Recent developments in circular colouring of graphs

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2005

Abstract

The study of circular chromatic number $\chi_c(G)$ of a graph G, which is a refinement of its chromatic number, has been very active in the past decade. Many nice results are obtained, new techniques are developed, and connections to other fields are established. This paper presents a glimpse of the recent progress on this subject. Besides presenting the results, some of the ideas and tools in the proofs are explained, although no detailed proofs are contained.

Key words: circular chromatic number, circular chromatic index, circular perfect graphs, circular flow number, graph homomorphism.

*This research was partially supported by the National Science Council under grant NSC93-2115-M-110-004

1 Definitions

The circular chromatic number of a graph can be defined in a few different but equivalent ways. We first list some of the commonly used definitions.

Circular r-colouring Suppose $G = (V, E)$ is a graph and $r \geq 1$ is real number. A *circular r-colouring* of G is a mapping $f : V \to [0, r)$ such that for any edge xy of G, $1 \leq |f(x) - f(y)| \leq r - 1$. We say a graph G is *circular r-colourable* if G has a circular r-colouring. The *circular chromatic number* $\chi_c(G)$ of G is defined as

$$\chi_c(G) = \inf\{r : G \text{ is circular } r\text{-colourable. }\}$$

It is known [11, 13, 134, 159] that the infimum in the definition above is always attained (even if G is an infinite graph), and hence can be replaced by the minimum. If $r = k$ is an integer, then a k-colouring is a circular k-colouring. Conversely, if f is a circular k-colouring of G then $g(v) = \lfloor f(v) \rfloor$ defines a k-colouring of G. So a graph G is circular k-colourable if and only if G is k-colourable. For this reason, a circular r-colouring of a graph G is usually simply called an r-colouring of G. Instead of saying G is circular r-colourable, we usually simply say G is r-colourable. It is obvious that if $r' \geq r$ and G is r-colourable then G is r'-colourable. This implies that for any graph G,

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G). \tag{1}$$

Inequality (1) shows that $\chi_c(G)$ contains more information about the structure of the graph G than $\chi(G)$ does. We say $\chi_c(G)$ is a refinement of $\chi(G)$ and $\chi(G)$ is an approximation of $\chi_c(G)$. If G is a finite graph, then $\chi_c(G) = p/q$ for some integers p, q (not necessarily relatively prime), such that G has a cycle of length p (unless G is a forest) and every vertex of G is contained in an independent set of size at least q. So if G has n vertices, then $\chi_c(G) = p/q$ for some $p \leq n$. Hence (1) can be improved to

$$(\chi(G) - 1)(1 + \frac{1}{n-1}) \leq \chi_c(G) \leq \chi(G). \tag{1'}$$

In a circular r-colouring of a graph G, the "colour set" consists of all the real numbers in the interval $[0, r)$. A better way of picturing the colour set is to identify 0 and r of the interval $[0, r]$ into a single point to obtain a circle of perimeter r. We denote this circle by S^r. So the colour set is the set of points on the circle S^r. For two points $a, b \in S^r$, the distance between a, b, denoted by $|a - b|_r$, is the length of the shorter arc of S^r connecting a and b. For a real number x and a positive real number r, we denote by $[x]_r$ the remainder of x dividing r, i.e., $[x]_r \in [0, r)$ is the unique number for which $x - [x]_r$ is a multiple of r. Then

$$|a - b|_r = \min\{[a - b]_r, [b - a]_r\} = \min\{|a - b|, r - |a - b|\}.$$

By this notation, a circular r-colouring of G is a mapping f which assigns to each vertex x of G a point $f(x) \in S^r$ such that for any edge xy of G, $|f(x) - f(y)|_r \geq 1$. This interpretation explains the name "circular colouring" and "circular chromatic number".

(p, q)-colouring Another useful definition of circular chromatic number uses the concept of (p, q)-colouring, where only finitely many colours are used. Suppose $p \geq q$ are positive integers. A (p, q)-colouring of a graph $G = (V, E)$ is a mapping $f : V \to \{0, 1, \ldots, p - 1\}$ such that for any edge xy of G, $q \leq |f(x) - f(y)| \leq p - q$. If f is a (p, q)-colouring of G, then the mapping $g(x) = f(x)/q$ defines a $\frac{p}{q}$-colouring of G. Conversely, if g is a $\frac{p}{q}$-colouring of G, then the mapping $f(x) = \lfloor g(x)q \rfloor$ defines a (p, q)-colouring. Therefore, for any graph G,

$$\chi_c(G) = \inf\{\frac{p}{q} : G \text{ has a } (p, q)\text{-colouring }\}.$$

If G is finite, then the infimum can be replaced by minimum. But infinite graphs G can have $\chi_c(G)$ equal to irrationals and the infimum cannot be replaced by the minimum.

Graph homomorphism Suppose G, H are graphs. A *homomorphism* of G to H is a mapping $f : V(G) \to V(H)$ such that for every edge xy of G, $f(x)f(y)$ is an edge of H. Two graphs G and H are *homorphically equivalent*, written as $G \sim H$, if each admits a homomorphism to the other. For positive integers $p \geq 2q$, let $K_{p/q}$ be the graph with vertex set $\{0, 1, \ldots, p-1\}$ in which ij is an edge if and only if $q \leq |i-j| \leq p-q$. Then a (p,q)-colouring of a graph G is simply a homomorphism of G to $K_{p/q}$.

Graph homomorphisms provide a unified language and useful tool for the study of different graph colouring problems [45, 55]. We write $G \preceq H$ if there is a homomorphism from G to H. Let $\mathcal{G}$ denote the set of (isomorphic classes of) graphs. Then $(\mathcal{G}/\sim, \preceq)$ is a partial order. The set $\mathcal{Z}_{\mathcal{G}} = \{K_1, K_2, \ldots\}$ of complete graphs form an infinite increasing chain in this partial order. This infinite increasing chain provides a scale that measures the 'colourability' of graphs. For a graph G, the chromatic number of G is the least k for which $G \preceq K_k$. The graphs $K_{p/q}$ play a special role in the study of circular colourings, and are called *circular complete graphs*. Let $\mathcal{Q}_{\mathcal{G}} = \{K_{p/q} : p \geq 2q, (p,q) = 1\} \cup \{K_1\}$. Then $\mathcal{Q}_{\mathcal{G}}$ is a superset of $\mathcal{Z}_{\mathcal{G}}$, and provides a finer scale that measures the colourability of graphs [152]. The circular chromatic number of G can be defined as

$$\chi_c(G) = \inf\{p/q : G \preceq K_{p/q}\}.$$

Orientation Another point of view of circular chromatic number is reflected in a result of Hoffman [57], relating the chromatic number of a graph G to orientations of G. An orientation D of a graph G is obtained from G by assigning to each edge an orientation. An edge with a given orientation is called an *arc*, and the set of arcs of D is denoted by $A(D)$. Suppose D is an orientation of G and C is a cycle of G. The *imbalance* of a cycle C of G with respect to D is $\mathrm{Imb}_D(C) = \max\{|C|/|C^+|, |C|/|C^-|\}$, where C^+ and C^- are the sets of forward arcs and backward arcs of C, respectively. The *Cycle Imbalance of* D is defined as $\mathrm{CycImb}(D) = \sup\{\mathrm{Imb}_D(C) : C$ is a cycle of $G\}$. Hoffman's Lemma says that for any graph G which is not a forest ,

$$\chi(G) = \inf\{\lceil \mathrm{CycImb}(D) \rceil, D \text{ is an acyclic orientations of } G\}.$$

It is proved in [39] (cf. [16]) that if the ceiling function is omitted, then one obtains the circular chromatic number of G, i.e.,

$$\chi_c(G) = \inf\{\mathrm{CycImb}(D), D \text{ is an acyclic orientations of } G\}.$$

So to prove a graph G has $\chi_c(G) \leq p/q$, it suffices to have an orientation D in which each cycle C has $|C|/|C^+| \leq p/q$ and $|C|/|C^-| \leq p/q$. Indeed, it is shown in [162] that it suffices to check those cycles C for which $[q|C|]_p \in \{1, 2, \ldots, 2q-1\}$.

Tension Suppose G is a graph and D is an orientation of G. A *tension* is a mapping $f : A(D) \to \mathbb{R}$ which assigns to each arc e of D a real number $f(e)$ such that for each cycle C of G,

$$\sum_{e \in C^+} f(e) = \sum_{e \in C^-} f(e).$$

For a real number $r \geq 2$, an r-*tension* of G is a tension f such that for each arc e, $1 \leq |f(e)| \leq r-1$. Each r-colouring ϕ of G corresponds to an r-tension f of G defined as $f(e) = \phi(y) - \phi(x)$, where $e = (x,y)$ is an arc. Conversely, if we have an r-tension f of G, then let x^* be a fixed vertex of G, and for each vertex x of G let W_x be an arbitrary x^*-x-walk. Then $\phi(x) = [\sum_{e \in W_x^+} f(e) - \sum_{e \in W_x^-} f(e)]_r$ defines an r-colouring of G. Therefore,

$$\chi_c(G) = \min\{r : \text{ there is an } r\text{-tension of } G\}.$$

The different points of view of looking at the circular chromatic number of graphs show that the parameter $\chi_c(G)$ is a very natural refinement of $\chi(G)$.

2 Circular colouring and periodic scheduling

Graph colouring is an ideal model for various scheduling problems. If the scheduling is periodic, then it is very likely that circular colouring of graphs provides a more accurate model. In [159], a traffic light problem (assigning the green light phases to the traffic flows) is used to motivate the definition of circular chromatic number. The traffic light problem is a typical periodic scheduling problem. However, such problems are usually not very complicated, and are usually solved by experience, without explicitly using the graph model. On the other hand, there are many periodic scheduling problems that are very complicated, and mathematical models are needed to analyze them. Computer science is a rich source for periodic scheduling problems, and sometimes circular colouring of graphs can be used in finding optimal solutions in such problems.

One problem studied extensively by computer scientists is the concurrency of *heavily loaded resource sharing systems*. Let V be a set of processes, and D be a set of data files. Each process x has access to a set $D(x) \subseteq D$ of data files. When a process x operates, it accesses all the files in $D(x)$. Therefore if processes x and y share a common data file, then x and y cannot operate at the same time. To ensure fairness, if x and y share a common data file, then x and y must alternate their turns to operate. In a heavily loaded resource sharing system, all the processes are constantly demanding access to all resources that they use. Subject to the constraints of fairness and that processes sharing a resource cannot operate at the same time, our task is to schedule the operating time of processes efficiently.

This problem is modeled by a graph, where each vertex represents a process, and two vertices are adjacent if the corresponding processes share a resource. A *scheduling* of G is a mapping f which assigns to each vertex x of G a subset $f(x)$ of $\{0, 1, \ldots\}$. The interpretation is that if $i \in f(x)$ then x operates at time i. A scheduling f of G is *valid* if for any edge xy of G, $f(x) \cap f(y) = \emptyset$. Moreover, if $i < i'$ and $i, i' \in f(x)$, then there is an integer $j \in f(y)$ such that $i < j < i'$. The efficiency $\sigma(f)$ of the scheduling f is $\sigma(f) = \liminf_{n\to\infty} \sum_{i=0}^{n} \frac{|f^{-1}(i)|}{n|V(G)|}$, which is the average portion of processes in operation. The *concurrency* of a graph G, denoted by $\xi^*(G)$, is defined as

$$\xi^*(G) = \sup\{\sigma(f) : f \text{ is a valid scheduling of } G\}.$$

For example, if $G = C_5$ is the 5-cycle with vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and edge set $\{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1\}$, then the mapping $f(v_1) = f(v_3) = \{3i : i = 0, 1, \ldots\}$, $f(v_2) = f(v_4) = \{3i + 1 : i = 0, 1, \ldots\}$ and $f(v_5) = \{3i + 2 : i = 0, 1, \ldots\}$ is a valid scheduling with efficiency $1/3$. In general, if G is a k-chromatic graph and c is a k-colouring of G, then $f(x) = \{i : [i]_k = c(x)\}$ is a valid scheduling with efficiency $1/k$.

One method for finding an optimal scheduling is developed by computer scientists. The method is called the *edge reversal method*. Given an acyclic orientation of G, the scheduling induced by the orientation is obtained by repeatedly applying the following step: Let all the sinks operate and reverse the orientation of those edges incident to sinks.

Consider the 5-cycle C_5, with initial orientation as in Figure 1(a). Then v_2, v_4 operate on the first

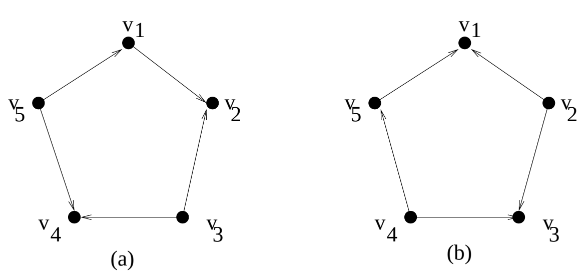

Figure 1: The initial and the second orientation of C_5

round and after the operation, the orientation will be changed to the one in Figure 1(b). Repeating the process, v_1, v_3 operate in the second round, v_2, v_5 the third round, v_1, v_3 the 4th round, v_2, v_5 the 5th round, and so on. The efficiency of this scheduling is $2/5$, better than the scheduling derived from the 3-colouring of C_5.

It is proved by Barbosa and Gafni [9] that given an acyclic orientation D of G, the efficiency of the scheduling derived from D is equal to the reciprocal of the imbalance of D. Moreover, for any graph G, there is an optimal scheduling of G which is derived from an acyclic orientation of G. In other words, $1/\xi^*(G) = \max\{\mathrm{Imb}(D) : D \text{ is an acyclic orientation of } G\}$. This implies that for any graph G,

$$\chi_c(G) = 1/\xi^*(G).$$

The study of circular chromatic number of graphs by graph theorists is apparently originated from the paper [134] by Vince, published in 1988. It follows from the result above that $\xi^*(G)$ is just the reciprocal of $\chi_c(G)$. The parameter $\xi^*(G)$ is defined in [7], and the results were consequently published in conference and journal [9]. The edge reversal method has been studied by computer scientists in many papers, however, the parameter $\xi^*(G)$ did not attract as much attention as $\chi_c(G)$.

A *fractional colouring* of G is a mapping f which assigns to each independent set I of G a non-negative weight $f(I)$ so that for each vertex x, $\sum_{x \in I} f(I) = 1$. If $\sum f(I) = r$ (where the summation is over all independent sets I of G), then f is called an *r-fractional colouring* of G. The *fractional chromatic number* of G is defined as $\chi_f(G) = \inf\{r : G \text{ has an } r\text{-fractional colouring}\}$. It is known to both computer scientists and graph theorists [10, 134] that for any graph G, $\chi_f(G) \leq 1/\xi^*(G) \leq \chi(G)$. However, the relation that $\lceil \chi_c(G) \rceil = \chi(G)$ is not obvious from the definition of $\xi^*(G)$, and the question whether $\xi^*(G)$ determines $\chi(G)$ was posed as an open problem in [8].

Although computer scientists and graph theorists are both interested in $\chi_c(G)$, they studied the parameter independently, using different languages, and without knowing the existence of the other side. It is until recently that the connection between the two sides has been revealed [144]. In [144], a formula is given that directly transform a periodic valid scheduling of a graph G into a circular colouring of G and vice versa.

Some other problems studied in computer sciences as well as in operations research can also be modeled as circular colouring of graphs, or circular colouring of edge weighted digraphs.

3 Circular colouring of digraphs

Let S^r be a circle of perimeter r as defined before. For two points p, p' of S^r, let $d(p, p') = [p' - p]_r$ be the length of the arc of S^r from p to p' along the clockwise direction.

An r-colouring of a digraph G is a mapping $f : V(G) \to S^r$ such that for each arc (x, y) of G, $d(f(x), f(y)) \geq 1$. We say G is r-colourable if there is an r-colouring of G. The circular chromatic number of a digraph G is defined as

$$\chi_c(G) = \inf\{r : G \text{ is } r\text{-colourable}\}.$$

By viewing an undirected graph G as a symmetric digraph, in which each edge $e = xy$ of G corresponds to two opposite arcs (x, y) and (y, x), the above definition of an r-colouring of the symmetric digraph is equivalent to the original definition of an r-colouring of the undirected graph. So the r-colouring of digraphs, introduced in [12], is a very natural generalization of the r-colouring of undirected graphs. Many results concerning r-colouring of undirected graphs generalize to r-colouring of digraphs without difficulties. There are also differences between undirected graphs and digraphs. One difference is that the infimum in the definition of the circular chromatic number of a digraph cannot be replaced by the minimum. For example, if G is an acyclic digraph, then for any $r > 1$, there is an r-colouring of G. So $\chi_c(G) = 1$. However, there is no 1-colouring of G. One may argue that this reflects one subtle aspect of the definition of distance between points on the circle S^r. Given a point p on S^r, what should be the distance $d(p, p)$? Should it be 0? Or should it be r? It is not a good idea to have $d(p, p) = r$, for otherwise one may colour all the vertices of a digraph G by the same colour to conclude that all digraphs are 1-colourable. On the other hand, we have $\lim_{p' \to p} d(p, p') = r$, if the limit is taken in the appropriate direction. To take this into consideration, the following definition is introduced in [12]:

An weak circular r-colouring of a digraph G is a mapping $f : V(G) \to S^r$ such that for each arc (x, y), either $f(x) = f(y)$ or $d(f(x), f(y)) \geq 1$. Moreover, for any point p of S^r, $f^{-1}(p)$ induces an acyclic subgraph of G

Observe that the definition of weak circular colouring applied to symmetric digraphs is also equivalent to the original definition. Using the weak circular r-colourability of digraphs, the circular chromatic number of a digraph G can be proved to be

$$\chi_c(G) = \min\{r : \ G \text{ is weak circular } r\text{-colourable}\}.$$

The circular chromatic number of an undirected graph is either 1 or at least 2. But for any rational $r \geq 1$, there is a finite digraph G with $\chi_c(G) = r$.

Theorem 3.1 *Suppose $p \geq q$. Let $\vec{K}_{p/q}$ be the digraph with vertex set $\{0, 1, \dots, p-1\}$ in which (i, j) is an arc if and only if $[j - i]_p \geq q$. Then $\chi_c(\vec{K}_{p/q}) = p/q$. In particular, for every rational $p/q \geq 1$, there exists a finite digraph with circular chromatic number p/q.*

Indeed, the digraphs $\vec{K}_{p/q}$ play the same role in the study of circular colouring of digraphs as the circular complete graphs $K_{p/q}$ in the study of circular colouring of undirected graphs. We define an *acyclic homomorphism* of a digraph G to a digraph G' as a mapping $f : V(G) \to V(G')$ such that for each arc (x, y) of G, either $f(x) = f(y)$ or $(f(x), f(y))$ is an arc of G'. Moreover, for each vertex v of G', $f^{-1}(x)$ induces an acyclic sub-digraph of G. Then we have the following result.

Theorem 3.2 *A digraph G has circular chromatic number at most p/q if and only if there exists an acyclic homomorphism of G to $\vec{K}_{p/q}$.*

Corollary 3.1 *If $p/q \leq p'/q'$ and G admits an acyclic homomorphism to $\vec{K}_{p/q}$ then G admits an acyclic homomorphism to $\vec{K}_{p'/q'}$.*

It is proved in [134] that if G is a finite undirected graph on n vertices then $\chi_c(G) = p/q$ for some $p \leq n$. The same conclusion holds for the circular chromatic number of digraphs. Given a weak circular r-colouring c of a digraph G, a cycle $C = (v_1, v_2, \dots, v_k, v_1)$ in the underlying graph of G is called a *tight cycle* if for each i, if (v_i, v_{i+1}) is an arc of C then $d(c(v_i), c(v_{i+1})) = 1$, otherwise (v_{i+1}, v_i) is an arc of C and $c(v_i) = c(v_{i+1})$, where additions in the indices are modulo k. The following result is a generalization of a result in [41], and is a strengthening of a result in [86].

Theorem 3.3 *A digraph G has $\chi_c(G) = r$ if and only if there is a weak circular r-colouring of G, and moreover, every weak circular r-colouring of G has a tight cycle.*

If C is a tight cycle of a weak circular r-colouring of G, then the weight $a(C)$ of C is the number of *forward edges* of C, i.e., number of indices i for which (v_i, v_{i+1}) is an arc. It follows from the definition of tight cycle that the weight $a(C)$ is a multiple of r, say $a(C) = qr$ for some positive integer q. As $a(C) = p$ is an integer less than or equal to $|C|$, we conclude that for any digraph G, $\chi_c(G) = p/q$ for some $p \leq |C| \leq |V(G)|$.

The definition of circular chromatic number of digraphs leads to the definition of *chromatic number* $\chi(G)$ of a digraph G to be the minimum integer k such that $V(G)$ can be partitioned into k acyclic subsets. In other words, $\chi(G)$ is the minimum integer r for which G has a weak circular r-colouring. Therefore for any digraph G we have

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G).$$

In case the digraph G is symmetric, then $\chi(G)$ coincides with the definition of chromatic number of undirected graphs.

Hell and Nesetril [54] proved that if H is non-bipartite, then it is NP-complete to decide if an arbitrary graph G admits a homomorphism to H. As a consequence, for any $r > 2$, it is NP-complete to determine if $\chi_c(G) \leq r$. For digraphs, it is easy to see that if H is acyclic, then it is polynomial to decide if an arbitrary digraph G admits an acyclic homomorphism to H. Feder, Hell and Mohar [32] proved that if H is not acyclic, then it is NP-complete to decide if an arbitrary digraph admits an acyclic homomorphism to H. It follows from this result that for any $r > 1$, it is NP-complete to decide if an arbitrary digraph G satisfies $\chi_c(G) \leq r$.

Given a digraph G, let $\alpha(G)$ be the maximum size of a subset of $V(G)$ that induces an acyclic sub-digraph. Then we have $\chi(G) \geq |V(G)|/\alpha(G)$ for any digraph G. By a probabilistic argument, it is proved in [12] that for any integer ℓ, there is a digraph G on n vertices with a digraph with digirth (i.e., the length of a shortest directed cycle) at least ℓ, and with $\alpha(G) \leq O(n^{1-\theta} \ln n)$ for a positive $\theta < 1/\ell$. So if n is large enough, then $\chi(G) \geq k$ for any given constant k. This implies the following result [12]:

Theorem 3.4 *For any integers k, ℓ, there is a digraph G with digirth at least ℓ and with circular chromatic number at least k.*

Almost all problems concerning the circular chromatic number of undirected graphs can be asked in terms of digraphs. The questions of possible values of the circular chromatic number of undirected planar graphs is answered by Moser [90] and this author [156, 155].

Theorem 3.5 *There is a finite undirected planar graph G with $\chi_c(G) = r$ if and only if $r = 1$ or r is a rational and $2 \leq r \leq 4$.*

Since, unlike the case for undirected graphs, the circular chromatic number of a digraph can be strictly between 1 and 2, one may wonder if every rational number between 1 and 2 is the circular chromatic number of a planar digraph. Soh [115] showed that the answer is yes.

Theorem 3.6 *There is a finite planar digraph G with $\chi_c(G) = r$ if and only if r is a rational and $1 \leq r \leq 4$.*

The maximum chromatic number of planar graphs (the Four Colour Problem) plays an important role in graph colouring theory. It is natural to ask what is the maximum acyclic chromatic number of an orientation of a planar graph. The following conjecture is proposed in [12]:

Conjecture 3.1 *If G is a planar digraph without 2-cycles, then the acyclic chromatic number of G is at most 2. I.e., $V(G)$ can be partitioned into $V_1 \cup V_2$ such that each V_i induces an acyclic sub-digraph of G.*

A more general version of circular colouring of digraphs is to consider edge weighted digraphs. Let $G = (V, E)$ be a digraph and $c : E \to \mathbb{R}^+ \cup \{0\}$ be the edge weights. An r-colouring of G is a mapping $f : V \to [0, r)$ such that for any arc $(x, y) \in E$, $[f(y) - f(x)]_r \geq c_{xy}$ (we use c_{xy} to denote the weight of arc (x, y)). The r-colouring of digraphs is the special case that $c_{xy} = 1$ for all $(x, y) \in E$. Basic properties of circular colouring of edge weighted digraphs is studied in [86] and it is shown there that circular colouring of edge weighted digraphs generalizes the earlier concept of circular colouring of vertex weighted graphs [25].

Circular colouring of edge weighted digraphs provides a model for parallel computations. Let $G = (V, E)$ be a digraph, and $c : E \to \mathbb{R}^+$ be the edge weight function. Let $T : E \to \mathbb{N}$ be a mapping which assigns to each arc (u, v) a number T_{uv} of *tokens*. The triple (G, c, T) is called a *timed marked graph.*

A timed marked graph (G, c, T) can be used to model the data movement in parallel computations. A vertex represents a task, an arc (u, v) represents a data channel. A token on arc (u, v) represents an input from u to v, and T is the initial assignment of tokens. When a vertex operates, it consumes one token from each of its in-arcs, and produces a token for each of its out-arcs. The weight c_{uv} represents the time required by task vertex u to place the result of its operation on (u, v). So if u operates at time t, then at time $t + c_{uv}$, a token is placed on (u, v) and becomes available to v. A scheduling for the timed marked graph determines, for each vertex v, the time pulses at which v operates. The scheduling is *admissible* if whenever a vertex v operates, each in-arc of v has at least one token available. Computer scientists are interested in periodic schedulings, in which each vertex v is assigned a single time pulse $\phi(u)$, and it operates at time pulses $\phi(u) + pk$ for $k = 0, 1, \ldots$ Here p is the period. An initial marking T of an edge weighted digraph (G, c) is *good* if for each directed cycle C, $\sum_{(u,v)\in C} T_{uv} > 0$ and for each arc (u, v), $T_{uv} + T_{vu} = 1$. Connection of circular colouring and periodic scheduling of timed marked graphs is studied in [143], where the following result is proved.

Theorem 3.7 *An edge-weighted symmetric digraph (G, c) has a circular p-colouring if and only if there is a good initial marking T for (G, c) for which the timed marked graph (G, c, T) admits a periodic admissible schedule with period p.*

4 Circular chromatic index

Given a graph G, the *line graph* of G, denoted by $L(G)$, has vertex set $E(G)$, in which e, e' are adjacent if e and e' have a common end-vertex. The *chromatic index* $\chi'(G)$ of G is defined as $\chi'(G) = \chi(L(G))$ and the *circular chromatic index* $\chi'_c(G)$ of G is defined as $\chi'_c(G) = \chi_c(L(G))$. So we have

$$\chi'(G) - 1 < \chi'_c(G) \leq \chi'(G).$$

If G is connected and $\Delta(G) = 2$, then G is either a cycle or a path. This implies that either $\chi'_c(G) = 2$ or $\chi'_c(G) = 2 + \frac{1}{k}$ for some positive integer k. Since graphs G with $\Delta(G) \geq 3$ have $\chi'_c(G) \geq 3$, 'most' of the rational numbers in the interval $(2, 3)$ are not the circular chromatic index of any graph.

Suppose G is a connected graph with $\Delta(G) = 3$. Then $3 \leq \chi'_c(G) \leq 4$. A natural question is that what are the possible values of the circular chromatic index of such a graph. It is well-known that the Four Color Theorem is equivalent to the statement that every 2-edge connected cubic planar graph G has $\chi'_c(G) = 3$. For nonplanar 2-edge connected cubic graphs, Jaeger [62] proposed the following conjecture (the Petersen Coloring Conjecture):

Conjecture 4.1 *If G is a 2-edge connected cubic graph, then one can colour the edges of G, using the edges of the Petersen graph as colours, in such a way that any three mutually adjacent edges of G are coloured by three edges that are mutually adjacent in the Petersen graph.*

Since the Petersen graph has circular chromatic index $11/3$, Conjecture 4.1 would imply that every 2-edge connected cubic graph G has $\chi'_c(G) \leq 11/3$. This consequence of Conjecture 4.1 is confirmed in [2].

Theorem 4.1 *The circular chromatic index of every 2-edge connected cubic graph G (parallel edges allowed) is less than or equal to $11/3$.*

Indeed, a more general result is proved in [2].

Theorem 4.2 *Suppose G is 2-edge connected and has maximum degree 3 (parallel edges are allowed). If $G \neq H_1, H_2$ (where H_1, H_2 are the graphs in Figure 2), then $\chi'_c(G) \leq 11/3$.*

Figure 2: (a): The graph H_1, (b): The graph H_2.

Corollary 4.1 *If G is graph (parallel edges allowed) of maximum degree 3 and G does not contain H_1 or H_2 as a subgraph, then $\chi'_c(G) \leq 11/3$.*

It is easy to verify that $\chi'_c(H_1) = \chi'_c(H_2) = 4$. Since graphs G with $\Delta(G) \geq 4$ have $\chi'_c(G) \geq 4$. Therefore we have the following:

Corollary 4.2 *There is no graph G with $11/3 < \chi'_c(G) < 4$.*

So the interval $(11/3, 4)$ is a gap for the circular chromatic indexes of graphs, and we do not know if there are other gaps. I propose the following conjecture.

Conjecture 4.2 *Let Ω be the set of all the circular chromatic indexes of graphs, i.e., $\Omega = \{\chi'_c(G) : G$ is a finite graph$\}$. There is no bounded strictly increasing infinite sequence in Ω.*

This conjecture is strong, and implies that there are gaps everywhere for the circular chromatic indexes of graphs. Here is a weaker conjecture.

Conjecture 4.3 *For each integer n, there is an $\varepsilon_n > 0$ such that there is no graph G with $\chi'_c(G) \in (n - \varepsilon_n, n)$.*

If $n = 2, 3, 4$, then ε_n exist and can be chosen as $\varepsilon_n = 1/(n-1)$. It might be true that ε_n can be chosen as $1/(n-1)$ for $n \geq 5$ as well, as conjectured in [2, 65] (see also http://www.math.nsysu.edu.tw/ zhu/open-problems).

It is conjectured by Jaeger and Swart [61] that every 2-edge connected cubic graph of large girth admits a nowhere zero 4-flow. This is equivalent to say that 2-edge connected cubic graph of large girth are 3-edge colourable. This conjecture (called the Girth Conjecture) is refuted by Kochol [69]. However, if one considers circular chromatic index instead of chromatic index, then the Girth Conjecture is almost true. The circular chromatic index of cubic graphs of large girth is studied by Kaiser, Král and Škrekovski [65]. They proved the following result.

Theorem 4.3 *For each $\varepsilon > 0$ there is an integer $n = n(\varepsilon)$ such that any cubic graph of girth at least n has $\chi'_c(G) < 3 + \varepsilon$.*

This result is generalized in [66].

Theorem 4.4 *For each $\varepsilon > 0$ and any integer Δ, there is an integer $n = n(\Delta, \varepsilon)$ such that if G is a graph with maximum degree Δ and with girth at least n has $\chi'_c(G) < \Delta + \varepsilon$.*

The circular chromatic index of some special classes of graphs have been studied in a few papers [37, 43, 89, 92, 63, 137]. Observe that if G has no parallel edges, then by Vizing Theorem, $\chi'(G) = \Delta(G)$

or $\Delta(G)+1$. Graphs G with $\chi'(G) = \Delta(G)$ are called *class 1* graphs, and graphs with $\chi'(G) = \Delta(G)+1$ are called *class 2* graphs. If G is class 1, then since $\omega(L(G)) = \Delta(G)$, we conclude that $\chi'_c(G) = \Delta(G)$. Thus for the study of the circular chromatic index of graphs, we are interested in class 2 graphs.

Cyclically 4-edge connected cubic graphs of class 2 are called *snarks*. One well-known infinite family of snarks is the family of flower snarks. The *flower snark* J_{2k+1} is obtained from the disjoint union of two cycles, $(a_0, a_1, \dots, a_{2k})$ and $(c_0, d_1, c_2, d_3, \dots, c_{2k}, d_0, c_1, d_2, c_3, \dots, c_{2k-1}, d_{2k})$, by adding vertices b_i and edges $a_i b_i, c_i b_i, d_i b_i$ for $i = 0, 1, 2, \dots, 2k$. The circular chromatic index of flower snarks are completely determined [37].

Theorem 4.5 *The circular chromatic index of flower snarks are as follows:* $\chi'_c(J_3) = 7/2$, $\chi'_c(J_5) = 17/5$ *and for* $k \geq 3$, $\chi'_c(J_{2k+1}) = 10/3$.

The circular chromatic index of the Cartesian product of graphs is studied in [137]. In particular, the circular chromatic index of the Cartesian product of two odd cycles is estimated.

Theorem 4.6 *For any* $m \geq k$,

$$\chi_c(C_{2k+1} \Box C_{2m+1}) \geq 4 + \frac{1}{\lfloor (3(2k+1))/4 \rfloor}.$$

Moreover, if $m \geq 3k+1$, *then equality holds.*

5 Graph products

Suppose $G = (V, E)$ and $G' = (V', E')$. The *categorical product* $G \times G'$ of G and G' has vertex set $V \times V'$ in which $(x, x') \sim (y, y')$ if $(x, y) \in E$ and $(x', y') \in E'$. It follows easily from the definition that $G \times G'$ admits homomorphisms to G and G'. Therefore

$$\chi(G \times G') \leq \min\{\chi(G), \chi(G')\}$$

and

$$\chi_c(G \times G') \leq \min\{\chi_c(G), \chi_c(G')\}.$$

Equality is conjectured to hold in both inequalities.

Conjecture 5.1 *[53] For any positive integer* n, *if* $\chi(G) = \chi(G') = n$, *then* $\chi(G \times G') = n$.

Conjecture 5.2 *[148] For any rational* $r \geq 2$, *if* $\chi_c(G) = \chi_c(G') = r$, *then* $\chi_c(G \times G') = r$.

Conjecture 5.2 is stronger than Conjecture 5.1. Both conjectures remain open, although the conjectures, especially Conjecture 5.1, have been studied extensively (see [111, 151]).

It is easy to see that Conjecture 5.1 is true if $n \leq 3$. The conjecture is also confirmed for $n = 4$ [29]. But this case is already quite difficult. For $n \geq 5$, Conjecture 5.1 remains open. Some other special cases of Conjecture 5.1 are proved: If $\chi(G) = \chi(G') = n$ and every vertex of G is contained in a clique of size $n-1$, then $\chi(G \times G') = n$ [17]. If $\chi(G) = \chi(G') = n$ and every pair of edges of G is connected by an edge of G, then $\chi(G \times G') = n$ [130]. If $\chi(G) = \chi(G') = n$ and G is obtained from K_n by a series of Hajós sums and at most one contraction of non-adjacent vertices, then $\chi(G \times G') = n$ [110]. If $\chi(G) = \chi(G') = n$, each of G and G' are connected, and $\omega(G) \geq n-1$ and $\omega(G') \geq n-1$, then $\chi(G \times G') = n$ [28, 136]. More generally, Larose and Tardif [71] proved the following result: If K is vertex transitive and projective, then whenever K is the retract of the product of two connected graphs, it is a retract of a factor (see Section 10 for the definition of projective graphs). El-Zahar and Sauer [29] proposed the following conjecture which is stronger than Conjecture 5.1: *If G, G' are connected graphs with $\chi(G) = \chi(G') = n$ and H and H' are $(n-1)$-chromatic subgraphs of G and G', respectively, then the subgraph of $G \times G'$ induced by $(G \times H') \cup (H \times G')$ has chromatic number n.* This conjecture is disproved in [125]. Generally speaking, there is no substantial positive results concerning Conjecture 5.1 in the past twenty years. Even the following question is still open: Let $f(n) = \min\{\chi(G \times G') : \chi(G) = \chi(G') = n\}$. Is it true that $\lim_{n\to\infty} f(n) = \infty$? If Conjecture 5.1 is true, then we would have $f(n) = n$. It is proved by Poljak and Rödl [107] that $f(n)$ is either unbounded or is bounded by 16. This is strengthened in [108] (see also [151]) where it is proved that $f(n)$ is either unbounded or bounded by 9.

A graph K is called *multiplicative* if $G \not\to K$ and $H \not\to K$ implies that $G \times H \not\to K$. Conjecture 5.1 says that each complete graph K_n is multiplicative. Conjecture 5.2 says that the circular complete graphs $K_{p/q}$ are multiplicative. As mentioned above, K_1, K_2, K_3 are the only complete graphs that are known to be multiplicative. However, there are some other graphs that are known to be multiplicative. The argument of El-Zahar and Sauer [29] is generalized by Häggkvist, Hell, Miller and Neumann Lara [44] to prove that odd cycles are multiplicative. Recently, Tardif [124] found an interesting graph operation that constructs new multiplicative graphs from old ones. This leads to a breakthrough in the study of Conjecture 5.2.

Suppose G is a graph. Then $\mathbb{P}_3^{-1}(G)$ is the graph defined as follows. For two subsets A, B of the vertex set of G, we write $A \bowtie B$ if every vertex of A is joined to every vertex of B. The graph $\mathbb{P}_3^{-1}(G)$ has vertex set $V(\mathbb{P}_3^{-1}(G)) = \{(u, A) : u \in V(G), \emptyset \neq A \subseteq N(u)\}$. Two vertices (u, A) and (v, B) are adjacent in $\mathbb{P}_3^{-1}(G)$ if $u \in B$, $v \in A$ and $A \bowtie B$.

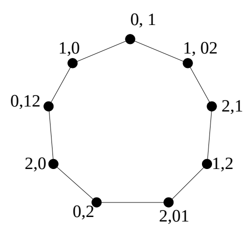

Figure 3: The graph $\mathbb{P}_3^{-1}(K_3)$

The graph operation $\mathbb{P}_3^{-1}$ is the right inverse of a simpler graph operation: $\mathbb{P}_3$. Given a graph G, $\mathbb{P}_3(G)$ has vertex set $V(\mathbb{P}_3(G)) = V(G)$ in which xy is an edge if and only if there are vertices u, v such that xu, uv, vy are edges of G. I.e., G has an x-y-walk of length 3. Observe that if xy is an edge of G, then (x, y, x, y) is an x-y-walk of length 3, so xy is an edge of $\mathbb{P}_3(G)$. Thus G is a subgraph of $\mathbb{P}_3(G)$. If G has a triangle induced by x, y, z, then (x, y, z, x) is an x-x-walk of length 3. So $\mathbb{P}_3(G)$ has a loop. Conversely, $\mathbb{P}_3(G)$ has a loop only if G contains a triangle. Since we are only interested in loopless graphs, we shall apply the operator $\mathbb{P}_3$ to triangle free graphs only.

Lemma 5.1 *For any graph G, $\mathbb{P}_3(\mathbb{P}_3^{-1}(G))$ and G are homomorphically equivalent.*

However, in general, $\mathbb{P}_3^{-1}\mathbb{P}_3(G)$ and G are quite different. For example, $\mathbb{P}_3^{-1}\mathbb{P}_3(C_5) = \mathbb{P}_3^{-1}(K_5)$ is not homomorphically equivalent to C_5. Indeed, $\mathbb{P}_3^{-1}(K_5)$ is 5-chromatic, but C_5 is 3-chromatic. Nevertheless, we have the following lemma.

Lemma 5.2 *For any graphs G, H,*

1. *$\mathbb{P}_3(G \times H) = \mathbb{P}_3(G) \times \mathbb{P}_3(H)$.*
2. *$\mathbb{P}_3(G) \to H$ if and only if $G \to \mathbb{P}_3^{-1}(H)$.*
3. *$G \to H$ if and only if $\mathbb{P}_3^{-1}(G) \to \mathbb{P}_3^{-1}(H)$.*

4. $\mathbf{P}_3^{-1}(G \times H)$ *and* $\mathbf{P}_3^{-1}(G) \times \mathbf{P}_3^{-1}(H)$ *are homomorphically equivalent.*

Using Lemmas 5.2 and 5.2, one can easily prove the following:

Theorem 5.1 *A graph K is multiplicative if and only if $\mathbf{P}_3^{-1}(K)$ is multiplicative.*

Now we have a graph operator that might be used to construct new multiplicative graphs from old ones. But, unfortunately, applying the operator $\mathbf{P}_3^{-1}$ to the presently known multiplicative graphs (i.e., K_1, K_2 and odd cycles) does not yield new multiplicative graphs. It is easy to verify that $\mathbf{P}_3^{-1}(K_1)$ is empty, $\mathbf{P}_3^{-1}(K_2) = K_2$, and for any cycle C_n, $\mathbf{P}_3^{-1}(C_n) = C_{3n}$. At this step, it seems that this approach is leading to a dead end. "*Blocked by mountains and waters, from left to right. Where can I go? No road is in sight. Wait! There lies a village, where willows are dark, flowers are bright*". Instead of applying the operator $\mathbf{P}_3^{-1}$ to graphs that are known to be multiplicative, we apply this operator to graphs G which are not known to be multiplicative. If the resulting graph $\mathbf{P}_3^{-1}(G)$ is multiplicative, then Theorem 5.1 implies that G itself is multiplicative. To which graphs should we apply the operator $\mathbf{P}_3^{-1}$? Conjecture 5.2 provides a natural family of candidates: the circular complete graphs.

Theorem 5.2 *If $p/q < 12/5$, then $\mathbf{P}_3^{-1}(K_{p/(3q-p)})$ and $K_{p/q}$ are homomorphically equivalent.*

Corollary 5.1 *If $p/q < 12/5$, then $K_{p/q}$ is multiplicative if and only if $K_{p/(3q-p)}$ is multiplicative.*

Since for every $k \geq 3$, $K_{(2k+1)/k}$ is multiplicative and $(2k+1)/k < 12/5$, we can apply Corollary 5.1 and conclude that $K_{(2k+1)/(3k-(2k+1))} = K_{(2k+1)/(k-1)}$ is multiplicative. If $k \geq 9$, then $(2k+1)/(k-1) < 12/5$ and we can apply Corollary 5.1 again, and conclude that $K_{(2k+1)/(k-4)}$ is multiplicative. Continue the process, we have the following corollary:

Corollary 5.2 *If $k \geq 1$, $0 \leq i \leq \log_3(k)$ and $d = k - (3^i - 1)/2 > \frac{2k+1}{4}$, then $K_{(2k+1)/d}$ is multiplicative.*

It can be verified that the set $P = \{\frac{2k+1}{k-(3^i-1)/2} : k \geq 1, 0 \leq i \leq \log_3(k)\}$ is dense in the interval $[2, 4]$. Therefore, we have the following result:

Theorem 5.3 *[124] For any rational $2 \leq p/q < 4$, $K_{p/q}$ is multiplicative.*

The approach above can be interpreted in a different way: Although in general, $\mathbf{P}_3^{-1}$ is not a left inverse of $\mathbf{P}_3(G)$, but for some graphs G it may happen that $\mathbf{P}_3^{-1}\mathbf{P}_3(G)$ is homomorphically equivalent to G. Indeed, since $\mathbf{P}_3^{-1}$ is the right inverse of $\mathbf{P}_3$, for any graph G, $\mathbf{P}_3^{-1}\mathbf{P}_3(\mathbf{P}_3^{-1}(G))$ is homomorphically equivalent to $\mathbf{P}_3^{-1}(G)$. This means that if restricted to the family of graphs $\mathcal{H} = \{\mathbf{P}_3^{-1}(G) : G \text{ is a graph}\}$, $\mathbf{P}_3^{-1}$ is also a left inverse of $\mathbf{P}_3(G)$. For such graphs G, if G is multiplicative, then Theorem 5.1 implies that $\mathbf{P}_3(G)$ is multiplicative. Theorem 5.2 shows that if $p/q < 12/5$, then $K_{p/q} \in \mathcal{H}$. So if $p/q < 12/5$, and $K_{p/q}$ is multiplicative, then $\mathbf{P}_3(K_{p/q}) = K_{p/(3q-p)}$ is also multiplicative.

For a digraph G, let $\underline{G}$ be the underline graph G, i.e., $\underline{G}$ is obtained from G by omitting the orientation of the edges. It is known that $\chi(\underline{G \times H})$ could be strictly less than $\min\{\chi(\underline{G}), \chi(\underline{H})\}$. Indeed, even if G and H are tournaments, we can still have $\chi(\underline{G \times H}) < \min\{\chi(\underline{G}), \chi(\underline{H})\}$. Analogous to the undirected case, we define $g(n) = \min\{\chi(\underline{G \times H}) : G, H \text{ are digraphs with } \chi(\underline{G}) = \chi(\underline{H}) = n\}$. It is known that $g(n)$ is either bounded by 3 or unbounded [108, 151]. Recently, it is proved in [126] that $g(n)$ is bounded if and only if $f(n)$ (defined in the 5th paragraph of this section) is bounded. For the product of tournaments, Tardif [122] defined $t(n) = \min\{\chi(\underline{G \times H}) : G \text{ and } H \text{ are } n\text{-tournaments}\}$, and proved that the sequence $t(n)/n$ tends to a limit λ, and $1/2 \leq \lambda \leq 2/3$.

There is also a conjecture analogue to Conjectures 5.1 and 5.2 for the fractional chromatic number of graphs [163].

Conjecture 5.3 *If $\chi_f(G) \geq r$ and $\chi_f(H) \geq r$ then $\chi_f(G \times H) \geq r$.*

For $r \geq 2$, let $\phi(r) = \min\{\chi_f(G \times H) : \chi_f(G) = \chi_f(H) = r\}$. Although we do not know if $\phi(r) = r$, it is proved recently by Tardif [123] that $\phi(r) \geq r/4$. Tardif also considered the relation between the chromatic number of the product graph and the fractional chromatic number of the factor graphs. The following result is proved in [121].

Theorem 5.4 *If $\chi_f(G), \chi_f(H) \geq 2n$ then $\chi(G \times H) \geq n$.*

There are some other graph products whose circular chromatic number have been studied. For the Cartesian product $G \Box H$ (in which (x, y) is adjacent to (x', y') in $G \Box H$ if and only if either ($x = x'$ and $(y, y') \in E(H)$) or ($(x, x') \in E(G)$ and $y = y'$)), it is trivial that $\chi_c(G \Box H) = \max\{\chi_c(G), \chi_c(H)\}$. For the lexicographic product $G[H]$ (in which (x, y) is adjacent to (x', y') in $G[H]$ if $(x, x') \in E(G)$ or $x = x'$ and $(y, y') \in E(H)$), it is known [148] that $\chi_f(G)\chi(H) \leq \chi_c(G[H]) \leq \chi_c(G)\chi(H)$, and both the upper and lower bounds are sharp. Recently, the circular chromatic number and chromatic number of the Cartesian sum (also known as the very strong product) of two graphs is studied in [82]. Suppose $G = (V, E)$ and $H = (V', E')$. The Cartesian sum $G \oplus H$ of G and H has vertex set $V \times V'$, in which (x, y) is adjacent to (x', y') in $G \oplus H$ if $(x, x') \in E(G)$ or $(y, y') \in E(H)$. The following result is a sharp upper bound for the chromatic number of $G \oplus H$ in terms of the circular chromatic number of G and H.

Theorem 5.5 *For any graphs G, H,*

$$\max\{\lceil \chi_f(G)\chi(H) \rceil, \lceil \chi(G)\chi_f(H) \rceil\} \leq \chi(G \oplus H) \leq \max\{\lceil \chi_c(G)\chi(H) \rceil, \lceil \chi(G)\chi_c(H) \rceil\}.$$

If $\chi_f(G) = \chi_c(G)$ and $\chi_f(H) = \chi_c(H)$, then the lower bound and upper bound above coincide. For general graphs G and H, it is known that $\max\{\chi_f(G)\chi(H), \chi(G)\chi_f(H)\} \leq \chi_c(G \oplus H)$, and it is conjectured in [82] that for any graphs G, H, $\chi_c(G \oplus H) \leq \max\{\chi_c(G)\chi(H), \chi(G)\chi_c(H)\}$. Some special cases of the conjecture are confirmed in [82], however, the general case is open.

6 Kneser, Schrijver and cone graphs (topological method)

Given positive integers $n \geq 2k$, the *Kneser graph* $K(n, k)$ has vertices all the k-subsets of $[n] = \{1, 2, \ldots, n\}$ and two vertices u, v are adjacent if u and v do not intersect (as subsets of $[n]$). A k-subset u of $[n]$ is called *stable* if, by viewing the elements of $[n]$ as cyclically ordered, u does not contain two consecutive elements. Namely if $i \in u$ then $i + 1 \notin u$, and if $n \in u$, then $1 \notin u$. The *Schrijver graph* $S(n, k)$ is the subgraph of $K(n, k)$ induced by those vertices which are stable k-subsets of $[n]$. The chromatic number of the Kneser graph $K(n, k)$ is conjectured by Kneser [68] to be equal to $n - 2k + 2$. The conjecture remained open for more than 20 years, before it is proved by Lovász [83] by an application of a topological method. Then Schrijver [112] defined the graphs $S(n, k)$ and proved that $\chi(S(n, k)) = \chi(K(n, k)) = n - 2k + 2$ and moreover $S(n, k)$ is χ-critical, i.e., removing any vertex of $S(n, k)$ decreases its chromatic number.

The circular chromatic number of Kneser graphs is first studied by Johnson, Holroyd and Stahl [64]. They conjectured that $\chi_c(K(n, k)) = \chi(K(n, k)) = n - 2k + 2$, and proved that the conjecture holds if $2k + 1 \leq n \leq 2k + 2$ or $k = 2$. The circular chromatic number of Schrijver graphs is first studied by Lih and Liu [75]. It is proved in [75] that if $k = 2$ and $n \neq 5$, then $\chi_c(S(n, k)) = \chi(S(n, k)) = 2 - 2k + 2$. Since $S(2k + 1, k)$ is the odd cycle C_{2k+1}, $\chi_c(S(2k + 1, k)) \neq \chi(S(2k + 1, k))$ for $k \geq 2$. Lih and Liu asked if for each k there is an integer $n(k)$ such that if $n \geq n(k)$ then $\chi_c(S(n, k)) = \chi(S(n, k))$. Hajiabolhassan and this author [47] proved Theorem 6.1 below which answers this question in the affirmative, and also provides strong support for the conjecture of Johnson, Holroyd and Stahl.

Theorem 6.1 *If $n \geq 2k^2(k-1)$, then $\chi_c(K(n,k)) = \chi(K(n,k)) = n-2k+2$. Moreover, for any integer k, there is an integer $n(k)$ such that if $n \geq n(k)$, then $\chi_c(S(n,k)) = \chi(S(n,k)) = n-2k+2$.*

Recently, Meunier [85] and Simonyi and Tardos [114], used the topological tools in the study of the circular chromatic numbers of Kneser graphs and Schrijver graphs. Here we give a sketch of this approach.

Suppose $G = (V, E)$ is a graph. The *box complex* $B_0(G)$ of G is a simplicial complex with vertex set $V \times \{1,2\}$, in which $(S \times \{1\}) \cup (T \times \{2\})$ is a simplex if $S \cap T = \emptyset$ and every vertex of S is adjacent to every vertex of T. We denote the set $(S \times \{1\}) \cup (T \times \{2\})$ by $S \uplus T$. For simplicity, the vertices $(x,1)$ and $(x,2)$ are denoted by $+x$ and $-x$, respectively.

As an example, we consider the box complex of complete graphs. Assume the vertex set of K_n is $\{v_1, v_2, \ldots, v_n\}$. Then $B_0(K_1)$ consists of two 0-simplices (i.e., two sets of a single vertex), $\{+v_1\}, \{-v_1\}$. The complex $B_0(K_2)$ consists of four 0-simplices $\{+v_1\}, \{+v_2\}, \{-v_1\}, \{-v_2\}$ and four 1-simplices $\{+v_1, +v_2\}$, $\{-v_1, -v_2\}$, $\{+v_1, -v_2\}$, $\{+v_2, -v_1\}$. The complex $B_0(K_3)$ has six 0-simplices, twelve 1-simplices and eight 2-simplices, as shown in Figure 4.

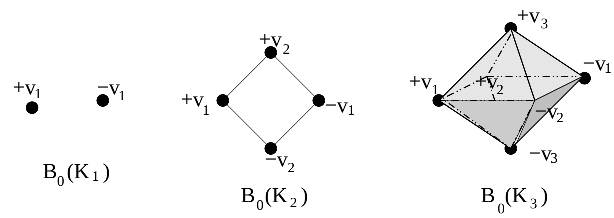

Figure 4: The complexes $B_0(K_1)$, $B_0(K_2)$ and $B_0(K_3)$

In general, for $n \geq 2$, the complex $B_0(K_n)$ is obtained from $B_0(K_{n-1})$ by adding two new vertices $+v_n, -v_n$ and for each simplex σ of $B_0(K_{n-1})$, add the simplex $\sigma \cup \{+v_n\}$ and the simplex $\sigma \cup \{-v_n\}$. This means that $B_0(K_n)$ is the suspension of $(B_0(K_{n-1}))$. As the suspension of S^d is homeomorphic to S^{d+1}, and $B_0(K_1) = S^0$, we conclude that $B_0(K_n)$ is homeomorphic to S^{n-1} for all $n \geq 1$.

We shall apply Borsuk-Ulam Theorem in proving the lower bounds for the chromatic number and the circular chromatic number of Kneser graphs and Schriver graphs. For a topological space X, a Z_2*-action* on X is a homeomorphism $\nu : X \to X$ such that for any $x \in X$, $\nu^2(x) = x$. A Z_2-action is free if $\nu(x) \neq x$ for all x. A natural Z_2-action on the sphere S^d is the antipodal map $\nu(x) = -x$. The pair (X, ν) is called a Z_2-space. Suppose (x, ν) and (Y, μ) are Z_2-spaces. A continuous map $f : X \to Y$ is called a Z_2-map if $f(\nu(x)) = \mu(f(x))$ for all $x \in X$. We write $(X, \mu) \to (Y, \mu)$ if there is a Z_2-map from (X, ν) to (Y, μ). When the Z_2-action ν is clear from the context (like the anitpodal map on the spheres or $\mathbb{R}^n$), we simply write X instead of (X, ν). The following is one of the many equivalent formulations of Borsuk-Ulam Theorem.

Theorem 6.2 *For $d \geq 0$, there is no Z_2-map from S^d to S^{d-1}.*

For a graph G, the box complex of G has Z_2-action ν defined as $\nu(S \uplus T) = T \uplus S$. (Note that this definition defines ν on the vertice of the simplicial complex $B_0(G)$, and this definition extends to a topological realization of the simplicial complex by convex combination.) The link between colouring of graphs and box complexes of graphs is reflected in the following lemma, whose proof is straightforward.

Lemma 6.1 *Suppose G, H are graphs and ϕ is a homomorphism from G to H. Then the mapping ϕ^* defined as $\phi^*(S \uplus T) = \phi(S) \uplus \phi(T)$ is a Z_2-map from $B_0(G)$ to $B_0(H)$.*

In particular, if $\chi(G) \leq k$, then there is a homomorphism of G to K_k. By Lemma 6.1, this implies that $B_0(G) \to B_0(K_k) \approx S^{k-1}$. Therefore by Borsuk-Ulam Theorem, there is no Z_2-map from S^k to $B_0(G)$. In other words, we have

Corollary 6.1 *If there is a Z_2-map from S^k to $B_0(G)$, then $\chi(G) > k$.*

A graph G is called *topologically t-chromatic* if there is a Z_2-map from S^{t-1} to $B_0(G)$. As $S^{n-2k+1} \to B_0(S(n,k))$, so $\chi(K(n,k)) \geq \chi(S(n,k)) \geq n-2k+2$. The proof of the fact that $S^{n-2k+1} \to B_0(S(n,k))$ is not trivial, and it can be found in [114] and in the references given there. Our interest is why this gives a lower bound for the circular chromatic number of graphs.

Although there is no Z_2-map from S^d to S^{d-1}, if we delete a pair of antipodal points from S^d, then there is a Z_2-map from the resulting space to S^{d-1}. Indeed, let $u = (0,\ldots,0,1)$ and $u' = (0,\ldots,0,-1)$ be the north pole and the south pole of S^d. Then $f : S^d \setminus \{u,u'\} \to S^{d-1}$ defined as $f((x_1 x_2,\ldots,x_{d+1})) = \frac{1}{\sqrt{1-x_{d+1}^2}}(x_1,x_2,\ldots,x_d)$ is a Z_2-map.

Theorem 6.3 *If a graph G is topologically k-chromatic and k is even, then $\chi_c(G) \geq k$.*

Proof. Assume $S^{k-1} \to B_0(G)$ and $k = 2t$ is odd. To prove that $\chi_c(G) \geq 2t$, by Theorem 3.3, it suffices to show that for any $2t$-colouring f of G, there is a cycle $C = (x_0,x_1,\ldots,x_{2t-1})$ such that $f(x_{i+1}) = [f(x_i)+1]_{2t}$ for $i = 0,1,\ldots,2t-1$, where the summation in the indices are modulo $2t$.

Let f be a $2t$-colouring of G, which is viewed as a homomorphism from G to K_{2t}. Consider the induced Z_2-map $f^* : B_0(G) \mapsto B_0(K_{2t})$, which is a simplicial map. If f^* is not onto, then there is a simplex σ of $B_0(K_{2t})$ which is not the image of any simplex of $B_0(G)$ Thus f^* is a Z_2-map from $B_0(G)$ to $B_0(K_{2t}) \setminus \{\sigma,-\sigma\}$.

Since the deletion of a pair of antipodal points from S^{2t-1} results in a space which admits a Z_2-map to S^{2t-2}, and since $B_0(K_{2t})$ is homeomorphic to S^{2t-1}, we have $B_0(K_{2t}) \setminus \{\sigma,-\sigma\} \to S^{2t-2}$. This implies that

$$S^{2t-1} \to B_0(G) \to B_0(K_{2t}) \setminus \{\sigma,-\sigma\} \to S^{2t-2},$$

in contrary to Borsuk-Ulam Theorem. Therefore f^* must be onto. In particular, there is a simplex $S \uplus T$ of $B_0(G)$ which is mapped to $A \uplus A'$, where $A = \{1,3,5,\ldots,2t-1\}$ and $A' = \{2,4,\ldots,2t\}$. So S contains vertices $v_1,v_3,\ldots,v_{2t-1}$ and T contains vertices $v_2,v_4,\ldots,v_{2t}$ with $f(v_i) = i$ for all i. As every vertex of S is adjacent to every vertex of T, $(v_1,v_2,\ldots,v_{2t})$ is a cycle in $D_f(G)$. ∎

Since $S(n,k)$ and $K(n,k)$ are topologically $(n-2k+2)$-chromatic and $\chi(S(n,k)) = \chi(K(n,k)) = n-2k+2$, we have the following corollary.

Corollary 6.2 *If $\chi(S(n,k))$ is even, then $\chi_c(K(n,k)) = \chi_c(S(n,k)) = \chi(S(n,k)) = \chi(K(n,k))$.*

Cone graphs is another class of graphs where the lower bound on its chromatic number is due to a topological reason. Suppose $G = (V,E)$ is a graph and $m \geq 1$ is an integer. The *m-cone* $\Delta_m(G)$ of G has vertex set

$$(V \times \{0,1,\ldots,m\}) \cup \{u\},$$

and edge set

$$\{(x,0)(y,0) : xy \in E\} \cup \{(x,i)(y,i+1) : xy \in E, i = 0,1,\ldots,m-1\} \cup \{(x,m-1)u : x \in V\}.$$

We shall refer the m-cone over G for any $m \geq 0$ as a *cone* over G. Note that $\Delta_0(G)$ is the graph obtained from G by adding a universal vertex u and $\Delta_1(G) = M(G)$ is the Mycielskian of G. The cone over G is also called the *generalized Mycielskian of* G. The vertex u is called the *root* of $\Delta_m(G)$. For $\vec{m} = (m_1,m_2,\ldots,m_t)$, $\Delta_{\vec{m}}(G) = \Delta_{m_1}(\Delta_{m_2}(\ldots\Delta_{m_t}(G)\ldots))$. Figure 6 depicts the graph $\Delta_2(C_5)$.

For $\vec{m} = (m_1,m_2,\ldots,m_t)$, $\Delta_{\vec{m}}(G) = \Delta_{m_1}(\Delta_{m_2}(\ldots\Delta_{m_t}(G)\ldots))$. It is easy to see that for any graph G, for any integer $m \geq 0$, $\chi(G) \leq \chi(\Delta_m(G)) \leq \chi(G)+1$. If $m \leq 1$, then $\chi(\Delta_m(G)) = \chi(G)+1$. For $m \geq 2$, it is possible that $\chi(\Delta_m(G)) = \chi(G)$. However, the following result is proved in [119] (see also [42]).

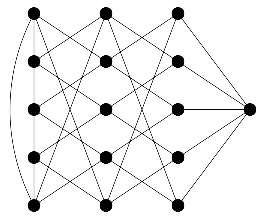

Figure 5: The graph $\Delta_2(C_5)$

Theorem 6.4 *If $\vec{m} = (m_1, m_2, \ldots, m_t)$, then $\chi(\Delta_{\vec{m}}(K_2)) = t+2$.*

The graph $\Delta_{\vec{m}}(K_2)$ is shown [119] to have chromatic number $t+2$ also for a "topological reason". The topological reason considered in [119] is different from the box complex argument, however, it is shown in [5] and [114] that that topological reason implies that $\Delta_{\vec{m}}(K_2)$ is topologically $(t+2)$-chromatic. Hence we have the following corollary [114].

Corollary 6.3 *If $\vec{m} = (m_1, m_2, \ldots, m_t)$, and t is even, then $\chi_c(\Delta_{\vec{m}}(K_2)) = t+2$.*

On the other hand, the following result is proved by Lam, Lin, Gu and Song [70]:

Theorem 6.5 *If n is even and $m \geq 0$ is an integer, then $\chi_c(\Delta_m(K_n)) = n + 1/t$, where $t = \lceil (2m+2)/n \rceil$.*

Let $\vec{m'} = (m_2, m_3, \ldots, m_t)$ and $\vec{m} = (m_1, m_2, \ldots, m_t)$. Since $\Delta_{\vec{m'}}(K_2)$ admits a homomorphism to K_{t+1}, we conclude that $\Delta_{\vec{m}}(K_2)$ admits a homomorphism to $\Delta_{m_1}(K_{t+1})$. Therefore we have the following consequence.

Corollary 6.4 *If t is odd, then $t+1 < \chi_c(\Delta_{\vec{m}}(K_2)) \leq t+1+1/s$, where $s = \lceil (2m_1+2)/(t+1) \rceil$. In particular, for any $\varepsilon > 0$, there is an integer n such that if t is odd, $\vec{m} = (m_1, m_2, \ldots, m_t)$ and $m_1 > n$, then $t+1 < \chi_c(\Delta \vec{m}(K_2)) < t+1+\varepsilon$.*

This result means that the condition that t be even in Corollary 6.3 is essential. The same is true for the circular chromatic number of Schrijver graphs. The following result is proved by Simonyi and Tardos [114]:

Lemma 6.2 *For any integers $t, m \geq 1$, if n is large enough, and $n - 2k + 2 = t$, then $S(n,k)$ admits a homomorphism to $\Delta_m(K_{t-1})$.*

As a consequence, we have the following corollary.

Corollary 6.5 *For any integers $t, m \geq 1$, if n is large enough and $2k = n - 2t + 1$, then $2t < \chi_c(S(n,k)) < 2t + \varepsilon$.*

7 Circular perfect graphs

Suppose G is a graph. The *clique number* of G is $\omega(G) = \max\{k : K_k \preceq G\}$, and the *circular clique number* of G is $\omega_c(G) = \max\{p/q : K_{p/q} \preceq G\}$. By this definition, for any graph G, we have

$$\omega(G) \leq \omega_c(G) \leq \chi_c(G) \leq \chi(G), \ \omega(G) = \lfloor \omega_c(G) \rfloor, \text{ and } \chi(G) = \lceil \chi_c(G) \rceil.$$

A graph G is *perfect* if for every induced subgraph H of G we have $\chi(H) = \omega(H)$. A graph G is called *circular perfect* if for every induced subgraph H of G we have $\chi_c(H) = \omega_c(H)$. Circular perfect graphs are first defined in [168], which was written in 2000, but published much later than some other papers on this subject.

As $\omega_c(H)$ and $\chi_c(H)$ are sandwiched between $\omega(H)$ and $\chi(H)$, every perfect graph is circular perfect. On the other hand, it is proved in [168] that circular complete graphs are circular perfect. In particular, the odd cycles and their complements are circular perfect. So the family of circular perfect graphs is strictly larger than the family of perfect graphs. Some necessary conditions and sufficient conditions for a graph to be circular perfect are proved in [166, 168].

Theorem 7.1 *If G is a circular perfect graph, then for any vertex x of G, $N(x)$ induces a perfect subgraph of G.*

Theorem 7.2 *Suppose G is a graph such that for every vertex x of G, $N[x]$ induces a perfect graph and $G - N[x]$ is a bipartite graph with no induced path on 5 vertices. Then G is circular perfect.*

Observe that if $p/q \geq 3$, then $K_{p/q}$ satisfies the conditions of Theorem 7.2. So the circular perfectness of such a circular complete graph $K_{p/q}$ follows from Theorem 7.2. However, if $p/q < 3$, even the circular complete graph $K_{p/q}$ may not satisfy the condition of Theorem 7.2. If p/q is close to 2, the subgraph of $K_{p/q}$ induced by $K_{p/q} - N[x]$ may have a long induced path. By a close examination, one can extract a property that is shared by the induced paths contained in $K_{p/q} - N[x]$. Namely, none of these paths is badly linked with respect to x, where a 'badly linked path' is defined as follows:

Given an induced path $P_n = (p_0, p_1, \ldots, p_n)$ of $G - N[x]$, we say P_n is badly-linked *with respect to x if one of the following holds:*

1. *There are three indices $i < j < k$ of the same parity such that $N(p_i) \cap N(x) \not\subseteq N(p_j)$ and $N(p_k) \cap N(x) \not\subseteq N(p_j)$.*
2. *There are three indices $i < j < k$ of the same parity such that $N(p_j) \cap N(x) \not\subseteq N(p_i)$ and $N(p_j) \cap N(x) \not\subseteq N(p_k)$.*
3. *There are two even indices $i < j$ and two odd indices $i' < j'$ such that $N(p_i) \cap N(x) \not\subseteq N(p_j)$ and $N(p_{i'}) \cap N(x) \not\subseteq N(p_{j'})$.*

It turns out that the non-existence of badly linked path with respect to x in $G - N[x]$ is indeed crucial for the circular perfectness of a graph.

Theorem 7.3 *Suppose G is a triangle free graph such that for every vertex x of G, $G - N[x]$ is a bipartite graph with no induced C_n for $n \geq 6$, and no badly linked induced path with respect to x. Then G is circular perfect.*

A graph G is a convex round graph if its vertices can be cyclically ordered in such a way that the neighbourhood of each vertex is a 'consecutive segment' of the vertex set. Convex round graphs, which are shown to be circular perfect in [6], can be shown to satisfy the conditions of Theorem 7.2 or Theorem 7.3.

Theorems 7.2 and 7.3 are crucial in the proof of an analogue of Hajós Theorem for circular chromatic number. Hajós Theorem says that for any positive integer n, the family of graphs G with $\chi(G) \geq n$ is the minimal family of graphs which contains K_n and is closed under three operations: (1) taking the Hajós sum of two graphs, (2) identifying non-adjacent vertices, and (3) adding vertices and edges. The question of finding an analogue of Hajós Theorem for circular chromatic number is asked in [154]. To find an analogue of Hajós Theorem for circular chromatic number, it amounts to find a set of graph operations that do not decrease the circular chromatic number, and can be used to construct all graphs G with $\chi_c(G) \geq p/q$, by starting from the single circular complete graph $K_{p/q}$. The operation of identifying non-adjacent vertices and the operation of adding vertices and edges do not decrease the circular chromatic number. However, taking the Hajós sum of two graphs with circular chromatic number p/q may results in a graph with circular chromatic number less than p/q. For example, the Hajós sum of two copies of K_3 is a C_5, where K_3 has circular chromatic number 3, and C_5 has circular chromatic number $5/2$. So we need to find some other operations to replace the Hajós sum. It is proved in [158], by using Theorem 7.2, that if $p/q \geq 3$, there is a set of three graph operations that can be used to replace the Hajós sum. The three graph operations are as follows:

The circular Hajós sum

Take $2k+1$ ($k \geq 1$) graphs $G_0, G_1, G_2, \ldots, G_{2k}$, with $e_i = x_i y_i$ be an edge of G_i. Remove the edges e_i for $i = 0, 1, 2, \ldots, 2k$. Identify all the x_i's into a single vertex x. Adding edges $y_0 y_1, y_1 y_2, \ldots, y_{2k-1} y_{2k}, y_{2k} y_0$. The resulting graph G is the *circular Hajós sum* of $G_0, G_1, G_2, \ldots, G_{2k}$.

The wheel join

Take n graphs $G_0, G_1, G_2, \ldots, G_{n-1}$ ($n \geq 4$), with $e_i = x_i y_i$ be an edge of G_i. Remove the edges e_i, and identify y_i with x_{i+1}, for $i = 0, 1, 2, \ldots, n-1$. Add edges to connect x_i and x_{i+2} for $i = 0, 1, 2, \ldots, n-1$, and add a vertex u and connect u to x_i for $i = 0, 1, 2, \ldots, n-1$. (The additions in the indices are modulo n.) The resulting graph is the *wheel join* of $G_0, G_1, G_2, \ldots, G_{n-1}$.

The pentagon join

Take graphs $G_1, G_2, \ldots, G_7$, with $e_i = x_i y_i$ be an edge of G_i. Remove the edges e_i, and identify y_1, y_2, y_3, y_4, y_5 into a single vertex y. Identify x_6 with x_2, y_6 with x_5, x_7 with x_1, y_7 with x_4. Add edges $x_1 x_2, x_2 x_3, x_3 x_4, x_4 x_5$. The resulting graph is the *pentagon join* of $G_1, G_2, \ldots, G_7$.

Figure 6 is an illustration of these operations, where each shaded area represents a $G_i - e_i$.

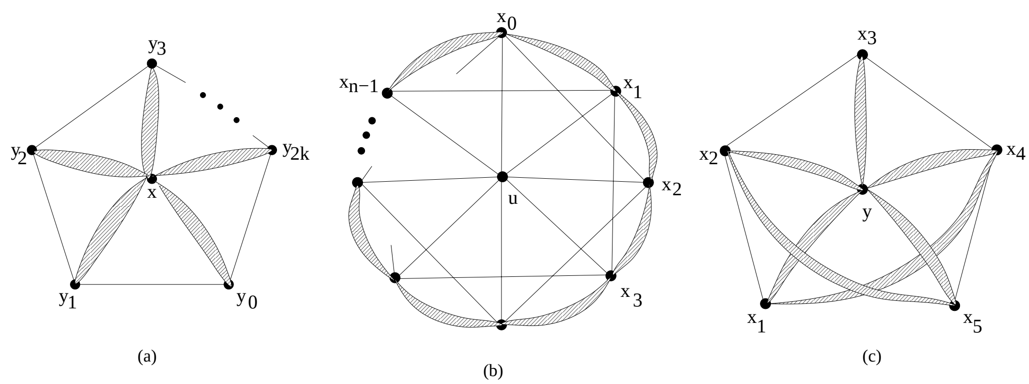

Figure 6: (a) The circular Hajós sum (b) The wheel join (c) The pentagon join

It is not difficult to show that the three operations defined above do not decrease the circular chromatic number, i.e., if each G_i has $\chi_c(G_i) \geq r$, then the circular Hajós sum of the G_i's, the wheel join of the G_i's and the pentagon join of the G_i's all have circular chromatic number at least r (for the pentagon join, we need to assume that $r \geq 3$). However, it is non-trivial to show that these operations (together with identifying non-adjacent vertices and adding vertices and edges) are enough to construct all graphs G with $\chi_c(G) \geq p/q$ (where $p/q \geq 3$), starting from the single graph $K_{p/q}$. This is done in [158], where a crucial step relies on Theorem 7.2.

Theorem 7.4 *Suppose $p/q \geq 3$. Then the family of graphs G with $\chi_c(G) \geq p/q$ is the smallest family of* *graphs that contains $K_{p/q}$ and are closed under the operations of circular Hajós sum, wheel join, pentagon join, adding vertices and edges and identifying non-adjacent vertices.*

Theorem 7.4 does not apply to rationals $p/q < 3$. The corresponding result for $p/q < 3$ is proved in [164], where four more graph operations are introduced. The descriptions of these four graph operations are technical. We omit the details.

An analogue of Hajós Theorem for circular chromatic number of edge weighted graphs is given in [87]. The *strong Hajós sum* of two edge weighted graphs G_1 and G_2 with respect to the edges $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$ is the graph obtained from the disjoint union of G_1 and G_2 by identifying u_1 and u_2, removing the edges u_1v_1 and u_2v_2, and adding an edge v_1v_2 with weight $a_{v_1v_2} = a_{u_1v_1} + a_{u_2v_2}$, where a_{uv} denotes the weight of edge uv. The following result is proved in [87]:

Theorem 7.5 *For any positive real number r, any edge weighted graph G of circular chromatic number at least r contains a subgraph H (in the sense that the weight of an edge in H is at most the weight of that edge in G) which can be constructed by a sequence of identifications of nonadjacent vertices and strong Hajós sums, starting from edge weighted complete graphs of circular chromatic number r whose edge-weights satisfy the ε-triangle inequality: $a_{uv} \leq a_{uw} + a_{wv} + \varepsilon$.*

Unlike in Theorem 7.4, where three graph operations are used to in place of the Hajós sum, a single graph operation, the strong Hajós sum, is used in Theorem 7.5 in place of the original Hajós sum. As the compensation, Theorem 7.5 starts the construction not with a single graph, but with all complete edge weighted graphs (which is an infinite family of graphs) with circular chromatic number r whose edge-weights satisfy the ε-triangle inequality. Observe that the requirement that the edge-weights of the complete graphs satisfy the ε-triangle inequality is important, as any edge weighted graph can be made into a complete edge weighted graph by assigning 0 weight to the non-edges. On the other hand, it is unclear whether the conclusion remains true if the ε-triangle inequality is replaced by triangle inequality. Although the ε-triangle inequality is a small relaxation of the triangle inequality, however, in some sense, the strong Hajós sum allows the ε's to be accumulated.

A homomorphism from a graph G to a graph H is called an *H-colouring* of G. A generalization of Hajós theorem to H-colourings is obtained in [96], where a generalized Hajós sum is used to replace the Hajós sum to construct all non-H-colourable graphs, starting from a finite set of non-H-colourable graphs.

Parallel to the research on perfect graphs, one naturally wonders if there is an appealing circular perfect graph conjecture. A graph G is *circular imperfect* if it is not circular perfect. A graph G is *minimal circular imperfect* if G is circular imperfect but every proper induced subgraph of G is circular perfect. Can we characterize the family of all minimal circular imperfect graphs? Presently, we do not have a 'circular perfect graph conjecture'. A few families of graphs are shown to be minimal circular imperfect [106, 105, 140, 141, 104], which suggests that a characterization of all minimal circular imperfect graphs might be difficult.

Theorem 7.6 *The following graphs are minimal circular imperfect:*

1. *The complement of $K_{(3k+1)/3}$, i.e., the graph with vertex set $\{0, 1, \ldots, 3k\}$ in which ij is an edge if $j = i + 1$ or $i + 2$ (addition modulo $3k + 1$).*
2. *The graph obtained from the odd wheel W_{2q+1} by subdividing each edge on the outer cycle into a path of length $2l - 1$, where q, l are positive integers with $q + l \geq 3$.*
3. *The graph obtained from $K_{(2q+1)/2}$ by adding a universal vertex, and the graph obtained from $K_{(2q+1)/q}$ by adding a universal vertex.*

It was asked in [106] whether every minimal circular imperfect graph G has $\min\{\alpha(G), \omega(G)\} \leq 3$. The answer is negative. It is shown in [104] that there are minimal circular imperfect graphs G with $\min\{\alpha(G), \omega(G)\}$ arbitrarily large.

Circular perfectness is not closed under complement. We call a graph G *strongly circular perfect* if both G and its complement are circular perfect. Perfect graphs are strongly circular perfect, odd cycles and their complements are strongly circular perfect. Strongly circular perfect graphs are studied in [24, 23, 142]. The following theorem follows from results in [142].

Theorem 7.7 *If G is strongly circular perfect, then for any induced subgraph H of G, $\chi_c(H)$ is either an integer, or equal to $2 + 1/k$ or $k + 1/2$.*

Triangle-free strongly circular perfect graphs are characterized in [24]. An *interlaced odd hole* is a graph obtained from an odd cycle C_{2k+1} as follows: Selecting two subsets A, B of $V(C_{2k+1})$ such that $A \cap B = \emptyset$ and B is an independent set. For each vertex v of B, replace v by a set S_v with $|S_v| \geq 2$ (so each vertex $x \in S_v$ is adjacent to the two neighbours of v in C_{2k+1}). For each vertex $v \in A$, add a set S_v of vertices so that each vertex of S_v is connected to v by an edge.

Theorem 7.8 *A triangle-free graph G is strongly circular perfect if and only if G is an interlaced odd hole.*

8 Colouring subgraphs

Suppose G is a graph with $\chi_c(G) = p/q$. What can we say about the circular chromatic number of subgraphs of G? Some simple questions of this type remain open. For example, we do not know if it is true that any graph G with $\chi_c(G) = 8/3$ has a subgraph H with $\chi_c(H) = 5/2$. However, there are some progress in the study of the circular chromatic number of subgraphs.

The following result is proved in [48].

Theorem 8.1 *Suppose G is a graph and e is an edge of G. Then $\chi_c(G - e) \geq \lceil \chi_c(G) \rceil - 1$.*

Corollary 8.1 *If n is a positive integer and G is a graph with $\chi_c(G) > n$, then G has a subgraph H with $\chi_c(H) = n$.*

Corollary 8.1 provides an easy way to construct large girth graphs G with $\chi_c(G)$ equal to a given positive integer. One only needs to construct a graph G of large girth and with $\chi(G) > n$, then by removing some edges from G, one can obtain a graph G' (which still has large girth) with $\chi_c(G') = n$.

Corollary 8.2 *If $\chi(G) = n + 1$ and $\chi(G - e) = n$ for an edge e of G, then $\chi_c(G - e) = n$.*

Corollary 8.3 *If $\chi(G) = n$ and G has a vertex x such that for every n-colouring f of G, $|f(N[x])| = n$, then $\chi_c(G) = n$.*

Corollary 8.3 follows from Corollary 8.2 by considering the graph G' obtained from G by adding a vertex x' and connecting x' to x and to each neighbour of x by an edge. The condition implies that $\chi(G') = \chi(G) + 1$, and hence $\chi_c(G) = \chi_c(G' - xx') = \chi(G') - 1 = \chi(G)$.

The question whether every graph G has a vertex x with $\chi_c(G - x) \geq \chi_c(G) - 1$ is asked in [159]. This question is answered in negative in [165].

Theorem 8.2 *There is an infinite family of graphs G such that $\chi_c(G) = 4$ and for any vertex x of G, $\chi_c(G-x) = 8/3$.*

Figure 7 is an example of a graph G with $\chi_c(G) = 4$ and $\chi_c(G-x) = 8/3$ for each vertex x. The integers besides the vertices show a $(8,3)$-colouring of $G-w$. Note that the graph G is not vertex transitive. The automorphism group of G has three orbits. To prove that $\chi_c(G-x) \le 8/3$ for every vertex v, we need to find a $(8,3)$-colouring for $G-v$ and $G-u$ as well.

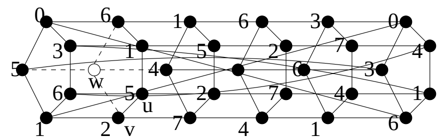

Figure 7: A graph G with $\chi_c(G) = 4$ and $\chi_c(G-x) = 8/3$ for each vertex x

Although there is an infinite family of graphs G for which $\chi_c(G-x) < \chi_c(G)-1$ for each vertex x, all these example graphs have similar structure. In particular, all the graphs have circular chromatic number 4. It is not clear if 4 is an exceptional integer. Properties of graphs G for which $\chi_c(G-x) < \chi_c(G)-1$ for each vertex x are studied in [139].

Question 8.1 *Does there exist a graph G with $\chi_c(G) \neq 4$ and for every vertex x of G, $\chi_c(G-x) < \chi_c(G) - 1$?*

Given a fraction p/q with $(p,q) = 1$. The fraction p'/q' with $p' < p$ and $pq' - p'q = 1$ is the fraction precedes p/q in the Farey sequence. It is known [13, 148] that for any vertex x of the circular complete graph $K_{p/q}$, $\chi_c(K_{p/q} - x) = p'/q'$. The following question remains open.

Question 8.2 *Suppose G is a graph with $\chi_c(G) = p/q$, where $(p,q) = 1$. Is it true that G has a subgraph H with $\chi_c(H) = p'/q'$?*

Theorem 8.1 implies that if p/q is an integer, then the answer to Question 8.2 is positive. For some fractions p/q, say $p/q \neq 4$, it might be true that every graph G with $\chi_c(G) = p/q$ has an induced subgraph H with $\chi_c(H) = p'/q'$.

All known graphs having a vertex v with $\chi_c(G-v) \le \chi_c(G)-1$ satisfy $\chi_c(G) = \chi(G)$. This motivates the following question:

Question 8.3 *Suppose G is a graph with $\chi_c(G) < \chi(G)$. Is it true that for any vertex x of G, $\chi_c(G-x) \ge \chi(G) - 1$?*

9 K_n-minor free graphs

In [134], Vince showed that for any rational $p/q \ge 2$, there is a graph G, namely $G = K_{p/q}$, which has circular chromatic number p/q. (The notation for the graph $K_{p/q}$ has been evolved from $G(q,p)$ [134] to G_p^q [13], and hopefully settled down to $K_{p/q}$. The graph $K_{p/q}$ can also be defined as the complement of powers of cycles $\overline{C_p^q}$, and are known as *antiwebs* in the literature [113, 135].) If we want to construct a graph with some special properties to have a given circular chromatic number, or to prove there is no such graph, the problem can be very difficult.

A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting some edges. A graph G is called *H-minor free* if H is not a minor of G. The well-known Hadwiger's conjecture says that K_n-minor free graphs are $(n-1)$-colourable. If this conjecture is true, then every K_n-minor free graph has circular chromatic number at most $n-1$. It is natural to ask if every rational number $r \in [2, n-1]$ is the circular chromatic number of some K_n-minor free graphs. The answer is positive if $n \geq 5$ and negative for $n = 4$ [56, 74, 157]. There are two key ideas in the construction of K_n-minor free graphs with given circular chromatic number. These ideas are useful elsewhere, for example, in the construction of graphs with given circular flow number and the construction of uniquely colourable graphs with given circular chromatic number. One of the ideas is to use Farey sequence that constructs the fractions between two consecutive integers in a special order. Given a non-negative integer n, the fractions in the interval $[n, n+1]$ can be partially ordered as follows: First we have two fractions $n/1$ and $(n+1)/1$. Suppose a/b and a'/b' are two fractions with $a'b - ab' = 1$, then we can construct a new fraction $p/q = (a+a')/(a+b')$. We call a/b and a'/b' the *lower parent* and the *upper parent* of p/q, respectively, and denoted by $a/b = p_l(p/q)$ and $a'/b' = p_u(p/q)$. Starting from $n/1$ and $(n+1)/1$, all the fractions $p/q \in [n, n+1]$ can be constructed in this way. If we want to construct, for a given $p/q \in [n, n+1]$, a K_{n+2}-minor free $\chi_c(G) = p/q$, we shall use induction, by first constructing K_{n+2}-minor free graphs H and H' with $\chi_c(H) = p_l(p/q)$ and $\chi_c(H') = p_u(p/q)$. Then graphs H and H' are used as building blocks in the construction of G.

The second idea is the labeling method. Suppose G is a graph and e^* is an edge of G. The pair (G, e^*) is called a *rooted graph* with *root edge* e^*. For convenience, we usually say G is a rooted graph with root edge e^*. Suppose G is a rooted graph with root edge $e^* = xy$ and $r \geq 2$ is a real number. Let D be an orientation of G. For convenience, the arc obtained by assigning an orientation to e^* is also denoted by e^*. A *rooted r-tension* of D is a tension f on D such that for each arc $e \neq e^*$, $1 \leq |f(e)| \leq r-1$. In other words, a rooted r-tension f is almost an r-tension except that there is no constraint on the value of f on the root arc.

Suppose G is a rooted graph with root edge e^* and $r \geq 2$ is a real number. Let D be an orientation of G. The *r-tension label set* of (G, e^*), denoted by $L_T^r(G, e^*)$, is defined as

$$L_T^r(G, e^*) = \{t \in [0, r), \exists \text{ a rooted } r\text{-tension } f \text{ of } D \text{ with } f(e^*) = t\}.$$

Although the definition of $L_T^r(G, e^*)$ needs to refer to an orientation of G, but any orientation defines the same label set.

We usually write $L_T^r(G)$ for $L_T^r(G, e^*)$. The root edges are usually clear from the context (or is irrelevant, such as in K_n or C_n). As an example, it follows from the definition that $L_T^r(C_2) = [1, r-1]_r$, where C_2 is the cycle with two edges.

The r-tension label set of a rooted graph G with root edge e^* contains information of possible r-tensions of $G - e^*$. For example, it is easy to see from the definition that

$$\begin{aligned} \chi_c(G) \leq r &\quad\Leftrightarrow\quad [1, r-1] \cap L_T^r(G) \neq \emptyset \\ \chi_c(G - e^*) \leq r &\quad\Leftrightarrow\quad L_T^r(G) \neq \emptyset \\ \chi_c(G/e^*) \leq 0 &\quad\Leftrightarrow\quad 0 \in L_T^r(G). \end{aligned}$$

In general, it is difficult to calculate the r-tension label set of a rooted graph. However, if a rooted graph G is constructed from some simple building blocks by some simple graph operations, then it might be easy to calculate $L_T^r(G)$. Here by 'simple graph operations', we mean the series join, the parallel join, the extended series joins and the diamond constructions. Among these, the easiest are series join and parallel join.

Suppose G and G' are two rooted graphs, with root edges $e = xy$ and $e' = x'y'$, respectively.

Series join The series join of G and G', denoted by $S(G, G')$, is the rooted graph obtained from the disjoint union of G and G' by removing e and e', identifying y and x' into a single vertex z, and adding an edge $e'' = xy'$, where e'' is the root edge of G''.

Parallel join The parallel join of G and G', denoted by $P(G, G')$, is the rooted graph obtained from the disjoint union of G and G' by removing e and e', identifying x and x' into a single vertex x'', identifying y and y' into a single vertex y'', and adding an edge $e'' = x''y''$, where e'' is the root edge of G''.

For two subsets A, B of $[0, r)$, let $A + B = \{[a + b]_r : a \in A, b \in B\}$. Then we have the following lemma:

Lemma 9.1 *For any rooted graphs G, G', we have*

$$\begin{aligned} L_T^r(S(G,G')) &= L_T^r(G) + L_T^r(G') \\ L_T^r(P(G,G')) &= L_T^r(G) \cap L_T^r(G'). \end{aligned}$$

Now we ask the following question: Let $\mathcal{F}$ be a family of graphs (usually $\mathcal{F}$ consists of a single or very few small graphs). Let $r \geq 2$ be a rational. Can we construct a graph G with $\chi_c(G) = r$ by repeatedly applying the series join and parallel joins to a graphs in $\mathcal{F}$?

Instead of constructing a graph with a given circular chromatic number, we may construct a graph with a given r-tension label set. This seemingly more difficult task turns out to be easier. Indeed, we may completely forget about graphs, and simply construct their label sets. Formally, fix a real number r and an initial label sets $\mathcal{L}$ (of subsets of $[0, r)$). We define *constructible label sets* (with respect to r) as follows: *If $A \in \mathcal{L}$, then A is constructible. If A, B are constructible label sets, then $A \cap B$ and $A + B$ are constructible label sets.* Note that $A + B$ is taken as the sum two subsets of $[0, r)$.

It follows from Lemma 9.1 that if $\mathcal{L} = \{L_T^r(G) : G \in \mathcal{F}\}$, then a label set A is constructible if and only if there is a graph G which can be constructed by repeatedly applying the series join and the parallel join to graphs in $\mathcal{F}$, and for which we have $L_T^r(G) = A$.

Theorem 9.1 *Suppose $\mathcal{L}$ is an initial label set and $\mathcal{A}$ is the set of label sets constructible from $\mathcal{L}$. Suppose $p/q > 4$ and $p_l(p/q) < r < p_u(p/q)$. If $[n, r-n]_r, [1, r-1]_r \in \mathcal{L}$, then*

$$[p - 1 - (q-1)r, qr - p + 1]_r \in \mathcal{A}_r.$$

Observe that for any $r > 2$, $L_T^r(C_2) = [1, r-1]_r$. Suppose H is a graph and e is an edge of H. It is not difficult to prove that if $\chi_c(H) = n+1$ and $\chi_c(H-e) = n$, then for any $r \in (n, n+1)$, $L_T^r(H) = [n, r-n]_r$. Thus as a consequence of Theorem 9.1, we have the following result:

Theorem 9.2 *Let $\mathcal{H}$ be a family of rooted graphs which contains C_2 and are closed under series joins and parallel joins. Let $n \geq 4$ be an integer. If there is a rooted graph $(H, e) \in \mathcal{H}$ such that $\chi_c(H) = n+1$ and $\chi_c(H-e) = n$, then for every rational $r \in (n, n+1)$, there is a rooted graph $G \in \mathcal{H}$ such that $\chi_c(G) = r$.*

As for $n \geq 4$, we have $\chi_c(K_{n+1}) = n+1$ and $\chi_c(K_{n+1} - e) = n$ for any edge e, we have the following result, which is proved in [74], also answering a question in [159]:

Theorem 9.3 *If $n \geq 5$, then for any rational number $r \in [n-2, n-1]$, there is a K_n-minor free graph G with $\chi_c(G) = r$.*

It is proved earlier in [90] and [155] that for any rational $r \in [2, 4]$, there is a planar graph (and hence a K_5-minor free graph) G with $\chi_c(G) = r$. Combined with the results above, we have

Corollary 9.1 *If $n \geq 5$, then for any rational $r \in [2, n-1]$, there is a K_n-minor free graph G with $\chi_c(G) = r$.*

For K_4-minor free graphs, the following result is a combination of results in [56] and [102].

Theorem 9.4 *If G is a K_4-minor free graph, then either $\chi_c(G) = 3$ or $\chi_c(G) \leq 8/3$. Moreover, for any rational $r \in [2, 8/3] \cup \{3\}$, there is a K_4-minor free graph G with $\chi_c(G) = r$.*

In the construction of planar graphs with circular chromatic number r for $r \in [2, 4]$, the series joins and parallel joins are not enough. Two other graph operations have been used: diamond construction and extended series joins. We refer readers to [74] for discussions about these two operations.

10 Graphs of large girth

A classical result of Erdős says that graphs of large girth can have large chromatic number. Erdős' result has been strengthened in different ways. One very strong result in this aspect is the following result of Müller [91].

Theorem 10.1 *Let ℓ be an integer, let A be a finite set, and let $f_1, f_2, \ldots, f_t$ be distinct n-colourings of the elements of A. Then there exists a graph $G = (V, E)$ such that the following hold:*

- *A is a subset of V.*
- *Each n-colouring of G is an extension of some f_i, and for each $i = 1, 2, \ldots, t$, there exists a unique n-colouring g_i of G which is an extension of f_i.*
- *G has girth at least ℓ.*

If $t = 1$, then the graph G as in Theorem 10.1 is uniquely n-colourable, and hence has chromatic number n.

Theorem 10.1 can be generalized to circular colourings, and more generally to H-colourings. Two r-colourings f and g of a graph G are equivalent if there are constants $c \in [0, r)$ and $\tau \in \{-1, 1\}$ such that for any vertex x of G, $f(x) = [\tau g(x) + c]_r$. A graph G is called *uniquely r-colourable* if G has exactly one r-colouring, up to equivalence. Two H-colourings f, g of a graph G are equivalent if there is an automorphism σ of H such that $f = \sigma \circ g$. A graph G is called uniquely H-colourable if, up to equivalence, there is exactly one onto H-colouring of G. If $r = p/q \geq 2$ and $(p, q) = 1$, then a graph G is uniquely r-colourable if and only if G is uniquely $K_{p/q}$-colourable. It is easy to see that if a graph G is uniquely r-colourable, then $\chi_c(G) = r$. So the existence of uniquely r-colourable graphs with a certain property implies the existence of graphs G with $\chi_c(G) = r$ with that property. One question of interest is the existence of graphs of large girth with given circular chromatic number [1, 97, 98, 117, 150, 153]. For any rational $r \geq 2$, for any integer g, does there exist a graph G of girth at least g and with $\chi_c(G) = r$? This leads to the study of the existence of uniquely r-colourable graphs of large girth. A graph G is a *core* if it does not admit a homomorphism to a proper subgraph. It is proved in [150] that for any core graph H and for any integer g, there is a graph G of girth at least g such that G is uniquely H-colourable. This implies that for any $r \geq 2$ and for any integer g, there is a graph G of girth at least g and with $\chi_c(G) = r$. In [98], the following question is considered: Let H be a graph. Given any integers t, g, does there exist a graph G of girth at least g which has exactly t H-colourings, up to equivalence? In case $t = 1$, the question asks whether there exists a uniquely H-colourable graph G of girth at least g. It turns our the answer is positive in most cases. Indeed, in most cases, we can ask for more: not only there are exactly t H-colourings, but these t H-colourings are extensions of any previously given t mappings from a subset A of $V(G)$ to $V(H)$. To state the result, we need one definition. A graph H is called *t-projective* if every homomorphism $f : H^t \to H$ is equivalent to a projection, i.e., there is a projection p and an automorphism σ of H such that $f = \sigma \circ p$. Here H^t is the categorical product of t copies of H, and a projection from H^t to H is a homomorphism $p : H^t \to H$ of the form $p(x_1, x_2, \ldots, x_t) = x_i$ for a fixed i. A graph H is called *projective* if H is t-projective for every positive integer t. Note that a projective graph is necessarily a core graph. Projectivity of graphs is first defined in [71, 72], but studied by others in different contexts (see [84]). It is proved in [71, 72] that a graph H is projective if and only if H is 2-projective. In [84], it is proved that most graphs are projective. The following result is proved in [98]:

Theorem 10.2 *Let H be a projective graph with k vertices. Let ℓ be an integer, let A be a finite set, and let $f_1, f_2, \ldots, f_t$ be distinct mappings from A to $V(H)$. Then there exists a graph $G = (V, E)$ such that the following hold:*

- *A is a subset of V.*
- *For every $i = 1, 2, \ldots, t$, there exists a unique homomorphism $g_i : G \to H$ such that $g_i|_A = f_i$.*
- *For every homomorphism $f : G \to H$, there exists an index i, $1 \leq i \leq t$, and an automorphism σ of H such that $f = \sigma \circ g_i$.*
- *G has girth at least ℓ.*

It follows from some general results in [71, 72, 73] that the circular complete graphs are projective. A direct proof of this fact can also be found in [97]. Thus we have the following corollary:

Corollary 10.1 *For any rational $r \geq 2$, for any positive integers t, ℓ, there is a graph G of girth at least ℓ such that G has exactly t r-colourings, up to equivalence. In particular, for any $r \geq 2$ and for any integer ℓ, there is a graph G of girth at least ℓ which is uniquely r-colourable and hence it has $\chi_c(G) = r$.*

The proof of Theorem 10.2 in [98] uses the probabilistic method. A constructive proof of Corollary 10.1 for $r \geq 3$ is given in [97], and a constructive proof of the corollary for $2 \leq r \leq 3$ is given in [103].

In [30], it is proved that there exist uniquely n-colourable graphs of large girth with bounded maximum degree. This result is generalized in [49], where it is proved that for any rational p/q, for any integer g, there is a uniquely p/q-colourable graph G of girth g and with $\Delta(G) \leq 5p^{13}$.

One interesting problem is the circular chromatic number of cubic graphs of large girth. The following question is due to Nešetřil [95]:

Question 10.1 *Is it true that cubic graphs G of sufficiently large girth have $\chi_c(G) \leq 5/2$?*

The answer is negative if $5/2$ is replaced by $7/3$ [51]. The following question is also open.

Is it true that cubic graphs G of sufficiently large girth have $\chi_c(G) \leq r$ for some $r < 3$?

On the other hand, it is proved by Hatami and this author [52] that cubic graphs G of girth at least 4 have $\chi_f(G) \leq 3 - \frac{3}{64}$.

11 $\chi_c(G)$ v.s. $\chi(G)$ and $\chi_f(G)$

It is known that for any graph G, we have $\chi_f(G) \leq \chi_c(G) \leq \chi(G)$. Graphs G with $\chi_c(G) = \chi(G)$ and graphs with $\chi_f(G) = \chi_c(G)$ are of special interest and have been studied in many papers. It is proved in [41] that it is NP-hard to determine if a graph G satisfies $\chi_c(G) = \chi(G)$. In [159], the author asked the question whether the problem is still NP-hard if $\chi(G)$ is known. This question is answered in [50].

Theorem 11.1 *The following decision problem is NP-hard:*
Instance: A graph G with $\chi(G) = n$;
Question: Is it true that $\chi_c(G) = \chi(G)$?

Some sufficient conditions are listed in [159] under which a graph G has $\chi_c(G) = \chi(G)$. For example, if G is uniquely n-colourable, then $\chi_c(G) = \chi(G)$ [150]; if the complement of G is disconnected, then $\chi_c(G) = \chi(G)$ [1, 148]. The following is a new sufficient condition found by Fan [31].

Theorem 11.2 *If the complement of G is not Hamiltonian, then $\chi_c(G) = \chi(G)$.*

The proof of Theorem 11.2 is quite easy. Assume $\chi_c(G) = p/q$, where $(p, q) = 1$ and $q \geq 2$. We shall prove that the complement $\overline{G}$ of G has a Hamilton cycle. Let f be a (p, q)-colouring of G. Then for each i, $f^{-1}(i) \neq \emptyset$. Let X_i be an arbitrary ordering of the vertices of $f^{-1}(i)$, then the cyclic concatenation $X_0X_1 \dots X_{p-1}$ of the X_i's is a Hamilton cycle of $\overline{G}$.

Theorem 11.2 is used in [31] to study the circular chromatic number of iterated Mycielskian of complete graphs. It is known that $\chi(M(G)) = \chi(G) + 1$, but $\chi_c(M(G))$ could be strictly smaller than $\chi(G) + 1$. However, for any integer $k \geq 3$, $\chi_c(M(K_n)) = n + 1$ [20]. For an integer $t \geq 2$, the t-th iterated Mycielskian $M^t(G)$ of G is defined as $M^t(G) = M(M^{t-1}(G))$. The circular chromatic number of the iterated Mycielskian of complete graphs is studied in [20]. The following conjecture is proposed in [20].

Conjecture 11.1 *If $t \leq n - 2$, then $\chi_c(M^t(K_n)) = \chi(M^t(K_n))$.*

As $M(G) = \Delta_1(G)$, Corollary 6.3 implies that $\chi_c(M^t(K_n)) = \chi(M^t(K_n))$ if $t + n$ is even. On the other hand, it is shown in [20] that if $t = n - 1$, then $\chi_c(M^t(K_n)) \leq \chi(M^t(G)) - 1/2$. So the requirement that $t \leq n - 2$ in Conjecture 11.1 is needed.

For $n = 3, 4$, Conjecture 11.1 is verified [20]. For $n = 3$, a stronger result is proved in [3]: *If $n \geq 3$ and G uniquely n-colourable, then $\chi_c(M(G)) = \chi(M(G))$.* Theorem 11.2 provides a simple proof for the $n = 3$ case of Conjecture 11.1, as $\overline{M(K_n)}$ (for $n \geq 3$) is easily seen to be non-Hamiltonian. By using Theorem 11.2, it is also proved in [31] that if G has n vertices and $\chi(G) \geq (n + 3)/2$, then $\chi_c(M(G)) = \chi(M(G))$.

For $n \geq 5$, Conjecture 11.1 remains open. However, it is proved in [46] that if $n \geq 2^t + 2$, then $\chi_c(M^t(K_n)) = \chi(M^t(K_n))$. This result is improved in [81], where it is proved that if $m \geq 2^{t-1} + 2t - 2$ and $t \geq 2$, then $\chi_c(M^t(K_m)) = \chi(M^t(K_m))$.

Graphs G with $\chi_f(G) = \chi_c(G)$ are called *star extremal*. Star extremal graphs have some nice properties. For example, if G, H are star extremal, then the upper and lower bounds in Theorem 5.5 coincide, and hence the chromatic number of $G \oplus H$ is determined. It is likely that it is NP-hard to determine if an arbitrary graph G is star extremal, but as far as this author knows, there is no proof yet. The study of star extremal graphs has been concentrated on circulant graphs and distance graphs. Some classes of circulant graphs and distance graphs are proved to have this property [19, 21, 36, 58, 76, 77, 78, 79, 80, 138, 149, 161]. A useful tool in the study of the circular chromatic number of circulant graphs and distance graphs is the *regular colouring method*. Suppose n is a positive integer and D is a set of positive integers $i \leq n/2$. The circulant graph $G(n, D)$ has vertex set $\{0, 1, \dots, n-1\}$ in which ij is an edge if $[i-j]_n \in D$ or $[j-i]_n \in D$. Given an integer k, let $\lambda_k(D) = \min\{|ki|_n : i \in D\}$, and let $\lambda(D) = \max\{\lambda_k(D) : k = 1, 2, \dots, n\}$. It is known [36] that $\chi_c(G(n, D)) \leq n/\lambda(D)$. In many cases, this bound is sharp. For example, if $G(n, D)$ has maximum degree 3, then $\chi_c(G(n, D)) = n/\lambda(D) = \chi_f(G(n, D))$ [36]; if $D = \{i, i + 1, i + 2, \dots, i'\}$ and $i' \geq 6i/5$, then $\chi_c(G(n, D)) = n/\lambda(D) = \chi_f(G(n, D))$ [76].

For a finite set D of positive integers, the distance graph $G(Z, D)$ has vertex set Z in which ij is an edge if $|i - j| \in D$. For any integers k, n, let $\lambda^n_k(D) = \min\{|ki|_n : i \in D\}$, and let $\lambda^n(D) = \max\{\lambda^n_k(D) : k = 1, 2, \dots, n\}$. It is also known that $\chi_c(G(Z, D)) \leq n/\lambda^n(D)$ for any integer n. In many cases, this upper bound is also sharp for an appropriately chosen n.

12 Circular choosability

A k-list assignment of a graph G is a mapping L which assigns to each vertex x a set $L(x)$ of k colours. A G is called k-choosable if for each k-list assignment L, there is a colouring f of G such that for each vertex x, $f(x) \in L(x)$. The *choosability* (or the *list chromatic number*) of G is the least k for which G is k-choosable. Choosability of graphs is an extensively studied graph parameter. The concept of choosability of graphs can be naturally extended to circular colouring. Suppose G is a graph and $p \geq 2q$ are positive integers. A (p, q)*-list assignment* L is a mapping which assigns to each vertex v of G a subset $L(v)$ of $\{0, 1, \ldots, p-1\}$. An L-(p,q)*-colouring* of G is a (p, q)-colouring f of G such that for any vertex v, $f(v) \in L(v)$. A *list-size assignment* is a mapping $\ell : V \to [0, p/q]$. Given a list-size assignment ℓ, an ℓ-(p,q)-list assignment is a (p, q)-list assignment L such that for each vertex v, $|L(v)| \geq \ell(v)q$. A graph G is called ℓ-(p,q)*-choosable* if for any ℓ-(p,q)-list assignment L, G has an L-(p,q)-colouring.

The circular list chromatic number of a graph is defined through list-size assignments which are constant mappings. Suppose $t \geq 1$ is a real number and $p \geq 2q$ are positive integers. A t-(p,q)*-list assignment* is a (p, q)-list assignment L such that for every vertex v, $|L(v)| \geq tq$. We say G is *circular* t-(p,q)*-choosable* if for any t-(p,q)-list assignment L, G has an L-(p,q)-colouring. We say G is *circular* t*-choosable* if G is circular t-(p,q)-choosable for any positive integers $p \geq 2q$. The *circular list chromatic number* (or the *circular choosability*) of G is defined as

$$\chi_{c,l}(G) = \inf\{t : G \text{ is circular } t\text{-choosable}\}.$$

The circular list chromatic number of a graph can also be defined through circular r-colourings using colours from the circle S^r. A subset U of S^r is said to be *assignable* if it is the union of finitely many disjoint open arcs on S^r. The *length of an assignable set* U, denoted by $\text{length}(U)$, is the sum of the lengths of the open arcs of U. If $G = (V, E)$ is a graph, then a function L that assigns to each vertex v of G an assignable subset $L(v)$ of S^r is called an *circular list assignment* (with respect to r). If for each vertex v of G, $L(v)$ has length at least t, then L is called a t*-circular list assignment* (with respect to r). A *circular* L*-colouring* of G is a mapping c from V to S^r such that $c(v) \in L(v)$ for each vertex v of G and for every pair u, v of adjacent vertices of G, $|c(u) - c(v)|_r \geq 1$.

It is proved in [167] that for any graph G, for any real number t, if G is circular t-choosable then for $\varepsilon > 0$ and for any $(t + \varepsilon)$-circular list assignment L, G has an circular L-colouring. Conversely, if for any t-circular list assignment L, G has an circular L-colouring, then G is circular t-choosable. Therefore the circular list chromatic number of G can be defined as

$$\chi_{c,l}(G) = \inf\{t : \text{ for any } t\text{-circular list assignment } L, G \text{ has a circular } L\text{-colouring}\}.$$

It is easy to see that for any integer n, if G is not n-choosable, then G is not circular $(n-1)$-choosable. As a consequence, we have $\chi_{c,l}(G) \geq \chi_l(G) - 1$. However, it is surprising that the circular list chromatic number of a graph can be much large than its list chromatic number. It is proved in [167] that for any positive integer k, for each $\varepsilon > 0$, there is a k-degenerated graph G which is not circular $(2k - \varepsilon)$-choosable. Since every k-degenerated graph is $(k+1)$-choosable, it follows that the difference $\chi_{c,l}(G) - \chi_l(G)$ can be arbitrarily large. On the other hand, we have the following result:

Theorem 12.1 *Suppose G is a finite k-degenerated graph and L is a $2k$-circular list assignment of G. Then there is a mapping f which assigns to each vertex v of G an open interval $f(v) \subseteq L(v)$ of positive length such that if $v \sim v'$, then for any $x \in f(v)$ and $x' \in f(v')$, $|x - x'|_r \geq 1$.*

As a corollary, every k-degenerated graph G has $\chi_{c,l}(G) \leq 2k$. For an arbitrary graph G, it is unknown if $\chi_{c,l}(G)$ is bounded by a multiple of $\chi_l(G)$.

Question 12.1 *Is there a constant α such that for any graph G, $\chi_{c,l}(G) \leq \alpha\chi_l(G)$? If such a constant exists, what is the smallest α?*

For an arbitrary graph G, it is difficult to determine $\chi_{c,l}(G)$. The question is open even for circular complete graphs.

Question 12.2 *What is the circular list chromatic number of $K_{p/q}$? Is it true that $\chi_{c,l}(K_{p/q}) = p/q$?*

The circular list chromatic number of odd cycles is determined in [167]. But the question for even cycles is still open.

Theorem 12.2 *The odd cycle C_{2k+1} has $\chi_{c,l}(C_{2k+1}) = 2 + \frac{1}{k}$.*

Although a k-degenerated graph can have circular list chromatic number almost as large as $2k$, $\chi_{c,l}(G)$ is bounded by 1 plus its maximum degree.

Theorem 12.3 *[167] Suppose G has maximum degree k. Then G is circular $(k+1)$-choosable.*

Circular list colouring can be used as a tool in inductive proofs in solving circular colouring problems. In an inductive proof, we may need to prove that any (p,q)-colouring of a subset X of $V(G)$ can be extended to a (p,q)-colouring of G. This would be equivalent to prove that for a certain list-size assignment ℓ, the subgraph $G - X$ is ℓ-(p,q)-choosable. For the purpose of application in such inductive proofs, one may need to consider list-size-assignments that are not constant mappings. It is a more difficult problem to characterize all colour-size-lists ℓ for which a graph G is ℓ-(p,q)-choosable. However, if G is a tree, such a characterization is given in [109].

Theorem 12.4 *Suppose T is a tree, $p \geq 2q$ are positive integers and $\ell : V(T) \to \{0,1,2,\ldots,p\}$ is a colour-size-list. Then T is ℓ-(p,q)-choosable if and only if for each subtree T' of T,*

$$\sum_{v \in T'} \ell(v) \geq 2(|V(T')| - 1)q + 1.$$

The corresponding problem for cycles is also studied in [109]. A sharp sufficient condition for ℓ is given in [109] under which a cycle G is ℓ-$(2k+1,k)$-choosable.

Theorem 12.5 *Let $k \geq 1$ be an integer, and let $X = (x_0, x_1, \ldots, x_{n-1})$ be a cycle of length $n \geq 2k+1$. Suppose $f : V(X) \to \{0,1,2,\ldots,2k+1\}$ is a colour-size-list for X. Then X is ℓ-$(2k+1,k)$-choosable if the following conditions hold:*

(1) For each interval $[j,j']_n$ of length m, $\sum_{t \in [j,j']_n} \ell(x_t) \geq 2(m-1)k + 1$.

(2) $\sum_{t=0}^{n-1} \ell(x_t) \geq 2nk + 1$.

Moreover, Condition (1) is necessary for X to be ℓ-$(2k+1,k)$-colourable, and in case X is an odd cycle, Condition (2) is sharp in the sense that there is a colour-size-list ℓ which satisfies (1), and $\sum_{t=0}^{n-1} \ell(x_t) = 2nk$, but X is not ℓ-$(2k+1,k)$-choosable.

13 K_4-minor free graphs

The class of K_4-minor free graphs have a very simple structure. Each block of a K_4-minor free graph can be obtained from C_2 by repeatedly apply series joins and parallel joins, and is called a *series-parallel graph*. The class of K_4-minor free graphs is also known as the class of partial 2-tree. Due to the simplicity of its structure, many difficult problems becomes easy when restricted to K_4-minor free graphs. One question of interest is the relation between circular chromatic number and the girth of graphs [1, 14, 18, 22, 35, 49, 56, 66, 67, 97, 98, 99, 100, 147, 150, 153]. By Corollary 10.1, in general, graphs of large girth can have arbitrary given circular chromatic number. However, if restricted to special classes of graphs, large girth graphs may be forced to have small circular chromatic number. The following result is proved in [35].

Theorem 13.1 *For any integer $n \geq 4$, for any $\varepsilon > 0$, there is an integer g such that every K_n-minor free graph G of girth at least g has $\chi_c(G) \leq 2 + \varepsilon$.*

Observe that K_3-minor free graphs are forests. So Theorem 13.1 holds trivially for $n = 3$ (the requirement that $n \geq 4$ is to exclude the trivial case). Given an integer $n \geq 4$ and an $\varepsilon > 0$, let $g(n, \varepsilon)$ be the smallest integer such that every K_n-minor free graph G of girth at least g has $\chi_c(G) \leq 2 + \varepsilon$. To determine $g(n, \varepsilon)$ is very difficult problem for $n \geq 5$. For example, the best known lower and upper bound for $g(5, 1/2)$ is $9 \leq g(5, 1/2) \leq 13$. As ε gets smaller, the gap between the best known upper and lower bounds becomes larger. However, $g(4, \varepsilon)$ is completely determined. It turns out that what really matters in bounding the circular chromatic number of a K_4-minor free graph is the odd girth of G. The following results are proved in [99, 100].

Theorem 13.2 *Suppose G is a K_4-minor free graph and $k \geq 1$ is an integer.*

1. *If G has odd girth at least $6k - 1$ then $\chi_c(G) \leq 8k/(4k - 1)$;*

2. *If G has odd girth at least $6k + 1$ then $\chi_c(G) \leq (4k + 1)/2k$;*

3. *If G is has odd girth at least $6k + 3$ then $\chi_c(G) \leq (4k + 3)/(2k + 1)$.*

Theorem 13.2 strengthens a result proved in [22], and the bounds given above are tight.

Theorem 13.3 *Let $k \geq 1$ be an integer, and let $\varepsilon > 0$.*

1. *There exists a series-parallel graph G of girth $6k - 1$ with $\chi_c(G) > 8k/(4k - 1) - \varepsilon$;*

2. *There exists a series-parallel graph G of girth $6k + 1$ with $\chi_c(G) > (4k + 1)/2k - \varepsilon$;*

3. *There exists a series-parallel graph G of girth $6k + 3$ with $\chi_c(G) > (4k + 3)/(2k + 1) - \varepsilon$.*

As a consequence of Theorems 13.2 and 13.3, we have the following

Corollary 13.1 *Suppose $1 > \varepsilon > 0$. If $1/(2k - 1) > \varepsilon \geq 2/(4k - 1)$, then $g(4, \varepsilon) = 6k - 1$. If $1/(2k - 2) > \varepsilon \geq 1/(2k - 1)$, then $g(4, \varepsilon) = 6k - 3$. If $2/(4k - 5) > \varepsilon \geq 1/(2k - 2)$, then $g(4, \varepsilon) = 6k - 5$.*

As mentioned in Section 9 that for a rational r, there is a K_4-minor free graph G with $\chi_c(G) = r$ if and only if $r \in [2, 8/3] \cup \{3\}$. We say a class $\mathcal{C}$ of graphs is *universal* if for each countable partial order P, there is an injective mapping $\phi : P \to \mathcal{C}$ such that $x \leq y$ if and only if there is a homomorphism of $\phi(x)$ to $\phi(y)$. For a rational $r \in [2, 8/3] \cup \{3\}$, let $\mathcal{C}_r$ be the family of K_4-minor free graphs G with $\chi_c(G) = r$. Nesetril and Nigussie [94] proved that $\mathcal{C}_r$ is universal if and only if $r \in (2, 5/2) \cup (5/2, 8/3)$.

14 Circular flow

Suppose G is a graph and D is an orientation of G. A *flow* of G with respect to D is a mapping $f : E(D) \to \mathbb{R}$ which assigns to each arc $e = (x, y)$ of D a real number $f(e)$ such that for each cut B of G,

$$\sum_{e \in B^+} f(e) = \sum_{e \in B^-} f(e).$$

Here a cut B is the set of arcs between a subset S of G and $\overline{S} = V(G) \setminus S$, B^+ and B^- denote the sets of arcs from S to $\overline{S}$ and arcs from $\overline{S}$ to S, respectively. For a real number $r \geq 2$, an *r-flow* of G with respect to an orientation D is a flow f such that for each arc e, $1 \leq |f(e)| \leq r - 1$. If f is an r-flow of G with respect to an orientation D, and D' is the orientation obtained from D by replacing an arc $e = (x, y)$ with its opposite arc $e^{-1} = (y, x)$, then by letting $f(e^{-1}) = -f(e)$, we obtain an r-flow of G with respect to D'. So if one orientation of G has an r-flow, then every orientation has an r-flow, and we simply say G has an r-flow. The *circular flow number* $\Phi_c(G)$ of a bridgeless graph G is defined as

$$\Phi_c(G) = \min\{r : \ G \text{ has an } r\text{-flow}\}.$$

Equivalently, for integers $p \geq 2q \geq 2$, we define a (p, q)-flow to be a mapping $f : E(D) \to \{\pm q, \pm(q + 1), \ldots, \pm(p - q)\}$ such that for each cut B of G,

$$\sum_{e \in B^+} f(e) = \sum_{e \in B^-} f(e).$$

Then the circular flow number of a bridgeless graph is $\Phi_c(G) = \min\{p/q : G \text{ has a } (p, q)\text{-flow}\}$. If $q = 1$, then a $(p, 1)$-flow f is called a *nowhere zero p-flow*. The *flow number* $\Phi(G)$ of a bridgeless graph G is defined as

$$\Phi(G) = \min\{p : \text{ there is a nowhere zero } p\text{-flow of } G\}.$$

It follows from the definition that for any graph G,

$$\Phi(G) - 1 < \Phi_c(G) \leq \Phi(G).$$

The circular flow number of a graph can also be defined through orientations. Suppose D is an orientation of G. The *imbalance* of a cut B with respect to D is $\mathrm{Imb}_D(B) = \max\{|B|/|B^+|, |B|/|B^-|\}$. The *Cut Imbalance of D* is defined as $\mathrm{CutImb}(D) = \max\{\mathrm{Imb}_D(B) : B \text{ is a cut of } G\}$. It is proved by Goddyn, Tarsi and Zhang [39] that

$$\Phi_c(G) = \min\{\mathrm{CutImb}(D), D \text{ is an acyclic orientations of } \ G\}.$$

15 Coloring-flow duality of embedded graphs

Coloring and flow are dual concepts in graph theory. It is proved by Tutte [131] that if G is a planar graph and G^* is the geometrical dual of G, then G is k-colourable if and only if G^* admits a nowhere zero k-flow. This result can be easily extended to circular chromatic number and circular flow number: *if G is a planar graph and G^* is the geometrical dual of G, then $\chi_c(G) = \Phi_c(G^*)$.* It is natural to ask if similar results exist for graphs embedded in other surfaces.

An embedded graph G is a triple $(V(G), E(G), F(G))$, where $V(G), E(G)$ and $F(G)$ are the vertex set, edge set and face set of G, respectively. Associated with each face $R \in F(G)$, is a *boundary walk*, which is a list $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$ of vertices and edges, with $v_0 = v_k$ and with v_{i-1}, v_i be the end vertices of e_i. There are two conditions that the set of face boundaries should satisfy. First, every edge occurs precisely twice among all the face boundaries, either once in two distinct face boundaries, or twice in one. Second, for each vertex v, the edges incident with v can be enumerated $e_0, e_1, e_2, \ldots, e_{d-1}$ in such a way that for each $i \in \{0, 1, \dots d-1\}$, e_i, e_{i+1} (summation in the index modulo d) are two consecutive edges of a face boundary. For an embedded graph G, we can construct a topological space denoted by $|G|$ as follows. If R is a face with boundary walk $v_0, e_1, v_1, e_2, \dots, e_k, v_k$, then R is associated with a regular k-gon $\pi(R) \subseteq \mathbb{R}^2$, the vertices and edges of $\pi(R)$ correspond to those in the boundary of R. By the requirements on face boundaries, each edge of G occurs twice in the polygons $\pi(R)$ ($R \in F(G)$). The space $|G|$ is obtained from the disjoint union of these polygons by identifying both copies of every edge e. By the requirements on the face boundaries, the space $|G|$ is a *surface*, i.e., a compact 2-manifold without boundary. The *surface dual* graph G^* of G is a graph embedded in the same surface, whose vertices are the faces of G and for every $e \in E(G)$ there corresponds an edge $e^* \in E(G^*)$ connecting the two (possibly identical) faces $R, R' \in V(G^*)$ whose boundaries contain e.

The boundary of a face is a closed walk (called a *facial walk*) of G, without specific reference to the direction or the origin of the walk. By *an orientation of the faces*, we mean for each facial walk, choose one of the two directions of traversal as the positive direction of that facial walk. If the faces can be oriented in such a way that each edge e is traversed in opposite directions by the two (not necessarily distinct) facial walks incident to e, then the surface is an *orientable surface*. Otherwise the surface is *non-orientable*. The surface classification theorem states that every orientable surface is homeomorphic to S_i for some $i \geq 0$, and every non-orientable surface is homeomorphic to N_j for some $j \geq 1$, where S_i is obtained from the sphere by adding i handles, and N_j is obtained from the sphere by adding j crosscaps.

If G is embedded in an orientable surface, then an orientation of G can be transferred into an orientation of G^* as follow: Choose an orientation of the faces so that each edge e is traversed in opposite directions by the two facial walks incident to e. An edge e^* of G^* incident to R (R a vertex of G^*, but a face of G) is oriented away from R (respectively, towards R) if the direction of the corresponding edge e of G agrees (respectively, disagree) with the direction of R. For this orientation of G^*, if f is an r-tension of G, then $f'(e^*) = f(e)$ is an r-flow on G^* (where e is the arc of G corresponding to the arc e^*). Thus for a graph G embedded in an arbitrary orientable surface Σ, we still have $\Phi_c(G^*) \leq \chi_c(G)$. However, the equality $\chi_c(G) = \Phi_c(G^*)$ does not hold in general. Indeed, $\chi_c(G)$ can be arbitrarily large but $\Phi_c(G^*)$ is always bounded by 6. However, if the graph G is 'nearly' planar, then a 'relaxed duality' still exists. If a simple closed curve separates the surface Σ into two parts and one part is homeomorphic to a disc, then the curve is *contractible*. Otherwise the curve is *non-contractible*. We say a cycle of G is contractible (respectively, non-contractible) if the corresponding closed curve in Σ is contractible (respectively, non-contractible). The *edge-width* of an embedded graph G is the length of a shortest non-contractible cycle. The following result is proved in [27].

Theorem 15.1 *Suppose Σ is an orientable surface. For any $\varepsilon > 0$, there is an integer M such that the following holds: If G is a graph embedded in Σ with edge-width at least M and G^* is the surface dual of G, then*

$$\Phi_c(G^*) \leq \chi_c(G) \leq \Phi_c(G^*) + \varepsilon.$$

Note that if the surface is the sphere, then there is no non-contractible cycle. Hence $\Phi_c(G^*) \leq \chi_c(G) \leq \Phi_c(G^*) + \varepsilon$ holds for every $\varepsilon > 0$ and hence $\chi_c(G) = \Phi_c(G^*)$. When M is large, the condition that every non-contractible cycle has length at least M means that the graph is locally planar: to detect that the surface is not a sphere, one has to travel a long distance in the graph. So intuitively, Theorem 15.1 says that if a graph G embedded in an orientable surface is locally planar, then $\chi_c(G)$ and $\Phi_c(G^*)$ are close.

In Theorem 15.1, it is important that the surface is orientable. For example, the graph in Figure 7 can be viewed as a 5×6 grid on the Kleine bottle. The grid can be extended to a $(2k+1) \times n$ grid on the Kleine bottle, which then has edge-width $\min\{2k+1, n\}$. However, such a graph still has circular chromatic number 4, while its surface dual is an Eulerian graph and hence has circular flow number 2.

For a graph G embedded in a non-orientable surface, instead of considering flows in its surface dual G^*, it is more suitable to consider *biflows* in G^*. If G is embedded in an non-orientable surface, then no matter how the faces are oriented, there are some edges that are traversed in the same direction by both facial walks incident to it. Nevertheless, we still choose a face orientation, and then transfer an arbitrary orientation of G to an 'orientation' of G^*: An edge e^* of G^* incident to R is oriented away from R (respectively, towards R) if the orientation of corresponding edge e of G agrees with the direction of R (respectively, disagrees with the direction of R). Since the orientation of an edge e may agree (or disagree) with both facial walks incident to e, the 'orientation' defined above does not give us a directed graph. Instead, what we obtain is a *bidirected graph*, in which each edge constitutes of two half-edges (one for each end-vertex) and it may happen that the two half-edges of an edge are both directed away from (or into) its end-vertex. A *biflow* f of the bidirected graph G assigns to each edge e of G a real number $f(e)$ such that for each vertex v, $\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$, where $E^+(v)$ (respectively, $E^-(v)$) denotes the edges directed away from (respectively, towards) v. So a biflow is the same as a flow, except that for an edge $e = xy$, we may have $e \in E^+(x)$ and $e \in E^+(y)$ (or $e \in E^-(x)$ and $e \in E^-(y)$). Bidirected graphs and biflows are introduced in [15]. We define an *r-biflow* of a bidirected graph H to be a biflow f of H such that for each edge e, $1 \leq |f(e)| \leq r-1$, and define the *circular biflow number* $\Phi_c^B(H)$ of H to be the least r for which there is an r-biflow. It is conjectured that "*every bidirected graph which has a nowhere-zero biflow has a nowhere-zero* 6*-biflow*". This is equivalent to say that every bidirected graph which has a nowhere zero biflow has circular biflow number at most 6. It is proved by DeVos [26] that this statement is true if 6 is replaced by 12 (and in [15], the statement is proved to be true if 6 is replaced by 216). Observe that the circular biflow number is defined for bidirected graphs only. However, if G is an embedded graph, G^* is its surface dual, and the bidirected orientation on G^* is induced by an orientation of G, then different orientations of G and different choice of face orientations of G^* do not affect the circular biflow number. Indeed, the circular biflow number of G^* can be defined through the graph G. Suppose G is an oriented graph and $\phi : E(G) \to \mathbb{R}$ assigns to each edge of G a real number. A closed walk W of G is *weight balanced* if the sum of ϕ on the forward edges of W is equal to the sum of ϕ on the backward edges of W. Then a tension ϕ of G is a map $\phi : E(G) \to \mathbb{R}$ such that every closed walk is weight balanced. For an embedded graph G, a map $\phi : E(G) \to \mathbb{R}$ is a *local tension* if every facial closed walk is weight balanced. An *r-local tension* of G is a local tension ϕ of G such that for each arc e, $1 \leq |\phi(e)| \leq r-1$. The *local circular chromatic number* $\chi_{loc}(G)$ is defined as

$$\chi_{loc}(G) = \inf\{r : G \text{ admits a local } r\text{-tension}\}.$$

Then for any embedded graph G (either in orientable or non-orientable surface), $\chi_{loc}(G) = \Phi_c^B(G^*)$, where G^* is any bidirected graph obtained from the surface dual G^* of G induced by an orientation of G. So $\chi_{loc}(G)$ unifies both the circular flow number $\Phi_c(G^*)$ in the orientable case and the circular biflow number $\Phi_c^B(G^*)$ in the nonorientable case. With this notion, Theorem 15.1 holds for nonorientable surface as well [27].

Theorem 15.2 *For every surface Σ and every $\varepsilon > 0$, there exists an integer M so that every loopless Σ-embedded graph G with edge-width at least M satisfies*

$$\chi_{loc}(G) \leq \chi_c(G) \leq \chi_{loc}(G) + \varepsilon.$$

For an graph G embedded in an arbitrary surface, $\chi_{loc}(G)$ could be strictly smaller than $\chi_c(G)$. However, for sphere and projective plane, the two parameters coincide.

Theorem 15.3 *[27] For any graph G embedded on the sphere or projective plane we have $\chi_{loc}(G) = \chi_c(G)$.*

In 1996, Youngs discovered that every quadrangulation of the projective plane has chromatic number 2 or 4, but never 3. If we only consider the chromatic number of graphs, this is a quite isolated result. However, by considering circular chromatic number of graphs, it is shown in [27] that this bimodal behavior can be observed in two generic classes of embedded graphs.

Theorem 15.4 *Let G be an embedded graph.*

- *If G is even faced (i.e., each facial walk has even length) with maximum face length $2r$, then either $\chi_{loc}(G) = 2$ or $\chi_{loc}(G) \geq 2r/(r-1)$.*
- *If G is a triangulation (i.e., each facial walk has length 3), then either $\chi_{loc}(G) = 3$ or $\chi_{loc}(G) \geq 4$.*

It is known that for any surface Σ, there is an integer M such that every loopless Σ-embedded graph of edge-width at least M is 5-colourable [127]; and every Σ-embedded graph of edge-width at least M and of girth at least 4 is 4-colourable. Combine these results with Theorems 15.3 and 15.4, we have the following corollary.

Corollary 15.1 *For any surface Σ and any $\varepsilon > 0$, there exists an integer M such that, for every Σ-embedded graph G with edge-width at least M,*

- *if G is even-faced, with maximum face length $2r$, then $\chi_c(G) \in [2, 2+\varepsilon] \cup [2r/(r-1), 4]$.*
- *If G is a triangulation, then $\chi_c(G) \in [3, 3+\varepsilon] \cup [4, 5]$.*

Corollary 15.1 strengthens results on the chromatic number of quadrangulations [4, 93], even-faced graphs [60, 88], and Eulerian triangulations [59]. The upper bounds in Corollary 15.1 are sharp in the sense that there are quadrangulations G of the projective plane of arbitrarily large edge-width with $\chi_c(G) = 4$, and there are triangulations G of the projective plane of arbitrarily large edge-width with $\chi_c(G) = 5$. However, for graphs of girth at least 5, the bound can be improved. It follows from a result of Thomassen [129] that if a graph G embedded in a surafce has large edge-width and girth at least 5, then G is 3-colourable. If the surface is the projective plane or the Klein bottle, then the large edge-width condition can be omitted [126, 128].

By Tutte's 5-Flow Conjecture, if a loopless graph G is embedded in an orientable surface with cogirth at least 2, then $\chi_{loc}(G) \leq 5$. By Bouchet's 6-Flow Conjecture, if a loopless graph G is embedded in any surface with cogirth at least 2, then $\chi_{loc}(G) \leq 6$. If the embedded graph G has large edge-width, then it is likely that there are better upper bounds on $\chi_{loc}(G)$. A detailed discussion about such conjectures can be found in [27]. For a given surface Σ, let $\chi_w(\Sigma) = \lim_{w\to\infty} \sup\{\chi_c(G) : G \text{ is embedded in } \Sigma \text{ with edge-width at least } w\}$. It follows from the above mentioned results that if Σ is nonorientable, then $\chi_w(\Sigma) = 5$. It is conjectured by Goddyn [38] that if Σ is an orientable surface, then $\chi_w(\Sigma) = 4$. Equivalently, the conjecture says that for any orientable surface Σ, for any $\varepsilon > 0$ there is an integer M such that every Σ-embedded graph with edge-width at least M has $\chi_c(G) < 4 + \varepsilon$. By using Theorem 15.3, we can see that this conjecture is weaker than a conjecture of Grünbaum [40], which says that if Σ is an orientable surface, then every Σ-embedded graph of edge-width at least 3 has $\chi_{loc}(G) \leq 4$.

16 Circular flow number

The fundamental problems concerning flow number and circular flow number are the possible values of the flow number and the circular flow number of graphs v.s. edge connectivity. If G has an edge cut, then G has no r-flow for any r, and $\Phi(G)$, $\Phi_c(G)$ are not defined (or defined to be ∞). There are three famous conjectures of Tutte [132, 133, 118] relating the flow number of a graph to its edge connectivity.

5-Flow Conjecture: *Every bridgeless graph G has $\Phi(G) \leq 5$.*

4-Flow Conjecture: *Every Petersen minor free bridgeless graph G has $\Phi(G) \leq 4$.*

3-Flow Conjecture: *Every 4-edge connected graph G has $\Phi(G) \leq 3$.*

All the three conjectures remain open. On the other hand, the following is a combination of results in [90, 101, 155].

Theorem 16.1 *For any rational $r \in [2,5]$, there is a graph G with $\Phi_c(G) = r$. For any $r \in [2,4]$, there is a planar graph (and hence a Petersen minor free graph) G with $\Phi_c(G) = r$. For any $r \in [2,3]$, there is a 4-edge connected (planar) graph G with $\Phi_c(G) = r$.*

Another important conjecture, which generalizes both the 5-flow conjecture and 3-flow conjectrue, and which relates connectivity and circular flow number of a graph, is proposed by Jaeger [62]:

$(2+1/k)$-Flow Conjecture: *Every $4k$-edge connected graph G has $\Phi_c(G) \leq 2 + 1/k$.*

The conjecture says that graphs without small edge cut have small circular flow number. It seems that to ensure a graph to have small circular flow number, what really matters is that there are no small odd edge cut. Define the *odd edge connectivity* of a graph G to be the smallest odd number k for which there is an edge cut of cardinality k. Based on this observation, Zhang [146] modified Jaeger's conjecture to the following stronger version:

$(2+1/k)$-Flow Conjecture (strong version): *Every graph G with odd edge connectivity at least $4k+1$ has $\Phi_c(G) \leq 2 + 1/k$.*

The $k = 1$ case of the conjecture is exactly the 3-flow conjecture. The $k = 2$ case implies the 5-Flow Conjecture. To see this, we assume the $(2+1/k)$-Flow Conjecture is true for $k = 2$. To prove that the 5-Flow Conjecture is true, it suffices to show that any 3-edge connected cubic graph G has a nowhere zero 5-flow [145]. Replace each edge of G by three parallel edges, the resulting graph G' is 9-edge connected, and hence has $\Phi_c(G') \leq 5/2$. Let D be an orientation of G and let D' be the orientation of G' obtained from D by replacing each arc of D with three arcs of the same direction. Let f be a $(5,2)$-flow on D'. For each arc a of D, let a_1, a_2, a_3 be the three parallel arcs in D' that replace a. Then $g(a) = [f(a_1) + f(a_2) + f(a_3)]_5$ is a nowhere zero Z_5-flow on D, which implies that $\Phi(G) \leq 5$ [145].

Although the 5-flow, 4-flow and 3-Flow Conjectures are open, their restrictions to planar graphs have been proved. For the $(2+1/k)$-Flow Conjecture, its restriction to planar graphs also remains open. For planar graphs, the circular flow number of G is equal to the circular chromatic number of its geometric dual graph G^*. The odd edge connectivity of G is equal to the odd girth (i.e., the length of a shortest odd cycle) of G^*. So the restriction of the strong version of the $(2+1/k)$-Flow Conjecture to planar graphs is equivalent to the following:

$(2+1/k)$-Flow Conjecture for planar graphs (strong version): *Every planar graph G with odd girth at least $4k+1$ has $\chi_c(G) \leq 2 + 1/k$.*

The circular chromatic number of planar graphs of large girth or large odd girth has been studied in [14, 35, 67, 33, 160]. The currently best known result is the following theorem proved in [14]:

Theorem 16.2 *If G is a planar graph of odd girth at least $\frac{20k-2}{3}$, then $\chi_c(G) \leq 2 + 1/k$.*

The following weaker version of $(2+1/k)$-Flow Conjecture is proposed by Seymour (cf. [146]):

$(2+1/k)$-Flow Conjecture (weak version): *For any $\varepsilon > 0$, there is an integer $n = n(\varepsilon)$ such that every graph G of girth (or odd girth) at least n has $\Phi_c(G) \leq 2 + \varepsilon$.*

This conjecture also remains open. Theorem 16.2 implies that the weak version of $(2+1/k)$-Flow Conjecture holds for planar graphs. Zhang [146] proved that for any given surface S, the weak version of $(2+1/k)$-Flow Conjecture holds for graphs embedded in S.

Theorem 16.3 *Let S be any given surface and ε be a positive real number. There is an integer $n = n_S(\varepsilon)$ such that any graph G with odd edge connectivity at least n has $\Phi_c(G) \leq 2 + \varepsilon$.*

Circular flow number of random graphs is discussed in [120]. It turns out that $(2+1/k)$-Flow Conjecture is true almost surely for random graphs.

Theorem 16.4 *In the random graph process which adds a uniformly chosen edge at each step, almost surely the graph has $\Phi_c(G) \leq 2 + 1/k$ as soon as the minimum degree of the graph is at least $2k$.*

Let $G(n,p)$ denote a random graph with n vertices in which each pair of vertices is joined by an edge with probability p.

Theorem 16.5 *Let $k \geq 1$ be an integer and let $\omega(n)$ be any function tending to infinity with n. Then*

- *if $p = (\ln n + (2k-1)\ln\ln n + \omega(n))/n$, then almost surely $\Phi_c(G(n,p)) \leq 2 + 1/k$;*
- *if $p = (\ln n + (2k-1)\ln\ln n - \omega(n))/n$, then almost surely $\Phi_c(G(n,p)) > 2 + 1/k$.*

Random regular graphs are also considered in [120]. When $d = 3$, it is known that an element of $G_{n,d}$ almost surely contains an odd cycle, and therefore has no nowhere zero 3-flow. It is shown in [120] that for odd $d \geq 11$, $G_{n,d}$ almost surely has a nowhere zero 3-flow. The corresponding result for $d = 5$ is posed as an open problem, for which Tutte's conjecture implies a positive answer.

It is also a long standing open question as whether there is an integer n such that every graph G of (odd) edge connectivity at least n has $\Phi(G) \leq 3$. It is known [145] 4-edge connected graphs has $\Phi(G) \leq 4$. For 6-edge connected graphs, there is a slightly better result [34].

Theorem 16.6 *If G is 6-edge connected, then $\Phi_c(G) < 4$.*

Possible values of the circular flow number of regular graphs are studied by Steffen [116]. The following result is proved:

Theorem 16.7 *If G is a $(2k+1)$-regular graphs, then either $\Phi_c(G) \geq 2 + \frac{2}{2k-1}$ or G is bipartite and in which case $\Phi_c(G) = 2 + \frac{1}{k}$.*

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