Perfect graphs for generalized colouring – circular perfect graphs

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Abstract

Suppose $\mathcal{F}$ is a family of graphs linearly ordered by homomorphisms. A graph G is perfect with respect to $\mathcal{F}$ if for every induced subgraph H of G, there exists a graph $K \in \mathcal{F}$ such that H and K are homomorphically equivalent. If $\mathcal{F} = \{K_1, K_2, K_3, \cdots\}$ consists of the complete graphs, then perfect graphs with respect to $\mathcal{F}$ are the perfect graphs defined by Berge. In this paper, we consider circular perfect graphs, which are perfect graphs with respect to the graph family $\mathcal{F} = \{K_{k/d} : k \geq d \geq 1\}$. We present a sufficient condition for a triangle free graph to be circular perfect. This sufficient condition is used in a companion paper [10] to prove an analogue of Hajós theorem for the circular chromatic number and for graphs with $2 < \chi_c(G) < 3$.

1 Introduction

Suppose G and H are graphs. A *homomorphism* from G to H is a mapping f from $V(G)$ to $V(H)$ such that $f(x)f(y) \in E(H)$ whenever $xy \in E(G)$. It is easy to see that a vertex colouring of a graph G with n-colours is equivalent to a homomorphism from G to K_n. Thus homomorphism can be viewed as a generalization of graph colouring. We write $G \preceq H$ if there exists a homomorphism from G to H. Then "$\preceq$" defines a partial order on the set

*This research was partially supported by the National Science Council under grant NSC 89-2115-M-110-003

of graphs. Two graphs G and G' are *homomorphically equivalent*, written as $G \leftrightarrow G'$, if $G \preceq G'$ and $G' \preceq G$.

Suppose $\mathcal{F}$ is a family of graphs linearly ordered by homomorphisms, i.e., under order relation "$\preceq$". A graph G is *perfect with respect to* $\mathcal{F}$ if for every induced subgraph H of G, there exists a graph $K \in \mathcal{F}$ such that $H \leftrightarrow K$. The set of complete graphs $\{K_1, K_2, K_3, \cdots\}$ forms an infinite increasing chain under "$\preceq$". The chromatic number $\chi(G)$ of a graph G is equal to the least k such that $G \preceq K_k$. The clique number $\omega(G)$ of G is the maximum k such that $K_k \preceq G$. Denote by $\mathcal{Z}_{\mathcal{G}}$ the set of complete graphs. Then a graph G is perfect with respect to $\mathcal{Z}_{\mathcal{G}}$ if and only if every induced subgraph H of G has $\chi(H) = \omega(H)$. Therefore a graph G is perfect with respect to $\mathcal{Z}_{\mathcal{G}}$ if and only if G is perfect (in the usual sense).

We may view $\mathcal{Z}_{\mathcal{G}}$ as a scale that measures a dimension of graphs, where each complete graph corresponds to a natural number. Just as the set of natural numbers can be extended to rational numbers, we can extend the set of complete graphs to a larger set of graphs. For a fraction k/d with $(k,d) = 1$ and $k \geq 2d$, let $K_{k/d}$ be the graph with vertex set $\{0, 1, \cdots, k-1\}$ in which ij is an edge if and only if $d \leq |i-j| \leq k-d$. Denote by $\mathcal{Q}_{\mathcal{G}}$ the set $\{K_{k/d} : (k,d) = 1 \text{ and } k \geq 2d\} \cup \{K_1\}$. Note that $K_{k/1} = K_k$, and hence $\mathcal{Q}_{\mathcal{G}}$ is indeed an extension of $\mathcal{Z}_{\mathcal{G}}$. Moreover, the set $\mathcal{Q}_{\mathcal{G}}$ is also linearly ordered. It was shown in [2, 4] that for any two fractions $k'/d',\ k/d \geq 2$, $k'/d' \leq k/d$ if and only if $K_{k'/d'} \preceq K_{k/d}$. Thus the set $\mathcal{Q}_{\mathcal{G}}$ together with the order $\preceq$ may be viewed as a representation of those rationals $r \geq 2$ or $r = 1$.

The *circular chromatic number* $\chi_c(G)$ of a graph G is defined as the infimum of those k/d for which G admits a homomorphism to $K_{k/d}$. It is known [8] that "infimum" in the definition is always attained and hence can be replaced by "minimum". Since $\mathcal{Q}_{\mathcal{G}}$ is an extension of $\mathcal{Z}_{\mathcal{G}}$ and $\mathcal{Q}_{\mathcal{G}}$ is also linearly ordered, it follows that for any graph G, we have

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G).$$

Therefore $\chi_c(G)$ is a refinement of $\chi(G)$ and it contains more information about the structure of G than $\chi(G)$ does. The parameter of circular chromatic number is a natural generalization of the concept of chromatic number from many different points of view, and has been studied extensively in the past decade. Readers are referred to [8] for a comprehensive survey on this subject.

The *circular clique number* of a graph G, introduced in [11], is defined as the supremum of those k/d for which $K_{k/d}$ admits a homomorphism to G. It was proved in [11] that the circular clique number of a finite graph G is equal to the maximum of those k/d for which $K_{k/d}$ is an induced subgraph of G.

Definition 1.1 *A graph G is called* circular perfect *if G is perfect with respect to $\mathcal{Q}_\mathcal{G}$.*

In other words, a graph G is circular perfect if for every induced subgraph H of G we have $\chi_c(H) = \omega_c(H)$. In [11], some basic properties of the circular clique number of a graph were proved. Some necessary conditions for a graph to be circular perfect were also given in [11]. It follows from the definition that every perfect graph is circular perfect. However, circular perfect graphs could be non-perfect. In particular, it was proved in [11] that for any $k \geq 2d$, the graph $K_{k/d}$ is circular perfect. If k/d is not an integer, then certainly $K_{k/d}$ is not perfect. The main result of [11] is the following sufficient condition for a graph to be circular perfect.

Theorem 1.1 *[11] If for each vertex x of G, $N_G[x]$ induces a perfect graph and $G - N_G[x]$ induces a bipartite graph which contains no induced P_4, then G is a circular perfect graph.*

This result was shown to be relatively tight [11] and useful [9].. However, for graphs G with $2 < \chi_c(G) < 3$, its application is very much restricted. Indeed, for $k/d < 3$, the fact that $K_{k/d}$ is a circular perfect graph does not follow from Theorem 1.1.

This paper presents a sufficient condition for a triangle free graph to be circular perfect. We shall prove the following theorem:

Theorem 1.2 *Suppose G is a triangle free graph such that for every vertex x of G, $G - N[x]$ is a bipartite graph with no induced C_n for $n \geq 6$, and any induced path of $G - N[x]$ is well-linked. Then G is circular perfect.*

The definition of well-linked paths is a little bit technical, and we defer it to Section 2. We remark that the graphs $K_{k/d}$ for $k/d < 3$ satisfy the condition of Theorem 1.2, and hence the fact that $K_{k/d}$ is circular perfect (which was proved in [11]) follows from this theorem.

Theorem 1.1 was used in [9] to prove an analogue of Hajós theorem for $k/d \geq 3$. In [10], we shall use Theorem 1.2 to prove an analogue of Hajós Theorem for $2 < k/d < 3$. Namely, we shall design a few graph operations and prove that for any $2 < k/d < 3$, starting from the graph $K_{k/d}$, one can construct all graphs of circular chromatic number at least k/d by repeatedly applying these graph operations. We remark that general Hajós type theorem for homomorphisms was given in [3], where some graph operations are designed to construct all graphs that do not admit a homomorphism to H. The result of [3] applies to $H = K_{k/d}$ as well. However, there are some genuine differences between the Hajós Theorem of [9, 10] and the Hajós type theorem of [3]. First of all, when applied to $H = K_{k/d}$, the operations in [3] constuct all graphs G with $\chi_c(G) > k/d$, where the operations in [9, 10] constuct all graphs G with $\chi_c(G) \geq k/d$. Secondly, all constructions need some graphs to start with. In [3], a finite set of graphs is needed to start the construction, but the finite set is not explicitly given. In [9, 10], one starts with a single graph $K_{k/d}$.

2 Some notation

The remainder of this paper is devoted to the proof of Theorem 1.2. First we introduce some notation.

We write $x \sim_G y$ to mean that x is adjacent to y in G. If the graph G is clear from the context, we write $x \sim y$ instead of $x \sim_G y$. For a vertex x of G, $N_G(x) = \{y \in V(G) : x \sim y\}$ denotes the set of neighbours of x, and $N_G[x] = N_G(x) \cup \{x\}$. We also use $N_G(x)$ and $N_G[x]$ to denote the subgraphs induced by these sets. When the graph G is clear from the context, we write $N(x)$ and $N[x]$ for $N_G(x)$ and $N_G[x]$. For a subset X of V, let $N(X) = \cup_{x \in X} N(x)$. We denote by $G - N[x]$ the subgraph of G induced by $V(G) - N[x]$.

For two sets A, B, we write $A \subseteq B$ to mean that A is a subset of B, and write $A \subset B$ to mean that A is a proper subset of B.

Given a graph G and a proper subgraph H of G. A homomorphism from G to H which fixes every vertex of H is called a *retraction*. If there is a retraction from G to H, then we say G *retracts* to H. A graph G is a *core* if G does not retract to any of its proper subgraphs (or equivalently, G admits no homomorphism to any of its proper subgraphs). A core of a graph G is a subgraph H of G such that H is a core and G retracts to H. It is well-known that each finite graph has a unique core (up to isomorphism). If H is the core of G then G and H have the same circular chromatic number and the same circular clique number. Thus for the calculation of circular chromatic number and circular clique number, we need only consider core graphs. It is easy to see that if G is a core graph, and x, y are two vertices x, y of G, then none of $N(x)$ and $N(y)$ is a subset of the other. For two vertices $u, v \in V - N[x]$, we frequently need to compare the neighbourhood of u, v in $N[x]$. We shall use the following notation:

- $u \leq^x v$ means $N(u) \cap N(x) \subseteq N(v) \cap N(x)$;
- $u <^x v$ means $N(u) \cap N(x) \subset N(v) \cap N(x)$;
- $u =^x v$ means $N(u) \cap N(x) = N(v) \cap N(x)$.

Definition 2.1 *Given an induced path $P_n = (p_0, p_1, \cdots, p_n)$ of $G - N[x]$, we say P_n is* badly-linked *with respect to x if one of the following holds:*

1. *There are three indices $i < j < k$ of the same parity such that $p_i \not\leq^x p_j$ and $p_k \not\leq^x p_j$.*
2. *There are three indices $i < j < k$ of the same parity such that $p_j \not\leq^x p_i$ and $p_j \not\leq^x p_k$.*
3. *There are two even indices $i < j$ and two odd indices $i' < j'$ such that $p_i \not\leq^x p_j$ and $p_{i'} \not\leq^x p_{j'}$.*

An induced path P of $G - N[x]$ is called well-linked *with respect to x if it is not badly-linked with respect to x.*

It is easy to see that if G satisfies the condition of Theorem 1.2, then any induced subgraph of G satisfies that condition. Therefore to prove Theorem 1.2, it suffices to prove the following:

Theorem 2.1 *Suppose G is a triangle free graph. If for every vertex x of G, $V - N[x]$ is a bipartite graph which contains no induced C_n for $n \geq 6$, and every induced path of $G - N[x]$ is well-linked, then $\chi_c(G) = \omega_c(G)$.*

3 Proof of Theorem 2.1

Assume that $G = (V, E)$ is a connected core graph which is a counterexample to Theorem 2.1. We shall derive a contradiction. For each $x \in V$ let H_x be the bipartite graph $G - N[x]$. The main part of the proof is to analyse the structure of H_x. The argument is a little complicated and divided into a few subsections.

3.1 Connectness of H_x

In this subsection, we prove that the bipartite graph H_x is connected. Note that H_x contains no isolated vertices because such vertices could be retracted to x, contradicting the fact that G is a core.

Lemma 3.1 *Assume H_x is not connected and $Q = A \cup B$ is a connected component of H_x. Then either $N(A) \cap N(x) = \emptyset$ or $N(B) \cap N(x) = \emptyset$.*

Proof. Assume to the contrary that there exist $a \in A$ and $b \in B$ such that $N(a) \cap N(x) \neq \emptyset$ and $N(b) \cap N(x) \neq \emptyset$. Let P be a path of Q connecting a and b. Let

$$u \in N(a) \cap N(x), \quad v \in N(b) \cap N(x).$$

Then $P \cup \{u, v, x\}$ induces a nonbipartite subgraph of G.

Let $Q' = A' \cup B'$ be another connected component of H_x. Then any element w of Q' is adjacent to at least one of u, v, for otherwise, H_w contains $P \cup \{u, v, x\}$ and hence is nonbipartite. On the other hand, w cannot be adjacent to both u and v, for otherwise, let w' be a neighbour of w in Q', then one of u, v is adjacent to both w, w', contrary to the fact that G contains no triangle. In particular, this shows that $u \neq v$.

Let $a'b'$ be an edge of Q', where $a' \in A'$ and $b' \in B'$. Assume a' is adjacent to u and b' is adjacent to v. (Note that a', b' cannot be adjacent to a same vertex as G is triangle free.) For any neighbour c' of b' in Q', c' must be adjacent to every neighbour u' of a, (because c' must be adjacent to one of u', v, and c' cannot be adjacent to v). Repeat this argument we conclude that

$$\forall a'' \in A', \quad N(a) \cap N(x) \subseteq N(a'') \cap N(x),$$

and similarly,

$$\forall b'' \in B', \quad N(b) \cap N(x) \subseteq N(b'') \cap N(x).$$

Let a', b' play the roles of a, b, we conclude that

$$\forall c \in A \cup A', \quad N(c) \cap N(x) = N(a') \cap N(x) = N(a) \cap N(x)$$

and

$$\forall d \in B \cup B', \quad N(d) \cap N(x) = N(b') \cap N(x) = N(b) \cap N(x).$$

So the mapping that sends all vertices of $A \cup A'$ to a' and all vertices of $B \cup B'$ to b' and fixes every other vertex is a retraction, contrary to the assumption that G is a core. ∎

Lemma 3.2 *For each $x \in V$, H_x is connected.*

Proof. Assume to the contrary that H_x is not connected and $Q = A \cup B$ is a connected component of H_x. By Lemma 3.1, we may assume that $N(A) \cap N(x) = \emptyset$. Then the map which sends B to x and sends A to a neighbour of x is a retraction, contrary to the assumption that G is a core. ∎

Since H_x is connected, there is a unique partition of H_x into two parts, which we denote by A_x and B_x. A vertex u of H_x is called a *$\leq^x$-maximum vertex* if $v \leq^x u$ for every other vertex v in the same part of H_x.

3.2 Ordering of A_x and B_x

In this subsection, we prove that all the vertices of A_x (respectively of B_x) are $\leq^x$-comparable. An induced path P of H_x is called a *maximal induced path* if P is not a proper subpath of an induced path P' of H_x. We denote by $\ell(P)$ the length (i.e., number of edges) of P.

Lemma 3.3 *If $P = (p_0, p_1, p_2, \cdots, p_{k-2}, p_{k-1}, p_k)$ is a maximal induced path of H_x, then $N_{H_x}(p_0) \subseteq N_{H_x}(p_2)$ and $N_{H_x}(p_k) \subseteq N_{H_x}(p_{k-2})$.*

Proof. Since P is a maximal induced path of H_x, $N_{H_x}(p_0) \subseteq \cup_{i=1}^{k} N_{H_x}(p_i)$. Assume to the contrary that there is a vertex $w \in N_{H_x}(p_0) - N_{H_x}(p_2)$. Let i be the minimum index such that $w \in N_{H_x}(p_i)$. As H_x is bipartite, i is even. Therefore $i \geq 4$. Now $(p_0, p_1, \cdots, p_i, w)$ is an induced cycle of H_x of length at least 6, contrary to our assumption. Similarly we can prove that $N_{H_x}(p_k) \subseteq N_{H_x}(p_{k-2})$. ∎

Lemma 3.4 *If P is an induced path of H_x which contains two $\leq^x$-incomparable vertices of A_x (or B_x), then $\ell(P) \leq 2$.*

Proof. Assume to the contrary that P is an induced path of H_x which contains two $\leq^x$-incomparable vertices of A_x (or B_x), and has length at least 3. As each induced path of H_x is contained in a maximal induced path, we may assume that P is a maximal induced path of H_x.

Suppose $P = (p_0, p_1, \cdots, p_k)$. By Lemma 3.3, $N_{H_x}(p_0) \subseteq N_{H_x}(p_2)$ and $N_{H_x}(p_k) \subseteq N_{H_x}(p_{k-2})$. As G is a core, we conclude that $p_0 \not\leq^x p_2$ and $p_k \not\leq^x p_{k-2}$. As P is well-linked with respect to x and $k \geq 3$, we conclude that k is odd, and

$$p_1 \leq^x p_3 \leq^x \cdots \leq^x p_{k-2} <^x p_k,$$

$$p_{k-1} \leq^x p_{k-3} \leq^x \cdots \leq^x p_2 <^x p_0.$$

This is in contrary to the assumption that P contains two $\leq^x$-incomparable vertices of A_x (or B_x). ∎

Corollary 3.1 *If $a, a' \in A_x$ are $\leq^x$-incomparable, then $N_{H_x}(a) = N_{H_x}(a')$.*

Proof. For otherwise H_x contains an induced path of length at least 3 which contains both a and a', contrary to Lemma 3.4. ∎

Suppose $x \in V$, we now define a graph Q with vertex set A_x as follows: $a \sim_Q a'$ if and only if a and a' are $\leq^x$-incomparable.

Corollary 3.2 *Let A be a connected component of Q. Let $B = N_{H_x}(A)$. Then the subgraph H of H_x induced by $A \cup B$ is a complete bipartite graph.*

Lemma 3.5 *Assume H_x is not complete. Then the vertices of A_x (respectively of B_x) are pairwise $\leq^x$-comparable.*

Proof. Assume to the contrary that H_x is not complete, and A_x contains two $\leq^x$-incomparable vertices. Then the graph Q defined above has a component A which contains at least two vertices. Let $B = N_{H_x}(A)$. By Corollary 3.2, the union $A \cup B$ induce a complete bipartite subgraph H of H_x. Since H_x is not complete, each edge of H_x is contained in an induced path of length at least 3. Let $a \in A$ and $b \in B$ and P be a longest induced path containing the edge ab. Assume

$$P = (\cdots, a_0, b_0, a, b, a_1, b_2 \cdots).$$

Note that a or b_0 could be the initial vertex of P. In this case, a_0 does not exist. Also b could be the terminal vertex of P. In this case, a_1 does not exist. However, since P has length at least 3, at least one of a_0, a_1 exists and does not belong to A. (Recall that all the vertices of A has the same neighbourhood in H_x.) Without loss of generality, we assume that a_1 exists and does not belong to A. Then a_1 is $\leq^x$-comparable to every vertex of A.

Assume $a, a' \in A$ are two $\leq^x$-incomparable vertices. Let

$$u \in (N(a) - N(a')) \cap N(x), \quad v \in (N(a') - N(a)) \cap N(x).$$

Fig. 1 below is a depiction of the named vertices (not all edges are drawn).

If $a_1 \leq^x a$ then $a_1 \leq^x a'$. Then $a_1 \not\sim u, a_1 \not\sim v$, hence a, u, x, v, a' induce a P_4 in H_{a_1}. Since

$$b \in N(a_1) \cap N(a) \cap N(a') \text{ and } b \notin N(x),$$

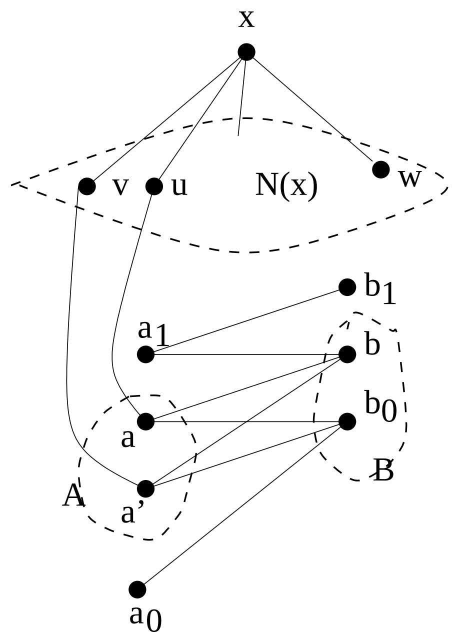

Figure 1: A depiction of $x, N(x), H_x$ for the proof of Lemma 3.5

we conclude that $a \not\leq^{a_1} x$ and $a' \not\leq^{a_1} x$. Hence the path induced by a, u, x, v, a' is not well-linked with respect to a_1, contrary to our assumption.

Thus we assume that $a <^x a_1$ and $a' <^x a_1$. As G is a core, we know that $N(a) \not\subseteq N(a_1)$, hence $N_{H_x}(a) \not\subseteq N_{H_x}(a_1)$. By Lemma 3.3, P contains the vertex b_0. Because P is well-linked with respect to x, and $a <^x a_1$, we conclude that $b \leq^x b_0$.

Assume first that $b <^x b_0$. Let $w \in (N(b_0) - N(b)) \cap N(x)$. If w is not adjacent to a_1, then a', v, a_1, u, a is an induced path in H_w (recall that $N(x)$ is an independent set, because G is triangle free). Since

$$b_0 \in N(w) \cap N(a) \cap N(a') \text{ and } \ b_0 \notin N(a_1),$$

we have $a' \not\leq^w a_1$ and $a \not\leq^w a_1$. Hence the path induced by a', v, a_1, u, a is not well-linked with respect to w, contrary to our assumption. Thus we assume that w is adjacent to a_1. Let $w' \in N(x) - N(a_1)$ (w' exists because G is a core). If w' is adjacent to b_0, then a', v, a_1, u, a is an induced path in $H_{w'}$, which is not well-linked (for the same reason as above). If w' is not adjacent to b_0, then w, b_0, a, b, a_1 induce a pentagon in $H_{w'}$, again contrary to our assumption.

So we must have $b =^x b_0$, i.e., $N(b) \cap N(x) = N(b_0) \cap N(x)$. Then because $N(b_0) \not\subseteq N(b)$ (as G is a core). By Lemma 3.3, b_0 is not the initial vertex of the induced path P, i.e., P contains the vertex a_0 (note that if a_0 is adjacent to b_1, then H_x contains an induced C_6, contrary to our assumption). Because P is well-linked, we have $a_0 \leq^x a$, and hence $a_0 \leq^x a'$. This is symmetric to the case that $a_1 \leq^x a$ and $a_1 \leq^x a'$ which we discussed above. Indeed, in this case, it can be verified that a, u, x, v, a' is an induced path of H_{a_0} which is not well-linked with respect to a_0. ∎

3.3 A new ordering on H_x

This subsection introduces a new ordering $\leq_N^x$ on A_x and B_x. This ordering is a refinement of the ordering $\leq^x$. Then we prove that each pair of vertices of A_x (as well as of B_x) are comparable with respect to $\leq_N^x$.

First we discuss for which $a, a' \in A_x$ we could have $a =^x a'$.

Lemma 3.6 *If $a, a' \in A_x$ and $a =^x a'$ then $N(a) \cap N(x) = N(a') \cap N(x) = \emptyset$.*

Proof. Assume to the contrary that $a, a' \in A_x$ are vertices such that $N(a) \cap N(x) = N(a') \cap N(x) \neq \emptyset$. Let

$$u \in N(a) \cap N(x) = N(a') \cap N(x).$$

Since G is a core, both sets $N_{H_x}(a) - N_{H_x}(a')$ and $N_{H_x}(a') - N_{H_x}(a)$ are nonempty. Let

$$b \in N_{H_x}(a) - N_{H_x}(a'), \quad b' \in N_{H_x}(a') - N_{H_x}(a).$$

(Since G is a core, b, b' exists). By applying Lemma 3.5, b and b' are $\leq^x$ comparable. Without loss of generality, we assume that $b \leq^x b'$. The named vertices are as depicted in Fig. 2 below.

If $N(b') \cap N(x) = \emptyset$, then let $w \in N(x) - N(a) = N(x) - N(a')$. If $b \sim w$, then a, b, w, x, u induce a pentagon in H_w. If $b \not\sim w$, then $P = (b, a, u, a', b')$ is an induced path of H_w which is not well-linked with respect to w, because $x \in N(w) \cap N(u)$ but $x \not\sim b, b'$, i.e., $u \not\leq^w b$ and $u \not\leq^w b'$.

Assume $N(b') \cap N(x) \neq \emptyset$, and let $w \in N(b') \cap N(x)$. If $w \not\sim b$, then $P = (b, a, u, a')$ is an induced path of H_w. As $x \in N(w) \cap N(u)$ but $x \not\sim b$, so $u \not\leq^w b$; $b' \in N(w) \cap N(a')$ but $b' \not\sim a$, so $a' \not\leq^w a$. Therefore P is not well-linked with respect to w.

If $w \sim b$ then $P = (x, u, a, b)$ is an induced path in $H_{b'}$, which is not well-linked with respect to b' (because $a' \in N(b') \cap N(u)$ but $a' \not\sim b$ which implies that $u \not\leq^{b'} b$, and $w \in N(x) \cap N(b')$ and $w \not\sim a$, which implies that $x \not\leq^{b'} a$). ∎

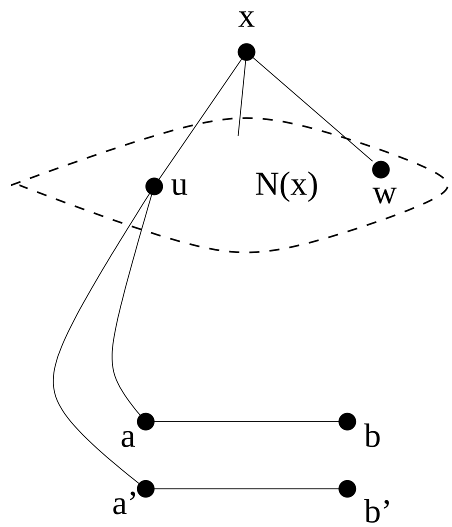

Figure 2: For the proof of Lemma 3.6

For $x \in V$, we let

$$A_x^* = \{a \in A_x : N(a) \cap N(x) \neq \emptyset\},$$

$$B_x^* = \{b \in B_x : N(b) \cap N(x) \neq \emptyset\}.$$

Lemma 3.6 says that if H_x is not a complete bipartite graph, then every two vertices of A_x^* (respectively of B_x^*) are strictly $\leq^x$-comparable (i.e., no two of them are equal with respect to $\leq^x$. However, all the vertices of $A_x - A_x^*$ are equal with respect to the order relation $\leq^x$. We shall introduce a refined order relation that distinguishes the vertices of $A_x - A_x^*$. Before introducing the refined order relation, we first prove that A_x^* and B_x^* are not empty.

Lemma 3.7 *If H_x is not complete then $A_x^* \neq \emptyset$ and $B_x^* \neq \emptyset$.*

Proof. If $N(a) \cap N(x) = \emptyset$ for all $a \in A_x$, then the mapping f which sends every vertex of B_x to x, and sends every vertex of A_x to a neighbour of x is a retraction, contrary to the assumption that G is a core. ∎

Corollary 3.3 *If H_x is not complete then each of A_x and B_x has a unique $\leq^x$-maximum vertex, belonging to A_x^* and B_x^*, respectively.*

Proof. Assume to the contrary that A_x has two $\leq^x$-maximum vertices a, a'. Then $a =^x a'$. By Lemma 3.6, we must have $N(a) \cap N(x) = N(a') \cap N(x) = \emptyset$, contrary to Lemma 3.7. ∎

For $a \in A_x - A_x^*$, let $\phi(x,a)$ be the distance in H_x from a to A_x^*. By Lemma 3.7, the function ϕ is well-defined, and $\phi(x,a) \geq 2$ for any $a \in A_x - A_x^*$. For $a \in A_x - A_x^*$, let $A_x^*(a)$ be the set of vertices $w \in A_x^*$ for which there is a path P in H_x of length $\phi(x,a)$ connecting a and w. Now we define a new ordering $\leq_N^x$ on the vertices of A_x. For $a, a' \in A_x$, we write $a <_N^x a'$ if and only if one of the following holds:

- $a <^x a'$;
- $a =^x a'$ and $\phi(x,a) > \phi(x,a')$ (note the reverse of the inequality, which is not a typo):
- $a =^x a'$, $\phi(x,a) = \phi(x,a')$ and $A_x^*(a) \subset A_x^*(a')$.

We write $a =_N^x a'$ if $a =^x a'$, $\phi(x,a) = \phi(x,a')$ and $A_x^*(a) = A_x^*(a')$; and write $a \leq_N^x a'$ if either $a <_N^x a'$ or $a =_N^x a'$. Similarly we define the orderings $<_N^x$, $\leq_N^x$ on B_x.

We shall prove in the following that every pair of vertices of A_x is $\leq_N^x$-comparable. In other words, if $a =^x a'$ and $\phi(x,a) = \phi(x,a')$ then one of the two sets $A_x^*(a)$, $A_x^*(a')$ contains the other.

Suppose P is a path of G and $u, v \in P$. Then we denote by $P[u,v]$ the subpath of P connecting u to v.

Lemma 3.8 *If $u, u' \in A_x$ and $u =^x u'$, then one of the two sets $A_x^*(u)$ and $A_x^*(u')$ contains the other. Similarly for any $v, v' \in B_x$, if $v =^x v'$ then one of the sets $B_x^*(v)$ and $B_x^*(v')$ contains the other.*

Proof. Suppose $u, u' \in A_x$ and $u =^x u'$. By Lemma 3.6, $u, u' \in A_x - A_x^*$. Assume to the contrary that there exist

$$w \in A_x^*(u) - A_x^*(u'), \quad w' \in A_x^*(u') - A_x^*(u).$$

Let P, P' be shortest paths of H_x connecting u to w, and u' to w' respectively. First we show that no vertex of P is adjacent to a vertex of P'. Assume to the contrary that $s \in P$ is adjacent to $s' \in P'$.

If $\ell(P[s,w]) \geq \ell(P'[s',w']) + 1$, then the concatenation of $P[u,s], ss', P'[s',w']$ is a path connecting u to w' which has length at most $\phi(x,u)$, contrary to our assumption that $w' \notin A_x^*(u)$. . So $\ell(P[s,w]) \leq \ell(P'[s',w'])$. By symmetry we have $\ell(P[s,w]) \geq \ell(P'[s',w'])$. Therefore

$\ell(P[s,w]) = \ell(P'[s',w'])$. However, this is impossible because $\ell(P[s,w])$ and $\ell(P'[s',w'])$ have different parity (note that both $w, w' \in B_x$ and $s \sim s'$, so one of the s, s' is in B_x and the other is in A_x).

Now let P'' be a shortest path (and hence an induced path) of H_x connecting u to u'. Since P'' is well-linked with respect to x, we conclude that $P'' \cap A_x^* = \emptyset$ (as the two ends of P'' are vertices in $A_x - A_x^*$). Now the union of P, P', P'' contains an induced path P^* connecting w and w'. If P^* contains any vertex, say w'', of A_x other than w, w', then because $w'' \in A_x - A_x^*$ (observe that $(P \cup P' \cup P'') \cap A_x^* = \{w, w'\}$), P'' is not well-linked with respect to x. Thus P^* contains no vertex of A_x, i.e., $P^* = (w, b, w')$ for some $b \in B_x$. If b is adjacent to any vertex of P other than w, say $b \sim s$ for $s \in P$, then the concatenation of $P[u,s]$ and (s,b,w) would be a shortest path of H_x connecting u to w. But this path contains a vertex b which is adjacent to a vertex w' of P', contrary to the previous paragraph. Similarly b is not adjacent to any vertex of P' other than w'. Hence the concatenation of $P[u,w], (w,b,w'), P[w',u']$ is an induced path of H_x. However, this path is not well-linked with respect to x, because $N(w) \cap N(x) \neq \emptyset$, and yet $N(u) \cap N(x)$ and $N(u') \cap N(x)$ are both emptyset. ∎

By Lemma 3.8, each pair of vertices of A_x are $\leq_N^x$-comparable, and similarly each pair of vertices of B_x are $\leq_N^x$-comparable.

3.4 Ordering B_x and B_a

Suppose a is a $\leq^x$-maximum vertex of A_x. This subsection and the next subsection discuss the relation between the two orderings $\leq_N^x$ and $\leq_N^a$. In this subsection, we prove that if a is the $\leq^x$-maximum vertex of A_x, then in the set $B_x \cap B_a$ which is contained in their common domain, these two ordering relations coincide with each other. Moreover, $B_a - B_x = \{x\}$ and x is the $\leq^x$-maximum vertex of B_a.

Lemma 3.9 *Suppose H_x is not complete and a is the $\leq^x$-maximum vertex of A_x. Then the two parts of H_a are $A_a = (A_x - \{a\}) \cup (N(x) - N(a))$ and $B_a = (B_x - N(a)) \cup \{x\}$. Moreover, the following are true:*

1. *H_a is also not complete;*

2. *x is the $\leq^a$-maximum vertex of B_a;*

3. *all $y \in A_a - A_x = N(x) - N(a)$ and $y' \in A_a \cap A_x$, then $y \leq^a y'$;*

4. if $y, y' \in A_a \cap A_x$ then $y <^x y'$ implies $y <^a y'$;

5. if $y, y' \in B_a \cap B_x$, then $y <^a y'$ implies $y <^x y'$.

Proof. Obviously we have

$$V(H_a) = (A_x - \{a\}) \cup (N(x) - N(a)) \cup (B_x - N(a)) \cup \{x\}.$$

It can be easily proved by induction on the length of the paths that any path in H_a starting from x with an odd length ends at a vertex of $(A_x - \{a\}) \cup (N(x) - N(a))$, and any path starting from x with an even length ends at a vertex of $(B_x - N(a)) \cup \{x\}$. As H_a is connected, so the two parts of H_a are as described in the lemma. Of course, the names A_a and B_a can be interchanged. However, in the following we shall always name A_a and B_a as described above.

Since x is not adjacent to any vertex of $A_x \cap A_a = A_x - \{a\}$, we know that H_a is not complete. (Note that $A_x - \{a\} \neq \emptyset$, for otherwise H_x is complete.) It is obvious that for any $b \in B_x$, $N(b) \cap N(a) \subseteq N(x)$. Therefore x is the $\leq^a$-maximum vertex of B_a. Next we prove that if $y \in A_a - A_x = N(x) - N(a)$ and $y' \in A_a \cap A_x$, then $y \leq^a y'$. Assume to the contrary that $w \in N(y) \cap N(a) - N(a') \cap N(a)$ for some $a' \in A_a \cap A_x$. Because a is $\leq^x$-maximum, by Lemmas 3.6 and 3.7, there exists $w' \in N(a) \cap N(x) - N(y') \cap N(x)$. Now w, y, x, w', a induce a nonbipartite graph in $H_{y'}$, contrary to our assumption. It remains to show that $\leq^a$ coincides with $\leq^x$ on their common domain. If $y, y' \in A_a \cap A_x$ and $y <^x y'$, then $N(y) \cap N(x) \subset N(y') \cap N(x)$. Since a is $\leq^x$-maximum, $N(y) \cap N(x) \subset N(a)$ and $N(y') \cap N(x) \subset N(a)$. Therefore $N(y) \cap N(a) \subset N(y') \cap N(a)$. If $y, y' \in B_a \cap B_x$ and $y <^a y'$, then $N(y) \cap N(a) \subset N(y') \cap N(a)$. Since $N(y) \cap N(a) \subset N(x)$ and $N(y') \cap N(a) \subset N(x)$. Therefore $N(y) \cap N(x) \subset N(y') \cap N(x)$. ∎

Lemma 3.9 basically says that $\leq^x$ and $\leq^a$ coincide in their common domain. Now we shall compare the refined orderings $\leq^x_N$ and $\leq^a_N$. First, for $u \in (B_a - B^*_a) \cap (B_x - B^*_x)$, we want to study the relation between $\phi(x, u)$ and $\phi(a, u)$.

Lemma 3.10 *Assume $a \in A_x$ is a $\leq^x$-maximum vertex of A_x. If $u \in B_x - (B^*_x \cup N(a))$ then for any shortest path P of H_x connecting u to B^*_x, $P \cap N(a) = \emptyset$.*

Proof. Assume $P = (u, a_1, b_1, \cdots, b_k)$ and b_k is adjacent to $w \in N(x)$. First we observe that $a \notin P$. Otherwise, say $a_j = a$ then $a_1 <^x a_j$, $u <^x$

b_k implies that P is not well-linked with respect to x. Assume now that $P \cap N(a) \neq \emptyset$. Let j be the least index such that $a \sim b_j$. If $j = k$ then $P' = (u, a_1, b_1, \cdots, b_k, a)$ is an induced path of H_x which is not well-linked (as $u <^x b_k$ and $a_1 <^x a$).

Assume $j < k$. Then $a \not\sim b_{j-1}$ and $a \not\sim b_{j+1}$, for otherwise we may replace a_j or a_{j+1} by a and obtain a shortest path in H_x connecting u to B_x^* which contains a, contrary to the conclusion above.

Let $w' \in N(a) \cap N(x) - (\cup_{i+1}^k N(a_i))$ (since a is the $\leq^x$-maximum vertex of A_x and every two vertices of A_x are $\leq^x$-comparable, the vertex w' exists). Consider the path

$$P'' = (b_{j-1}, a_j, b_j, a_{j+1}, b_{j+1}),$$

where b_{j-1} could be u. We must have $b_{j+1} \not\sim w'$, for otherwise $w, a, b_j, a_{j+1}, b_{j+1}$ induce a nonbipartite graph in $H_{b_{j-1}}$. So P'' is an induced path in $H_{w'}$ (note that $w' \not\sim b_j, b_{j-1}$, because $b_j, b_{j-1} \in B_x - B_x^*$). As $a \in N(w') \cap N(b_j)$ and $a \not\sim b_{j-1}$ $a \not\sim b_{j+1}$ which imply that $b_j \not\leq^x b_{j-1}$ and $b_j \not\leq^x b_{j+1}$, we conclude that P'' is not well-linked with respect to w'. ∎

Corollary 3.4 *Assume $a \in A_x$ is a $\leq^x$-maximum vertex of A_x. If $u \in B_x - (B_x^* \cup N(a))$ and $u \notin B_a^*$, then either*

- *$B_x^*(u) \cap B_a^* \neq \emptyset$ and $\phi(a, u) = \phi(x, u)$, or*
- *$B_x^*(u) \cap B_a^* = \emptyset$ and $\phi(a, u) = \phi(x, u) + 2$.*

Proof. Observe that $B_a^* \subseteq B_x^* \cup \{x\}$. Assume P is a shortest path of H_a connecting u to B_a^*. If P does not contain any vertex of $N[x]$, then P is also a path of H_x connecting u to B_x^*. If P does contain a vertex, say w, of $N(x)$, then let b' be the vertex of P preceding w, then $P[u, b']$ is a path of H_x connecting u to B_x^*. Therefore $\phi(a, u) \geq \phi(x, u)$.

If $B_x^*(u) \cap B_a^* \neq \emptyset$ then let P be a shortest path in H_x connecting u to a vertex $w \in B_x^*(u) \cap B_a^*$. By Lemma 3.10, P is also a path in H_a. Therefore $\phi(a, u) = \phi(x, u)$.

If $B_x^*(u) \cap B_a^* = \emptyset$ then any shortest path in H_x connecting u to B_x^* is not a path connecting u to B_a^*. Therefore $\phi(a, u) > \phi(x, u)$ and hence $\phi(a, u) \geq \phi(x, u) + 2$. Let P be a shortest path of H_x connecting u to $w \in B_x^*$, and let $v \in N(w) \cap N(x)$, then the concatenation of P and (w, v, x) is a path connecting u to x. As $x \in B_a^*$, we have $\phi(a, u) \leq \phi(x, u) + 2$ and hence $\phi(a, u) = \phi(x, u) + 2$. ∎

Now we are ready to study the ordering relation between two vertices $u, u' \in B_a \cap B_x$ with respect to the two orderings $\leq_N^x$ and $\leq_N^a$, where a is the $\leq^x$-maximum vertex of A_x. We shall eventually prove that $u \leq_N^x u'$ implies that $u \leq_N^a u'$. First we consider the case that $u \in B_x^*$.

Lemma 3.11 *Suppose $a \in A_x$ is a $\leq^x$-maximum vertex of A_x. Assume $u, u' \in B_x \cap B_a$, $u \neq u'$ and $u \leq^x u'$. If $u \in B_x^*$ then $u' \in B_a^*$.*

Proof. Since $u \in B_x^*$, $u \neq u'$ and $u \leq^x u'$, by Lemma 3.6), $u <^x u'$ and

$$\emptyset \neq N(u) \cap N(x) \subset N(u') \cap N(x).$$

Let

$$w \in N(u) \cap N(x), \quad w' \in N(u') \cap N(x) - N(u) \cap N(x).$$

Assume to the contrary of this lemma that $u' \notin B_a^*$. Then $w, w' \notin N(a)$. Therefore $w, w' \notin N(a')$ for any $a' \in A_x$. Since G is a core, $N(w') \not\subseteq N(w)$. Let $u'' \in N(w') - N(w)$. Then $u'' \in B_x$. But then none of $N(u'') \cap N(x)$ and $N(u) \cap N(x)$ contains the other, contrary to Lemma 3.5. ∎

Now we are ready to prove that $u \leq_N^x u'$ implies $u \leq_N^a u'$.

Lemma 3.12 *Suppose H_x is not complete and a is the $\leq^x$-maximum vertex of A_x. If $u, u' \in B_x \cap B_a$ and $u \leq_N^x u'$ then $u \leq_N^a u'$.*

Proof. Observe that

$$N(u) \cap N(a) \subseteq N(u) \cap N(x)$$

and

$$N(u') \cap N(a) \subseteq N(u') \cap N(x).$$

Since $u \leq_N^x u'$, we have $N(u) \cap N(x) \subseteq N(u') \cap N(x)$, which implies (by applying Lemma 3.5) that $N(u) \cap N(a) \subseteq N(u') \cap N(a)$.

If $N(u) \cap N(a) \subset N(u') \cap N(a)$ then $u <^a u'$ and hence $u <_N^a u'$, we are done.

Assume now that $N(u) \cap N(a) = N(u') \cap N(a)$. By Lemma 3.6,

$$N(u) \cap N(a) = N(u') \cap N(a) = \emptyset.$$

By Lemma 3.11, $u \notin B_x^*$, i.e., $N(u) \cap N(x) = \emptyset$. (For otherwise, we would have $u' \in B_a^*$, i.e., $N(u') \cap N(a) \neq \emptyset$.)

If $N(u') \cap N(x) \neq \emptyset$, then let $w \in N(u') \cap N(x)$. Then $P = (u', w, x)$ is a path from u to B_a^* (as $x \in B_a^*$). Therefore $\phi(a, u') = 2 \leq \phi(a, u)$. If $\phi(a, u) > 2$ then $u \leq_N^a u'$ and we are done. Assume $\phi(a, u) = 2$. We need to show that $B_a^*(u) \subseteq B_a^*(u')$. Since $N(u) \cap N(x) = \emptyset$, there is no path of length 2 in H_a connecting u to x, i.e., $x \notin B_a^*(u)$. Therefore, by applying Lemma 3.8, $B_a^*(u) \subset B_a^*(u')$, hence $u <_N^a u'$.

Assume now that $N(u') \cap N(x) = \emptyset$. Then $u =^x u'$, and $\phi(x, u) \geq \phi(x, u')$.

If $\phi(x, u) > \phi(x, u')$, then $\phi(x, u) \geq \phi(x, u') + 2$. By Corollary 3.4,

$$\phi(a, u) \geq \phi(x, u) \geq \phi(x, u') + 2 \geq \phi(a, u').$$

If $\phi(a, u) > \phi(a, u')$ then $u <_N^a u'$ and we are done. Assume that $\phi(a, u) = \phi(a, u')$. Then $x \in B_a^*(u') - B_a^*(u)$. By applying Lemma 3.8, we conclude that $B_a^*(u) \subset B_a^*(u')$. So $u <_N^a u'$.

Assume now that $\phi(x, u) = \phi(x, u')$. As $u \leq_N^x u'$, we know that $B_x^*(u) \subseteq B_x^*(u')$. By Corollary 3.4, either $B_x^*(u) \cap B_a^* \neq \emptyset$ and $\phi(a, u) = \phi(x, u)$ or $B_x^*(u) \cap B_a^* = \emptyset$ and $\phi(a, u) = \phi(x, u) + 2$.

In the former case, we have $B_x^*(u') \cap B_a^* \neq \emptyset$ (as $B_x^*(u) \subseteq B_x^*(u')$) and $\phi(a, u') = \phi(x, u')$. So $\phi(a, u') = \phi(a, u)$. As $B_x^*(u) \subseteq B_x^*(u')$, and $B_a^*(u) = B_x^*(u) \cap B_a^*$ and $B_a^*(u') = B_x^*(u') \cap B_a^*$ we conclude that $B_a^*(u) \subseteq B_a^*(u')$. So $u \leq_N^a u'$.

In the latter case, if $\phi(a, u') = \phi(x, u')$ then we have $\phi(a, u') < \phi(a, u)$ and hence $u \leq_N^a u'$; if $\phi(a, u') = \phi(x, u') + 2$ then we need to prove that $B_a^*(u) \subseteq B_a^*(u')$. Assume to the contrary that there is a vertex $b \in B_a^*(u) - B_a^*(u')$. Let

$$P = (u, \cdots, v, a', b)$$

be a shortest path of H_a connecting u to b. Since $b \notin B_a^*(u')$ we know that $v \notin B_x^*(u')$ and hence $v \notin B_x^*(u)$, because $B_x^*(u) \subseteq B_x^*(u')$. Therefore $N(v) \cap N(x) = \emptyset$. Let $v' \in B_x^*(u')$. Then $\phi(a, v) = \phi(a, v') = 2$. Note that $N(v') \cap N(b) = \emptyset$, for otherwise we would have $b \in B_a^*(u')$. Therefore $x \in B_a^*(v') - B_a^*(v)$ and $b \in B_a^*(v) - B_a^*(v')$, i.e., none of the sets $B_a^*(v')$ and $B_a^*(v)$ contains the other. This is in contrary to Lemma 3.8. This completes the proof. ∎

3.5 Ordering A_x and A_a

In this subsection, we compare the two orderings $\leq_N^x$ and $\leq_N^a$. We shall prove that if $v' \in A_x \cap A_a$ and $v \in A_a - A_x$ then $v \leq_N^a v'$. If $v, v' \in A_x \cap A_a$ and $v \leq_N^a v'$ then $v \leq_N^x v'$.

Lemma 3.13 *Suppose H_x is not complete and a is the $\leq^x$-maximum vertex of A_x. Then for any $v' \in A_x \cap A_a$ and $v \in A_a - A_x$ we have $v \leq^a_N v'$.*

Proof. Since a is the $\leq^x$-maximum vertex of A_x, $N(v') \cap N(x) = N(v') \cap N(a) \cap N(x)$. If $N(v') \cap N(x) \neq \emptyset$ then since $N(v) \cap N(x) = \emptyset$ (as G is triangle free), we have $N(v') \cap N(a) \not\subseteq N(v) \cap N(a)$. By Lemma 3.6, $N(v) \cap N(a) \subset N(v') \cap N(a)$. This implies that $v <^a v'$ and hence $v <^a_N v'$.

Assume now that $N(v') \cap N(x) = \emptyset$. First we show that $N(v) \cap N(a) \subseteq N(v') \cap N(a)$. Assume to the contrary that $b \in N(v') \cap N(a) - N(v) \cap N(a)$. By Lemma 3.3, there is a vertex $w' \in N(a) \cap N(x) - N(v')$. Then v, x, w', a, b induce a pentagon in $H_{v'}$, contrary to our assumption that $H_{v'}$ is bipartite.

If $N(v) \cap N(a) \subset N(v') \cap N(a)$, then $v <^a_N v'$ and we are done. Assume now that $N(v) \cap N(a) = N(v') \cap N(a)$. By Lemma 3.6,

$$N(v) \cap N(a) = N(v') \cap N(a) = \emptyset.$$

Let w be any vertex of $A^*_a(v)$ and let P be a shortest path of H_a connecting v to a vertex w. Let $w' \in N(a)$ be a vertex adjacent to w, and let $w'' \in N(x) \cap N(a)$. If v' is not adjacent to any vertex of the path P, then the concatenation of P and (w, w', a, w'', x, v) is an odd closed walk in H_v, contrary to our assumption. Thus v' is adjacent to some vertex of the path P. Therefore $\phi(a, v') \leq \phi(a, v)$. Moreover, if $\phi(a, v') = \phi(a, v)$ then $w \in A^*_a(v')$. As w is an arbitrary vertex of $A^*_a(v)$ we have $A^*_a(v) \subseteq A^*_a(v')$. So in any case we have $v \leq^a_N v'$. ∎

Lemma 3.14 *Suppose H_x is not complete and a is the $\leq^x$-maximum vertex of A_x. If $v, v' \in A_x \cap A_a$ and $v <^a v'$ then $v \leq^x_N v'$.*

Proof. If $v <^a v'$, then $N(v) \cap N(a) \subset N(v') \cap N(a)$. Let

$$w \in N(v') \cap N(a) - N(v) \cap N(a).$$

If $N(v') \cap N(x) \neq \emptyset$ then by Lemma 3.6 we know that $N(v) \cap N(x) \subset N(v') \cap N(x)$, hence $v <^x_N v'$ and we are done.

Assume

$$N(v') \cap N(x) = N(v) \cap N(x) = \emptyset.$$

So $v, v' \in A_x - A^*_x$. By Lemma 3.6, $v <^a v'$ implies that $N(v') \cap N(a) \neq \emptyset$. Let $w \in N(v') \cap N(a)$. Then (v', w, a) is a path of H_x connecting v' to A^*_x. Therefore $\phi(x, v') = 2$ (because $a \in A^*_x$).

If $\phi(x, v) > 2$ then $v <^x_N v'$ and we are done. Thus we assume that $\phi(x, v) = 2$. If $A^*_x(v) \subseteq A^*_x(v')$ then $v \leq^x_N v'$ and we are done. Thus we assume that there is a vertex $a' \in A_x^*(v) - A_x^*(v')$. Let (v, w', a') be a path in H_x. Observe that (v', w, a) is a path of H_x as well.

We must have $a' \not\sim w$ and $v' \not\sim w'$ for otherwise $a' \in A_a^*(v')$.

Let P be a shortest path of H_x connecting v to v'. Since the two end vertices of P are not in A_x^*, P does not contain any vertex of A_x^*, for otherwise P is not well-linked with respect to x. Now the concatenation of (a', w', v), P and (v', w, a) is a walk W in H_x connecting a' to a. It contains an induced path P' of H_x connecting a' and a. If P' contains any other vertex of A_x other than a, a', then P' is not well-linked with respect to x, because all the other vertices of $W \cap A_x$ is in $A_x - A_x^*$. Thus P' is of the form (a, b, a') for some $b \in P \cap B_x$. We have $b \not\sim v'$, for otherwise $a' \in A_x^*(v')$. But then the concatenation of $P[v, b]$ and (b, a, w, v') is a v, v'-path of H_x whose length is not longer than P. Hence P^* is a shortest v, v'-path, and hence is an induced path. But P^* is not well-linked with respect to x, because the two end vertices of P^* are not in A_x^*, and it contains the vertex a which is in A_x^*. This completes the proof of Lemma 3.14. ∎

Lemma 3.15 *Suppose H_x is not complete and a is the $\leq^x$-maximum vertex of A_x. If $v, v' \in A_x \cap A_a$ and $v \leq_N^a v'$ then $v \leq_N^x v'$.*

Proof. By Lemma 3.14, it suffices to consider the case that $v =^a v'$, i.e., $N(v) \cap N(a) = N(v') \cap N(a)$. By Lemma 3.6, $N(v) \cap N(a) = N(v') \cap N(a) = \emptyset$, i.e., $v, v' \in A_a - A_a^*$.

Claim *For $u \in A_a - A_a^*$, if $A_a^*(u)$ contains a vertex w such that $N(w) \cap N(x) \neq \emptyset$, then $\phi(a, u) = \phi(x, u)$. Otherwise $\phi(a, u) = \phi(x, u) - 2$.*

The proof of this claim is similar to that of Corollary 3.4, and we omit the details.

If $\phi(a, v') < \phi(a, v)$, then $\phi(a, v') \leq \phi(a, v) - 2$ and hence

$$\phi(x, v') \leq \phi(a, v') + 2 \leq \phi(a, v) \leq \phi(x, v).$$

Moreover, in case $\phi(x, v') = \phi(x, v)$, we have $\phi(x, v') = \phi(a, v') + 2$ which implies that $a \in A_x^*(v')$ and $\phi(x, v) = \phi(a, v)$ which implies that $a \notin A_x^*(v)$. So in this case $A_x^*(v) \subset A_x^*(v')$, and hence $v <_N^x v'$.

Assume $\phi(a, v') = \phi(a, v)$. As $v \leq_N^a v'$, we have $A_a^*(v) \subseteq A_a^*(v')$. By the claim we have $\phi(x, v') \leq \phi(x, v)$. If $\phi(x, v') < \phi(x, v)$, then $v <_N^x v'$ and we are done. Assume $\phi(x, v') = \phi(x, v)$. By the claim, either

- $\phi(x, v') = \phi(a, v')$ and $\phi(x, v) = \phi(a, v)$, or

- $\phi(x, v') = \phi(a, v') + 2$ and $\phi(x, v) = \phi(a, v) + 2$.

In the former case, since a is the $\leq^x$-maximum vertex of A_x which implies that $A_x^* - \{a\} \subseteq A_a^*$, we have $A_x^*(v') = A_a^*(v') \cap A_x^*$ and $A_x^*(v) = A_a^*(v) \cap A_x^*$. Therefore, $A_x^*(v) \subseteq A_x^*(v')$, hence $v \leq_N^x v'$ and we are done.

Assume now that

$$\phi(x, v') = \phi(a, v') + 2, \quad \phi(x, v) = \phi(a, v) + 2.$$

We need to prove that $A_x^*(v) \subseteq A_x^*(v')$. Assume to the contrary that there is a vertex $a' \in A_x^*(v) - A_x^*(v')$. Let

$$P = (v, \cdots, b, w, b', a')$$

be a path of H_x of length $\phi(x, v)$ connecting v to a'. Then $w \notin A_a^*$, for otherwise $w \in A_a^*(v) \subseteq A_a^*(v')$ and hence $a' \in A_x^*(v')$. Therefore $b' \not\sim a$.

Let $u \in A_a^*(v) \subseteq A_a^*(v')$, and let $b'' \in N(u) \cap N(a)$. Let

$$P' = (v, \cdots, u, b'', a)$$

be a path of H_x of length $\phi(x, v)$ connecting v to a. Now $b'' \not\sim w$, for otherwise we would have $w \in A_a^*$. Since

$$N(a') \cap N(x) \neq \emptyset$$

and

$$N(a') \cap N(x) \subseteq N(a) \cap N(x),$$

$a' \in A_a^*$, and there is a vertex $z \in N(a') \cap N(a) \cap N(x)$. Because $z \notin N(u)$ (for otherwise we would have $u \in A_x^*(v)$ and $\phi(x, v) = \phi(a, v)$), by Lemma 3.5, we know that $N(u) \cap N(a) \subset N(a') \cap N(a)$. Hence $b'' \sim a'$. So the concatenation of $P[v, a']$ and $P[v, b'']$ is a closed walk Z of H_x. The segment of this closed walk "near" b'' is as follows:

$$(\cdots, b, w, b', a', b'', u, \cdots).$$

Moreover, each of these six vertices occurs in the walk just once (as the paths P, P' are shortest paths from v to a', a). Note that $b \not\sim a'$, for otherwise there is a path of H_x of length $\phi(a, v) - 2$ connecting v to a', contrary to the definition of $\phi(x, v)$. We observed before that $b'' \not\sim w$. These imply that Z contains an induced cycle of length at least 6, contrary to our assumption. ∎

3.6 The final contradiction

Now we shall complete the proof of Theorem 2.1.

Let $V' = \{x \in V : H_x \text{ is not complete}\}$. If $V' = \emptyset$ then for each $x \in V$, $V - N[x]$ contains no induced P_4, hence G is circular perfect by Theorem 1.1.

Thus we may assume that $V' \neq \emptyset$. Define a graph Q with vertex set V', in which $x \sim y$ if and only if y is a $\leq^x$-maximum vertex of A_x or B_x. It follows from Lemma 3.9 and Corollary 3.3 that Q is a graph in which each vertex has degree 2. Let $C = (x_0, x_1, x_2, \cdots, x_{k-1})$ be a cycle which is a connected component of Q. Arbitrarily assign a direction of traversal to the cycle C, say $x_0 \to x_1 \to x_2 \to \cdots \to x_{k-1}$. For any x_i, the two neighbours x_{i-1} and x_{i+1} are the two $\leq^x$-maximum vertices of x_i. It follows from Lemma 3.9 that we can properly label the two parts of H_{x_i} so that $x_{i+1} \in A_{x_i}$ and $x_{i-1} \in B_{x_i}$, for all i (where summation and subtraction in the index are modulo k). By the moreover part of Lemma 3.9 and Lemma 3.15, for each i, there is a $d_i \geq 2$ such that $A_{x_i} = \{x_{i+1}, x_{i+2}, \cdots, x_{i+d_i-1}\}$. Indeed, x_{i+1} is the $\leq^x$-maximum vertex of A_{x_i}. Now x_{i+2} is the $\leq^x$-maximum vertex of $A_{x_{i+1}}$ and by Lemma 3.15, x_{i+2} is the $\leq^x$-maximum vertex of $A_{x_i} - \{x_{i+1}\}$, provided $A_{x_i} - \{x_{i+1}\} \neq \emptyset$. In general, if

$$A_{x_i} - \{x_{i+1}, x_{i+2}, \cdots, x_{i+t}\} \neq \emptyset,$$

then by Lemma 3.15, x_{i+t+1} is the $\leq^x$-maximum vertex of $A_{x_{i+t}}$ which is the $\leq^x$-maximum vertex of

$$A_{x_i} - \{x_{i+1}, x_{i+2}, \cdots, x_{i+t}\}.$$

By Lemma 3.13 and Lemma 3.15, for each i, there is a $d'_i \geq 2$ such that $B_{x_i} = \{x_{i-1}, x_{i-2}, \cdots, x_{i-d'_i-1}\}$.

By Lemma 3.9, $A_{x_{i+1}} = (A_{x_i} - \{x_{i+1}\}) \cup R$, where $R = N(x_i) - N(x_{i+1})$. Since $R \neq \emptyset$ (for otherwise $N(x_{i+1}) \subseteq N(x_i)$, contrary to the assumption that G is a core), we conclude that $|A_{x_{i+1}}| \geq |A_{x_i}|$ for all i (again summation in the index is modulo k). So

$$|A_{x_0}| \leq |A_{x_1}| \leq |A_{x_2}| \leq \cdots \leq |A_{x_{k-1}}| \leq |A_{x_0}|.$$

Therefore $|A_{x_i}| = |A_{x_j}|$ for all $0 \leq i, j \leq k-1$. So $d_i = d_j$ for all $0 \leq i, j \leq k-1$. Let $d = d_i$.

Similarly, by Lemma 3.9, $B_{x_{i+1}} = (B_{x_i} - N(x_{i+1})) \cup \{x_i\}$. As $N(x_{i+1}) \cap B_{x_i} \neq \emptyset$, we conclude that $|B_{x_{i+1}}| \leq |B_{x_i}|$ for all i (again summation in the index is modulo k). So

$$|B_{x_0}| \geq |B_{x_1}| \geq |B_{x_2}| \geq \cdots \geq |B_{x_{k-1}}| \geq |B_{x_0}|.$$

Therefore $|B_{x_i}| = |B_{x_j}|$ for all $0 \le i, j \le k-1$. So $d'_i = d'_j$ for all $0 \le i, j \le k-1$. Because $\Sigma_{i=0}^{k-1} d_i = \Sigma_{i=0}^{k-1} d'_i$, which is equal to the number of nonedges of the subgraph of G induced by $x_0, x_1, \cdots, x_{k-1}$, we conclude that $d'_i = d$ for all i.

Now for any $x_i \in C$, any vertex x of G not adjacent to x either belong to A_{x_i} or belong to B_{x_i}. So $x = x_j \in C$ for some j, and $d \le |i-j| \le k-d$. Therefore, the subgraph of G generated by $x_0, x_1, \cdots, x_{k-1}$ is isomorphic to $K_{k/d}$. Moreover, if $y \in V(G) - C$ then y is adjacent to every vertex of C.

If $V(G) \neq C$, then for any $y \in V(G) - C$, we have $C \subset N[y]$, contrary to the assumption that G is triangle free (observe that C contains at least one edge because H_x is connected). If $V(G) = C$, then $G = K_{k/d}$, contrary to our assumption that $\omega_c(G) \neq \chi_c(G)$. This completes the proof of Theorem 2.1.

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