = 2, 3$, then it is easy to see that $|f(y_1) - f(y_g)|_r < \tau(H)$. ∎

Corollary 4.1 *For any rational number $r = p/q \geq 2$ and for any integer g, there is an r-indicator of girth at least g.*

Proof. By Lemma 4.1, there is an r-semi-indicator. By Lemma 4.2, there is an r-semi-indicator (H, a) of girth at least g such that $\tau(H)$ is minimum among all r-semi-indicators of girth at least g. By Lemma 4.5, $\tau(H) = 0$. So (H, a) is an r-indicator. ∎

Combining Corollary 4.1 and Theorem 3.1, we have the following corollary.

Corollary 4.2 *For any rational $r \geq 2$ and for any integer g, there is an r-superedge (H, a) of girth at least g.*

Combining Corollary 4.1 and Lemma 3.2, we have the following corollary.

Corollary 4.3 *For any rational $r \geq 2$ and for any integer g, there is a graph G of girth at least g such that $\chi_c(G) = r$.*

As mentioned in Section 1, Corollary 4.3 was first proved in [11] by the probabilistic method. Note that Corollary 4.3 also follows from Theorem 1.1, as uniquely r-colourable graphs G have $\chi_c(G) = r$.

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