A simple proof of Moser's theorem

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Abstract

This paper gives a simple proof of a result of Moser which says that for any rational number r between 2 and 3, there exists a planar graph G whose circular chromatic number is equal to r.

1 Introduction

The circular chromatic number $\chi_c(G)$ of a graph G, introduced by Vince in 1988 (under the name "the star-chromatic number"), is a natural generalization of the chromatic number of a graph.

For a pair of integers k, d, a (k, d)-coloring of a graph G is a mapping c of $V(G)$ to the set $\{0, 1, \cdots, k-1\}$ such that for any adjacent vertices x, y of G, $d \leq |c(x) - c(y)| \leq k - d$. The *circular chromatic number* $\chi_c(G)$ of a graph G is the infimum of the ratios k/d for which there exists a (k, d)-coloring of G.

It is known (cf. [5]) that for any rational number $r \geq 2$, there exists a graph G with $\chi_c(G) = r$. However the question remains open whether or not for every rational number r between 2 and 4, there is a planar graph G with $\chi_c(G) = r$. Recently, with a brilliant construction and a complex argument, Moser [4] proved that for every rational number r between 2 and 3, there does exist a planar graph G with $\chi_c(G) = r$. This paper gives a simple proof of Moser's result. The construction of the graphs is the same as in [4], and the idea of using the Farey sequence and the definition of the alpha sequence of a fraction are also the same as in [4]. The different part is in the proof that determines the circular chromatic number of the constructed graphs.

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2 Preliminary results

Given any rational number p/q between 2 and 3 such that $(p, q) = 1$, we let p', q' be the unique positive integers such that $p' < p, q' < q$ and $pq' - qp' = 1$. Then it is straightforward to verify that $p'/q' < p/q$ and that p'/q' is the largest fraction with the property that $p'/q' < p/q$ and $p' \leq p$. Similarly, we let p'', q'' be positive integers such that $p'' < p', q'' < q'$ and $p'q'' - p''q' = 1$. Repeating this process of finding smaller and smaller fractions, we shall reach the fraction $2/1$ in a finite number of steps. Thus given any rational p/q between 2 and 3, there corresponds a unique sequence of fractions

$$\frac{2}{1} = \frac{p_0}{q_0} < \frac{p_1}{q_1} < \frac{p_2}{q_2} < \cdots < \frac{p_n}{q_n} = \frac{p}{q}.$$

The sequence $(p_i/q_i : i = 0, 1, \cdots, n)$ is called the *Farey sequence* of p/q (cf. [4]). It will also be convenient to define $P_{-1} = -1$ and $q_{-1} = 0$.

As $p_i q_{i-1} - p_{i-1} q_i = 1$ and $p_{i-1} q_{i-2} - p_{i-2} q_{i-1} = 1$, it follows that $p_{i-1}(q_i + q_{i-2}) = q_{i-1}(p_i + p_{i-2})$. As p_{i-1}, q_{i-1} are co-prime, for $i \geq 1$,

$$\alpha_i = \frac{p_i + p_{i-2}}{p_{i-1}} = \frac{q_i + q_{i-2}}{q_{i-1}}$$

is an integer, which is greater than 1, and hence is at least 2. The sequence $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ is called the *alpha sequence* of p/q (cf. [4]), which is obviously uniquely determined by p/q. The process of deducing the alpha sequence from the rational p/q can also be reversed. In other words, each sequence $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ with $\alpha_i \geq 2$ determines a rational p/q between 2 and 3. Indeed, given the alpha sequence $(\alpha_1, \alpha_2, \cdots, \alpha_n)$, the fractions p_i/q_i can be easily determined by solving the difference equations

$$p_i = \alpha_i p_{i-1} - p_{i-2}, \quad q_i = \alpha_i q_{i-1} - q_{i-2}, \qquad (*)$$

with the initial condition that $(p_{-1}, q_{-1}) = (-1, 0)$ and $(p_0, q_0) = (2, 1)$.

By repeatedly applying the equation $(*)$, we may express p_i (respectively q_i) in terms of p_{i-t} and p_{i-t-1} (respectively q_{i-t} and q_{i-t-1}) for any $2 \leq t \leq i$. Lemma 2.1 below gives the explicit expressions. For $1 \leq r \leq s \leq n$, we let

$$\Lambda_{r,s} = \det \begin{pmatrix} \alpha_r & 1 & 0 & \cdots & 0 & 0 \\ 1 & \alpha_{r+1} & 1 & \cdots & 0 & 0 \\ 0 & 1 & \alpha_{r+2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{s-1} & 1 \\ 0 & 0 & 0 & \cdots & 1 & \alpha_s \end{pmatrix}$$

Lemma 2.1 *For $2 \leq t \leq i$, we have*

$$p_i = p_{i-t}\Lambda_{i-t+1,i} - p_{i-t-1}\Lambda_{i-t+2,i}, \quad q_i = q_{i-t}\Lambda_{i-t+1,i} - q_{i-t-1}\Lambda_{i-t+2,i} \quad (**)$$

Proof. It suffices to prove the first equality. We shall prove it by induction on t. When $t=2$, by applying $(*)$ twice, we obtain $(**)$. Suppose $i \geq t > 2$, and that $(**)$ is true for any $t' < t$. Then by cofactor expansion,

$$\begin{aligned} p_{i-t}\Lambda_{i-t+1,i} - p_{i-t-1}\Lambda_{i-t+2,i} &= \alpha_i(p_{i-t}\Lambda_{i-t+1,i-1} - p_{i-t-1}\Lambda_{i-t+2,i-1}) \\ &\quad -(p_{i-t}\Lambda_{i-t+1,i-2} - p_{i-t-1}\Lambda_{i-t+2,i-2}) \\ &= \alpha_i p_{i-1} - p_{i-2} = p_i. \end{aligned}$$

The second equality uses the induction hypothesis. ∎

By letting $t = i$ in $(**)$, and by using the initial condition, we have

$$p_i = 2\Lambda_{1,i} + \Lambda_{2,i}, \quad q_i = \Lambda_{1,i}. \quad (***)$$

Lemma 2.2 *For* $2 \leq t \leq i$, $p_{i-t}q_i = p_i q_{i-t} - \Lambda_{i-t+2,i}$.

Proof. We prove it by induction on t. The case $t = 2$ is proved by applying twice the equality $(*)$. Assume $t > 2$ and that the lemma is true for $t-1$, i.e., $p_{i-t+1}q_i = p_i q_{i-t+1} - \Lambda_{i-t+3,i}$ Since $p_{i-t}q_{i-t+1} = p_{i-t+1}q_{i-t} - 1$, we have

$$\begin{aligned} p_{i-t}q_i &= \frac{(p_{i-t+1}q_{i-t} - 1)q_i}{q_{i-t+1}} \\ &= p_i q_{i-t} - \frac{q_i + q_{i-t}\Lambda_{i-t+3,i}}{q_{i-t+1}} \\ &= p_i q_{i-t} - \Lambda_{i-t+2,i}. \end{aligned}$$

The second equality uses the induction hypothesis, and the last equality follows from $(**)$. ∎

Lemma 2.3 *For any* $2 < t \leq i$, $\Lambda_{t,i} \leq \Lambda_{t-1,i}$.

We omit the proof, which is an induction, by noting that $\alpha_j \geq 2$.

Lemma 2.4 below was proved in [2] and also implicitly used in [5, 6]. Given a (k,d)-coloring c of a graph G, we define a directed graph $D_c(G)$ on the vertex set of G by putting a directed edge from x to y if and only if (x,y) is an edge of G and that $c(x) - c(y) = d \pmod{k}$.

Lemma 2.4 *For any graph* G, $\chi_c(G) = k/d$ *if and only if* G *is* (k,d)-*colorable, and for any* (k,d)-*coloring* c *of* G, *the directed graph* $D_c(G)$ *contains a directed cycle.*

A simple calculation shows that the length of the directed cycle in $D_c(G)$ is a multiple of k, and hence is at least k (under the assumption that $(k,d) = 1$). Thus we have the following corollary:

Corollary 2.1 *For any graph* G, *if* $\chi_c(G) = k/d$ *where* $(k,d) = 1$, *then* G *has a cycle of length at least* k. *In particular* $k \leq |V(G)|$.

3 The construction and the proof

Let $r = p/q$ be any rational number between 2 and 3, where $(p, q) = 1$, and let $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ be the alpha sequence of p/q, and let $(p_i/q_i : i = 1, 2, \cdots, n)$ be the Farey sequence of p/q.

We construct graphs F_i, H_i (for $i = 1, 2, \cdots, n$) recursively as follows:

F_1 is a singleton which is labelled with two labels u_1, v_1, H_1 is a path with $2\alpha_1$ vertices in which the two end vertices are labelled x_1, y_1 respectively. F_2 is a path with $2\alpha_1 - 2$ vertices whose two end vertices are labelled u_2, v_2 respectively.

For $i \geq 2$, the graph H_i is constructed as follows: Take α_i copies of F_{i-1}, in which the labelled vertices in the jth copy are labelled with u^j_{i-1} and v^j_{i-1}. Also take $\alpha_i - 1$ copies of H_{i-1}, in which the labelled vertices in the jth copy are labelled with x^j_{i-1} and y^j_{i-1}. Then H_i is obtained from these copies of H_{i-1} and F_{i-1} by joining x^j_{i-1} to both u^j_{i-1} and u^{j+1}_{i-1}, and joining y^j_{i-1} to both v^j_{i-1} and v^{j+1}_{i-1}. Also label the vertex u^1_{i-1} with the additional label x_i, and label $v^{\alpha_i}_{i-1}$ with additional label y_i.

For $i \geq 3$, the graph F_i is constructed as follows: Take $\alpha_{i-1} - 1$ copies of F_{i-2}, in which the labelled vertices in the jth copy are labelled with u^j_{i-2} and v^j_{i-2}. Take $\alpha_{i-1} - 2$ copies of H_{i-2}, in which the labelled vertices in the jth copy are labelled with x^j_{i-2} and y^j_{i-2}. Then F_i is obtained from these copies of H_{i-1} and F_{i-1} by joining x^j_{i-2} to both u^j_{i-2} and u^{j+1}_{i-2}, and joining y^j_{i-2} to both v^j_{i-2} and v^{j+1}_{i-2}. Also label the vertex u^1_{i-2} with the additional label u_i, and label $v^{\alpha_{i-1}-1}_{i-2}$ with additional label v_i.

Finally, let G_i be obtained from a copy of F_i and a copy of H_i by joining x_i to u_i, and y_i to v_i. Figure 1. illustrates the construction of F_i, H_i and G_i.

(Insert Figure 1 here.)

It follows from the construction that the graphs G_i are planar. We shall show that $\chi_c(G_i) = p_i/q_i$.

For $i \geq 1$, let $f_i = |F_i|, h_i = |H_i|$ and $g_i = |G_i|$. It follows from the construction that

$$g_i = f_i + h_i,\ f_1 = 1,\ f_2 = 2\alpha_1 - 2, h_1 = 2\alpha_1,$$

and for $i \geq 2$,

$$h_i = \alpha_i f_{i-1} + (\alpha_i - 1)h_{i-1},$$

for $i \geq 3$,

$$f_i = (\alpha_{i-1} - 1)f_{i-2} + (\alpha_{i-1} - 2)h_{i-2}.$$

Simple algebraic calculation shows that

$$h_i = \alpha_i g_{i-1} - h_{i-1},\ f_i = (\alpha_{i-1} - 1)g_{i-2} - h_{i-2} = h_{i-1} - g_{i-2}.$$

Hence

$$g_i = \alpha_i g_{i-1} - g_{i-2}.$$

Since $g_1 = p_1, g_2 = p_2$, and g_i, p_i satisfy the same difference equation, we conclude that $|G_i| = g_i = p_i$.

Now we observe that G_i has a unique Hamiltonian cycle, up to an isomorphism. Indeed, it is not difficult to see (or verify by induction) that each graph F_i has a unique Hamiltonian path from u_i to v_i, up to an isomorphism. Also each H_i has a unique Hamiltonian path from x_i to y_i, up to an isomorphism. Hence G_i has a unique Hamiltonian cycle, up to an isomorphism. A Hamiltonian cycle of G_i consists of the two edges (x_i, u_i) and (y_i, v_i), and a Hamiltonian path of F_i from u_i to v_i and a Hamiltonian path of H_i from x_i to y_i (cf. Figure 1). Thus any Hamiltonian cycle Q of G_i is of the form

$$(x_i, u_i, \cdots, [f_i - 2], \cdots, v_i, y_i, \cdots, [h_i - 2], \cdots, x_i).$$

The above notation means that there are $f_i - 2$ vertices between u_i and v_i, and $h_i - 2$ vertices between y_i and x_i.

Lemma 3.1 *Let $Q = (c_1, c_2, \cdots, c_{p_i}, c_1)$ be a Hamiltonian cycle of G_i. If (c_a, c_b) is an edge of G_i which is not an edge of the Hamiltonian cycle Q, then $|a - b| = p_t - 1$ or $p_i - (p_t - 1)$ for some $1 \leq t \leq i - 1$.*

We shall omit the proof, which is an easy induction and also quite obvious by referring to Figure 1.

Suppose $\chi_c(G_i) = p_i/q_i$, and that c is an (p_i, q_i)-coloring of G_i. It follows from Lemma 2.4 that there is a directed cycle of $D_c(G_i)$ of length at least p_i. Since $|G_i| = p_i$, we conlude that there is a Hamiltonian cycle, say $Q = (c_1, c_2, \cdots, c_{p_i}, c_1)$, of G_i such that $c(c_j) - c(c_{j-1}) = q_i \pmod{p_i}$. Therefore if we know the "distance" between two vertices x, y along the positive direction of the cycle Q, then the color of x determines the color of y, and the color y determines the color of x.

Lemma 3.2 *Suppose $\chi_c(G_i) = p_i/q_i$ for some i. Let c be any (p_i, q_i)-coloring of G_i. Then the colors of the two vertices x_i, y_i uniquely determine the colors of u_i, v_i. Conversely, the colors of u_i, v_i uniquely determine the colors of x_i, y_i.*

Proof. Let $Q = (c_1, c_2, \cdots, c_{p_i}, c_1)$ be the Hamiltonian cycle of G_i such that $c(c_j) - c(c_{j-1}) = q_i \pmod{p_i}$) for $j = 1, 2, \cdots, p_i$. As we noted before, the undirected Q (i.e., forget the direction of the edges of Q) is of the form

$$(x_i, u_i, \cdots, [f_i - 2], \cdots, v_i, y_i, \cdots, [h_i - 2], \cdots, x_i).$$

Thus if we know the colors of x_i, y_i, then the direction of the cycle Q is determined, and hence the colors of u_i, v_i are determined. Conversely, the colors of u_i, v_i determine the colors of x_i, y_i. ∎

Let T_i be the graph obtained from F_i and F_{i-1} by connecting u_i to u_{i-1} and v_i to v_{i-1}. We shall prove the following theorem by induction:

Theorem 3.1 *For each i, $\chi_c(G_i) = p_i/q_i$ and $\chi_c(T_i) > p_{i-1}/q_{i-1}$.*

Proof. First we shall show that $\chi_c(G_i) \le p_i/q_i$. Let $Q = (c_1, c_2, \cdots, c_{p_i}, c_1)$ be a Hamiltonian cycle of G_i. We color the vertex c_a by color $\phi(c_a) \equiv aq_i \pmod{p_i}$. Now we shall show that ϕ is indeed an (p_i, q_i)-coloring of G_i. In other words, we shall show that for each edge (c_a, c_b) of G_i, $q_i \le |\phi(c_a) - \phi(c_b)| \le p_i - q_i$. This is trivially true if $a - b = \pm 1$. Otherwise $a - b = p_{i-t} - 1 \pmod{p_i}$ for some $1 \le t \le i-1$, by Lemma 3.1. Then $|\phi(c_a) - \phi(c_b)| \equiv (p_{i-t} - 1)q_i \pmod{p_i}$. By applying Lemma 2.2, we have $p_{i-t}q_i \equiv p_i - \Lambda_{i-t+2,i} \pmod{p_i}$. Since $p_i = 2q_i + \Lambda_{2,i}$ (cf. $(***)$), and $\Lambda_{j,i} \le \Lambda_{2,i}$ (cf. Lemma 2.3), we conclude that $q_i \le |\phi(c_a) - \phi(c_b)| = p_i - q_i - \Lambda_{i-t+2,i} \le p_i - q_i$. Therefore ϕ is a (p_i, q_i)-coloring of G_i, and hence $\chi_c(G_i) \le p_i/q_i$.

Next we shall prove by induction on i that $\chi_c(G_i) = p_i/q_i$ and $\chi_c(T_i) > p_{i-1}/q_{i-1}$. For $i = 1$, this is trivially true. Suppose $\chi_c(G_i) = p_i/q_i$ and $\chi_c(T_i) > p_{i-1}/q_{i-1}$. We consider the graphs G_{i+1} and T_{i+1}.

First we show that $\chi_c(T_{i+1}) > p_i/q_i$. If $i = 2$, this is trivially true. Thus we assume that $i \ge 2$. Assume to the contrary that $\chi_c(T_{i+1}) \le p_i/q_i$. Since $|F_{i+1}| < |H_i|$, we have $|T_{i+1}| < |G_i| = p_i$. It follows from Corollary 2.1 that $\chi_c(T_{i+1}) = m/w < p_i/q_i$ for some integers m, w with $m < p_i$. As p_{i-1}/q_{i-1} is the largest fraction satisfying the property that $p_{i-1} < p_i$ and $p_{i-1}/q_{i-1} < p_i/q_i$, we conclude that $\chi_c(T_{i+1}) \le p_{i-1}/q_{i-1}$.

We consider two cases:

Case 1: $\alpha_i = 2$. In this case $F_{i+1} = F_{i-1}$, and hence $T_{i+1} = T_i$. By induction hypothesis, $\chi_c(T_i) > p_{i-1}/q_{i-1}$, which is a contradiction.

Case 2: $\alpha_i > 2$. In this case F_{i+1} consists of $\alpha_i - 1$ copies of F_{i-1}, say $F_{i-1}^1, \cdots, F_{i-1}^{\alpha_i - 1}$, and $\alpha_i - 2$ copies of H_{i-1}, say $H_{i-1}^1, \cdots, H_{i-1}^{\alpha_i - 2}$. For each $1 \le j \le \alpha_i - 2$, each of the two subgraphs induced by the sets $F_{i-1}^j \cup H_{i-1}^j$ and $H_{i-1}^j \cup F_{i-1}^{j+1}$ is a copy of G_{i-1}. By the induction hypothesis, $\chi_c(G_{i-1}) = p_{i-1}/q_{i-1}$. Therefore $\chi_c(T_{i+1}) = p_{i-1}/q_{i-1}$.

Let ϕ be a (p_{i-1}, q_{i-1})-coloring of T_{i+1}. The restriction of ϕ to $F_{i-1}^j \cup H_{i-1}^j$ and $H_{i-1}^j \cup F_{i-1}^{j+1}$ are (p_{i-1}, q_{i-1})-colorings of G_{i-1}. By Lemma 3.2, the colors of x_{i-1}^1, y_{i-1}^1 determine the colors of u_{i-1}^1, v_{i-1}^1 as well as the colors of u_{i-1}^2, v_{i-1}^2. Therefore u_{i-1}^1, u_{i-1}^2 have the same color, and v_{i-1}^1, v_{i-1}^2 have the same color. Repeating the same argument, we conclude that u_{i-1}^1 and $u_{i-1}^{\alpha_i - 1}$ have the same color, and v_{i-1}^1 and $v_{i-1}^{\alpha_i - 1}$ have the same color. Thus the restriction of

ϕ to $F_{i-1}^1 \cup F_i$ is indeed a (p_{i-1}, q_{i-1})-coloring of T_i, contrary to the induction hypothesis. (Note that the subgraph of T_{i+1} induced by the subset $F_{i-1}^1 \cup F_i$ is isomorphic to $T_i - e$, where $e = (v_{i-1}, v_i)$.) This completes the proof that $\chi_c(T_{i+1}) > p_i/q_i$.

Finally we show that $\chi_c(G_{i+1}) = p_{i+1}/q_{i+1}$. Assume to the contrary that $\chi_c(G_{i+1}) < p_{i+1}/q_{i+1}$. Then $\chi_c(G_{i+1}) \leq p_i/q_i$, because p_i/q_i is the largest fraction which is smaller than p_{i+1}/q_{i+1} and whose enumerator is not bigger than $|G_{i+1}|$.

Now H_{i+1} consists of α_{i+1} copies of F_i, say $F_i^1, \cdots, F_i^{\alpha_{i+1}}$, and $\alpha_{i+1} - 1$ copies of H_i, say $H_i^1, \cdots, H_i^{\alpha_{i+1}-1}$. For each $1 \leq j \leq \alpha_{i+1} - 1$, each of the two subgraphs induced by the sets $F_i^j \cup H_i^j$ and $H_i^j \cup F_i^{j+1}$ is a copy of G_i. By the induction hypothesis, $\chi_c(G_i) = p_i/q_i$. Therefore $\chi_c(G_{i+1}) = p_i/q_i$. Let ϕ be a (p_i, q_i)-coloring of G_{i+1}. The restriction of ϕ to $F_i^j \cup H_i^j$ and $H_i^j \cup F_i^{j+1}$ are (p_i, q_i)-colorings of G_i. With same argument as in the second previous paragraph, we may conclude that the restriction of ϕ to $F_i^1 \cup F_{i+1}$ is a (p_i, q_i)-coloring of T_{i+1}, contrary to the previous result. ∎

Remark The argument in this paper has been extended by the author in [7] to prove that for every rational number r between 3 and 4, there is a planar graph G with $\chi_c(G) = r$. Some related questions are studied or asked in [3, 7, 8].

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