Circular chromatic number and Mycielski construction

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Abstract

This paper gives a sufficient condition for a graph G to have its circular chromatic number equal its chromatic number. By using this result, we prove that for any integer $t \geq 1$, there exists an integer n such that for all $k \geq n$ $\chi_c(M^t(K_k)) = \chi(M^t(K_k))$.

Keywords: Circular chromatic number, Mycielski's graphs, chromatic number.

1 Introduction

All graphs considered in this paper are finite and simple. Suppose $G = (V, E)$ is a graph and $k \geq 2d$ are positive integers. A (k, d)-colouring of G is a mapping

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$c : V \to \{0, 1, \cdots, k-1\}$ such that for any edge xy of G, $d \le |c(x) - c(y)| \le k - d$. The *circular chromatic number* $\chi_c(G)$ of G is defined by

$$\chi_c(G) = \inf\{k/d : \text{ there exists a } (k,d)\text{-colouring of } G\}.$$

The circular chromatic number (also known as the star chromatic number) is a natural generalization of the chromatic number (note that a $(k, 1)$-colouring is simply a k-colouring), and has been studied extensively in the past decade, [1, 2, 4, 6, 11, 12, 13, 14, 15, 16]. It is known [12] that $\chi(G) - 1 < \chi_c(G) \le \chi(G)$ for any graph G, and there are graphs G with $\chi_c(G) = \chi(G)$ as well as graphs with $\chi_c(G)$ arbitrarily close to $\chi(G) - 1$. The question of which graphs G have $\chi_c(G) = \chi(G)$ has attracted some attention [1, 4, 6, 11, 12, 13, 14, 15]. It is NP-hard to determine if a given graph G has $\chi_c(G) = \chi(G)$ [6]. However, some sufficient conditions for graphs to have $\chi_c(G) = \chi(G)$ can be found in the literature [1, 4, 6, 11, 15, 13].

This paper gives another sufficient condition for graphs to have $\chi_c(G) = \chi(G)$. This condition is then applied to the study of the circular chromatic number of Mycielski's graphs, especially, the circular chromatic number of the iterated Mycielskian of complete graphs.

For a graph G with vertex set $V(G) = V$ and edge set $E(G) = E$, the *Mycielskian* $M(G)$ of G is the graph with vertex set $V \cup V' \cup \{u\}$, where $V' = \{x' : x \in V\}$, and edge set $E \cup \{x'y : xy \in E\} \cup \{x'u : x' \in V'\}$. The vertex x' is called the *twin* of the vertex x (and x is also called the twin of x'); and the vertex u is called the *root* of $M(G)$. If there is no ambiguity we shall always use u as the root of $M(G)$. For $t \ge 2$, let $M^t(G) = M(M^{t-1}(G))$.

There is a very simple formula for $\chi(M(G))$ in terms of $\chi(G)$, i.e., $\chi(M(G)) = \chi(G) + 1$ [10], as well as a very simple formula for $\chi_f(M(G))$ in terms of $\chi_f(G)$, i.e., $\chi_f(M(G)) = \chi_f(G) + 1/\chi_f(G)$ [8]. ($\chi_f(G)$ denotes the fractional chromatic number of G). However, there is no simple formula for $\chi_c(M(G))$ in terms of $\chi_c(G)$.

The problem of calculating the circular chromatic number of Mycielski's graphs has been investigated in [4, 3, 7]. It turns out that the circular chromatic number of $M(G)$ is not determined by the circular chromatic number of G alone. Rather, it depends on the structure of G. Even for graphs G with very simple structure, it is still difficult to determine $\chi_c(M(G))$.

The problem of determining the circular chromatic number of the iterated Mycielskian of complete graphs was discussed in [3]. It was conjectured in [3] that if $n \geq t+2 \geq 3$, then $\chi_c(M^t(K_n)) = \chi(M^t(K_n)) = n+t$. With complicated arguments, the special cases $t = 1, 2$ of this conjecture have been proved in [3, 4]. A simpler proof of these special cases was given in [4] (for $t = 2$, the result proved in [4] is slightly weaker, i.e., it was proved in [4] that for $n \geq 5$, $\chi_c(M^2(K_n)) = \chi(M^2(K_n)) = n + 2$).

We shall prove in this paper that for any integer $t \geq 1$, if $n \geq 2^t + 2$ then $\chi_c(M^t(K_n)) = \chi(M^t(K_n)) = n + t$.

2 A sufficient condition for $\chi_c(G) = \chi(G)$

For $k \geq 2d$ and $\gcd(k, d) = 1$, let G^k_d be the graph with vertex set $\{0, 1, \cdots, k-1\}$ and in which $j \sim j'$ if and only if $d \leq |j - j'| \leq k - d$. A *homomorphism* from a graph $G = (V, E)$ to a graph $G' = (V', E')$ is a mapping $f : V \to V'$ such that $f(x)f(y) \in E'$ whenever $xy \in E$. It is well known [16] and easy to see that if G admits a homomorphism to G' then $\chi_c(G) \leq \chi_c(G')$. It follows from the definition that a (k, d)-colouring of a graph G is simply a homomorphism from G to G^k_d. When considering a (k, d)-colouring of G (or equivalently, a homomorphism from G to G^k_d), we view the colours $0, 1, \cdots, k-1$ as cyclically ordered. For $a, b \in \{0, 1, \cdots, k-1\}$, the interval $[a, b]_k$ means the interval from a to b in this cyclic order. For example, $[2, 5]_k = \{2, 3, 4, 5\}$ and $[5, 2]_k = \{5, 6, \cdots, k-1, 0, 1, 2\}$. The following lemma is an easy consequence of a result of [13].

Lemma 1 *Suppose $G = (V, E)$ is a graph with $\chi_c(G) = k/d$ and that $c : V \to \{0, 1, \cdots, k-1\}$ is a (k, d)-colouring of G. Then for each $i \in \{0, 1, \cdots, k-1\}$, there exists a vertex x with $c(x) = i$ which is adjacent to a vertex y with $c(y) = i + d$. Here the addition is modulo k.*

Proof. Assume to the contrary that there exists an index i such that no vertex of colour i is adjacent to a vertex of colour $i + d$. Let $e = i(i + d)$. Then the colouring c is a homomorphism from G to $G^k_d - e$. However, it was proved in [13] that $\chi_c(G^k_d - e) < k/d$, which implies that $\chi_c(G) \leq \chi_c(G^k_d - e) < k/d$, contrary to our assumption. ∎

Suppose $G = (V, E)$ is a graph. For a vertex x of G, we denote by $\overline{N}_G(x)$ the set of non-neighbours of x, i.e., $\overline{N}_G(x) = \{y : y \not\sim x, \quad y \neq x\}$. For $X \subseteq V$, let $\overline{N}_G(X) = \cup_{x \in X} \overline{N}_G(x)$. With an abuse of notation, for any subset X of V, we shall also use X to denote the subgraph of G induced by X. We say $Y \subseteq \overline{N}_G(X)$ is a *pointwise-dominating set of* $\overline{N}_G(X)$ if for each $x \in X$, $\overline{N}_G(x) - Y$ is an independent set. We define a parameter $\beta_G(X)$ as follows:

$$\beta_G(X) = \min\{|Y| : \ Y \text{ is a pointwise-dominating set of } \overline{N}_G(X)\}.$$

Theorem 2 *Suppose $G = (V, E)$ is a graph and X is a clique of G. If $\chi_c(G) = k/d$ (with $gcd(k, d) = 1$), then*

$$|X|(d-1) \leq 2\beta_G(X).$$

Proof. Let $c : V \rightarrow \{0, 1, \cdots, k-1\}$ be a (k, d)-colouring of G. It is well known [2, 12] (and also follows from Lemma 1) that c must use every colour. Assume that $X = \{x_1, x_2, \cdots, x_m\}$ and $c(x_i) < c(x_{i+1})$.

For $i = 0, 1, \cdots, k-1$, let $A_i = c^{-1}(i)$, i.e., A_i is the set of vertices coloured by colour i. In the following, all the additions and subtractions of the indices are carried out modulo k.

Let Y be a pointwise-dominating set of $\overline{N}_G(X)$ with $|Y| = \beta_G(X)$.

Construct a bipartite graph H with X, Y as its two parts so that $x_i \sim y$ if and only if

$$c(y) \in [c(x_i) - d + 1, c(x_i) + d - 1]_k - \{c(x_i)\}.$$

We shall show that each vertex of X has degree at least $d-1$ in H and each vertex of Y has degree at most 2 in H.

Suppose $x_i \in X$ and $c(x_i) = a$. For each $\ell \in \{1, 2, \cdots, d-1\}$, $A_{a-d+\ell} \cup A_{a+\ell} \subseteq \overline{N}_G(x_i)$. By Lemma 1, there exists an edge joining a vertex of $A_{a-d+\ell}$ to a vertex of $A_{a+\ell}$. Since $\overline{N}_G(x_i) - Y$ is an independent set, we conclude that $A_{a-d+\ell} \cup A_{a+\ell}$ contains at least one vertex of Y. It follows that x_i has degree at least $d-1$ in H.

Suppose $y \in Y$ and $c(y) \in [c(x_i), c(x_{i+1})]_k$. Since X is a clique, we know that for any $x_j \in X$ where $j \neq i, i+1$,

$$[c(x_j) - d + 1, c(x_j) + d - 1]_k \cap [c(x_i), c(x_{i+1})]_k = \emptyset.$$

Therefore $y \not\sim x_j$ in H. So y has degree at most 2 in H. Hence

$$|X|(d-1) \leq |E(H)| \leq 2|Y|,$$

and the proof is complete. ∎

Corollary 3 *If G has a clique X such that $|X| > 2\beta_G(X)$, then $\chi_c(G) = \chi(G)$.*

Proof. Otherwise $\chi_c(G) = k/d$ for some k, d with $\gcd(k,d) = 1$ and $d \geq 2$. By Theorem 2, we have $|X| \leq |X|(d-1) \leq 2\beta_G(X)$, contrary to our assumption. ∎

Corollary 3 can be generalized to the following:

Theorem 4 *Suppose $G = (V, E)$ is a graph. If there exists a subset $X \subseteq V$ such that $\chi(X) > 2\beta_G(X)$, then $\chi_c(G) = \chi(G)$.*

Proof. Assume to the contrary that $\chi_c(G) = k/d$ for some $d \geq 2$ (and $(k,d) = 1$). Let c be a (k,d)-colouring of G. We choose a sequence $\{x_1, x_2, \cdots, x_s\}$ as follows: Let $x_1 \in X$ be a vertex for which $c(x_1) = \min\{c(x) : x \in X\}$. Suppose $x_i \in X$ has been chosen. If there is a vertex $x \in X$ with $c(x) \geq c(x_i) + d$, then let $x_{i+1} \in X$ be a vertex for which $c(x_{i+1}) = \min\{c(x) : x \in X, c(x) \geq c(x_i) + d\}$. Otherwise, let $i = s$ and the construction of the sequence is completed. Let $f : X \to \{1, 2, \cdots, s\}$ be defined as $f(x) = i$ if $c(x_i) \leq c(x) < c(x_i) + d$. Then f is a proper colouring of X. Therefore $s \geq \chi(X)$.

The remaining part is similar to the proof of Theorem 2, with $x_1, x_2, \cdots, x_s$ playing the roles of the vertices of the clique. In case $d = 2$, the argument is exactly the same. In case $d \geq 3$, we need to be careful, because in the bipartite graph H constructed in the proof of Theorem 2, those vertices of Y whose colours lie in the interval $[c(x_{s-1}), c(x_s)]_k$ and $[c(x_1, c(x_2)]_k$ may have degree greater than 2. So we do not have $\chi(X)(d-1) \leq 2\beta_G(X)$, but it is easy to show that $\chi(X) \leq 2\beta_G(X)$, contrary to our assumption. The argument is similar, and we omit the details. ∎

3 Mycielski's graphs

Suppose $G = (V, E)$ is a graph and $M(G) = (V \cup V' \cup \{u\}, E')$ is the Mycielskian of G. If X is a subset of V, then we also consider X as a subset of $V(M(G))$ (because V is a subset of $V(M(G))$). Recall that for $x \in V$, the twin of x is denoted by x'. For a subset X of V, we shall denote by X' the set of twins of vertices in X, i.e., $X' = \{x' : x \in X\}$.

Lemma 5 *Let $M(G)$ be the Mycielskian of G. Then for any subset X of $V(G)$, $\beta_{M(G)}(X) \leq 2\beta_G(X) + 1$.*

Proof. It is easy to see that for any vertex x, if $S = \overline{N}_G(x)$, then $\overline{N}_{M(G)}(x) = S \cup S' \cup \{x', u\}$. Let $Z = \overline{N}_G(X)$. Then

$$\overline{N}_{M(G)}(X) = Z \cup Z' \cup X' \cup \{u\}.$$

Let Y be a pointwise-dominating set of $\overline{N}_G(X)$ with $|Y| = \beta(X)$. To complete the proof of Lemma 5, it suffices to show that $Y \cup Y' \cup \{u\}$ is a pointwise-dominating set of $\overline{N}_{M(G)}(X)$.

Let $x \in X$ and $a, b \in \overline{N}_{M(G)}(x) - (Y \cup Y' \cup \{u\})$. We need to prove that $a \not\sim b$. Note that $\overline{N}_{M(G)}(x) \subseteq Z \cup Z' \cup \{x', u\}$. Assume to the contrary that $a \sim b$. Since $Z' \cup \{x'\}$ is an independent set, we have $\{a, b\} \not\subseteq Z' \cup \{x'\}$. Since Y is a pointwise-dominating set of $\overline{N}_G(X)$, it follows that $\{a, b\} \not\subseteq Z$. Thus we may assume that $a \in Z - Y$ and $b \in (Z' \cup \{x'\}) - Y'$. But x' is not adjacent to any vertex of $\overline{N}_G(x)$. Therefore $b \neq x'$. Assume that $b = c'$. Then $c \in Z - Y$. Now $a \sim c'$ implies that $a \sim c$, contrary to the assumption that Y a pointwise-dominating set of $\overline{N}_G(X)$. ∎

Corollary 6 *Suppose G is a graph and X is a subset of $V(G)$. Then for $t \geq 1$,*

$$\beta_{M^t(G)}(X) \leq 2^t \beta_G(X) + 2^t - 1.$$

Proof. By Lemma 5, $\beta_{M^t(G)}(X) \leq 2(\beta_{M^{t-1}(G)}(X)) + 1$. The conclusion follows by induction. ∎

Corollary 7 *Suppose G is a graph, $t \geq 1$ and $\chi_c(M^t(G)) = k/d$ (where gcd$(k, d) = 1$). If X is a clique of G then*

$$|X|(d-1) \leq 2^{t+1}\beta_G(X) + 2^{t+1} - 2.$$

Corollary 8 *If G has a clique X such that $|X| \geq 2^{t+1}\beta_G(X) + 2^{t+1} - 1$, then $\chi_c(M^t(G)) = \chi(M^t(G))$.*

In Corollary 8, the clique X can be replaced by any subgraph X of G, with $|X|$ be replaced by $\chi(X)$ (see Theorem 4).

Corollary 9 *Let G be a graph on n vertices and X the set of vertices of degree $n-1$. If $|X| \geq 2^{t+1} - 1$, then $\chi_c(M^t(G)) = \chi(M^t(G))$.*

Proof. Since each vertex of X has degree $n-1$, it follows that $\overline{N}_G(X) = \emptyset$. So X is a clique with $\beta_G(X) = 0$. The conclusion follows from Corollary 8. ∎

The special case $t = 1$ of Corollary 9 was proved in [4].

Corollary 10 *If $n \geq 2^{t+1} - 1$, then $\chi_c(M^t(K_n)) = \chi(M^t(K_n))$.*

4 Some improvements

For the circular chromatic number of the iterated Mycielskian of complete graphs, the following was conjectured in [3]:

Conjecture 1 *[3] If $n \geq t + 2$, then $\chi_c(M^t(K_n)) = \chi(M^t(K_n))$.*

For any integer $t \geq 1$, let $n(t)$ be the minimum integer such that for any $n \geq n(t)$, $\chi_c(M^t(K_n)) = \chi(M^t(K_n))$. Corollary 10 shows that for any $t \geq 1$, the integer $n(t)$ exists and $n(t) \leq 2^{t+1} - 1$. It was shown in [3] that $n(t) \geq t + 2$. Therefore we have

$$t + 2 \leq n(t) \leq 2^{t+1} - 1.$$

Conjecture 1 above is equivalent to saying that $n(t) = t + 2$.

For $t = 1$, the upper and lower bound for $n(t)$ coincide and we have $n(1) = 3$. For $t = 2$, Conjecture 1 was proved in [3]. Hence $n(2) = 4$. For $t \geq 3$, there is a big gap between the upper and lower bounds for $n(t)$. In this section, we shall slightly improve the upper bound for $n(t)$.

In Theorem 2, the cardinality of a minimum pointwise-dominating set Y of $\overline{N}_G(X)$ is used to bound the cardinality of X. A careful analysis of the proof of Theorem 2 shows that what really matters is the number of colour classes of a (k,d)-colouring of G that contain elements of Y. By finding a (k,d)-colouring that colours some pairs of elements of Y with the same colours, we prove the following result, which is a strengthening of Theorem 2.

Theorem 11 *Suppose G is a graph and X is a clique of G. If $\chi_c(M(G)) = k/d$ (where $gcd(k,d) = 1$), then*

$$(|X| - 3)(d - 1) \leq 2\beta_G(X).$$

Proof. Let Y be a pointwise-dominating set of $\overline{N}_G(X)$ with $|Y| = \beta_G(X)$. By the proof of Lemma 5, we know that $D = Y \cup Y' \cup \{u\}$ is a pointwise-dominating set of $\overline{N}_{M(G)}(X)$.

The remainder of the proof is similar to that of Theorem 2. However, instead of the cardinality of D, we count the number of colour classes that contain an element of D in a (k,d)-colouring of $M(G)$.

Let f be a (k,d)-colouring of $M(G)$. We construct a new (k,d)-colouring f' of $M(G)$ as follows:

$f'(x) = f(x)$ if $x \notin Y'$ or $x = v' \in Y'$ and $f(v) \in [f(u) - d + 1, f(u) + d - 1]_k$;

$f'(v') = f(v)$ if $v' \in Y'$ and $f(v) \notin [f(u) - d + 1, f(u) + d - 1]_k$.

It is easy to verify that f' is still a (k,d)-colouring of $M(G)$. By the definition, there are at most $\beta_G(X)$ colour classes $(f')^{-1}(i)$ such that $i \notin [f(u)-d+1, f(u)+d-1]_k$ and $(f')^{-1}(i)$ contains an element of D.

Let

$$T = \{i : 0 \leq i \leq k - 1, (f')^{-1}(i) \cap D \neq \emptyset\},$$

and let

$$T' = T - [f(u) - d + 1, f(u) + d - 1]_k.$$

The argument above shows that

$$|T'| \leq \beta_G(X).$$

Now we construct a bipartite graph H with vertex $X \cup T$ and in which $x \in X$ is adjacent to $i \in T$ if $i \in [f'(x) - d + 1, f'(x) + d - 1]_k$.

The same argument as in the proof of Theorem 2 shows that each vertex of X has degree at least $d - 1$, and each vertex in T has degree at most 2.

Since X is a clique, the colours of elements of X are far apart from each other (i.e., $f'(x) \notin [f'(y) - d + 1, f'(y) + d - 1]_k$ if $x, y \in X$ and $x \neq y$). Let

$$X' = \{x \in X : N_H(x) \cap [f'(u) - d + 1, f'(u) + d - 1]_k \neq \emptyset.$$

Easy calculations show that $|X'| \leq 4$.

If $|X'| \leq 3$, then we have

$$(|X| - 3)(d - 1) \leq 2\beta_G(X).$$

If $|X'| = 4$, then by calculating the distance between the colours of elements of X', one can show that there are two vertices $a, b \in X'$ such that

$$|([f'(a)-d+1, f'(a)+d-1]_k \cup [f'(b)-d+1, f'(b)+d-1]_k) \cap [f'(u)-d+1, f'(u)+d-1]_k| \leq d-2.$$

This would imply that there are at least d edges of H joining a and b to vertices of T'.

Therefore

$$2\beta_G(X) - d \geq (|X| - 4)(d - 1),$$

which is equivalent to

$$(|X| - 3)(d - 1) \leq 2\beta_G(X) - 1.$$

This completes the proof. ∎

Corollary 12 *Suppose G is a graph, $t \geq 1$ and $\chi_c(M^t(G)) = k/d$ (where $gcd(k,d) = 1$). If X is a clique of G then*

$$(|X| - 3)(d - 1) \leq 2^t \beta_G(X) + 2^t - 2.$$

Proof. Note that $M^t(G) = M(M^{t-1}(G))$ and X is a clique of $M^{t-1}(G)$. By Corollary 6, $\beta_{M^{t-1}(G)}(X) \leq 2^{t-1}\beta_G(X) + 2^{t-1} - 1$. The result follows from Theorem 11. ∎

For $t \geq 2$, the following are improvements of Corollaries 9 and 10, respectively.

Corollary 13 *Let G be a graph on n vertices and X the set of vertices of degree $n-1$. If $|X| \geq 2^t + 2$, then $\chi_c(M^t(G)) = \chi(M^t(G))$.*

Corollary 14 *If $n \geq 2^t + 2$, then $\chi_c(M^t(K_n)) = \chi(M^t(K_n))$.*

For $t = 3$, Corollary 14 asserts that $\chi_c(M^3(K_n)) = \chi(M^3(K_n))$ for $n \geq 10$. Conjecture 1 asserts that $\chi_c(M^3(K_n)) = \chi(M^3(K_n))$ for $n \geq 5$. By pushing further the technique used in the proof of Theorem 11 (with a more complicated argument), one can show that $\chi_c(M^3(K_n)) = \chi(M^3(K_n))$ for $n \geq 8$. It remains unknown if $\chi_c(M^3(K_n)) = \chi(M^3(K_n))$ for $n = 5, 6, 7$.

It was proved in [4] as well as in Section 2 (Corollary 9) that if an n-vertex graph G has 3 vertices of degree $n-1$ then $\chi_c(M(G)) = \chi(M(G))$. The following is an improvement of this.

Theorem 15 *Let $G = (V, E)$ be a graph on $n \geq 3$ vertices. If G has 2 vertices of degree $n-1$, then $\chi_c(M(G)) = \chi(M(G))$.*

Proof. Let a, b be two vertices of G of degree $n-1$. Let a', b' be the twins of a, b, respectively in $M(G)$, and let u be the root of $M(G)$. Assume to the contrary that $\chi_c(M(G)) = k/d$ for some $d \geq 2$ (and $\gcd(k,d) = 1$). Let f be a (k,d)-colouring of $M(G)$. Assume $f(a) = i$. Then none of $f^{-1}(i-d+1), \cdots, f^{-1}(i+d-1)$ is empty and each contains only vertices that are not adjacent to a. However, the only vertices of $M(G)$ not adjacent to a are a' and u. So we must have $d = 2$ and, say, $f^{-1}(i-1) = \{a'\}$, $f^{-1}(i) = \{a\}$ and $f^{-1}(i+1) = \{u\}$.

By symmetry, we have $f^{-1}(i+2) = \{b\}$ and $f^{-1}(i+3) = \{b'\}$. Since $n \geq 3$, G has vertices other than a and b. Now for each vertex $x \in V - \{a, b\}$, since $f(x) \notin \{f(u)-1, f(u), f(u)+1\}$, we can assume that $f(x') = f(x)$ (if $f(x) \neq f(x')$, we can recolour x' with the colour of x to obtain another (k,d)-colouring of $M(G)$). Since every vertex of $V - \{b\}$ is adjacent to b', and since $f(x) = f(x')$ for every $x \in V - \{a, b\}$, we conclude that $f^{-1}(i+4) = \emptyset$. But this is contrary to Lemma 1. ∎

Theorem 15 is sharp in the sense that there are graphs G with one universal vertex for which $\chi_c(M(G)) \neq \chi(M(G))$. Let W_{2n+1} be the odd wheel which is obtained from the odd cycle C_{2n+1} by adding a universal vertex. Then for $n \geq 2$, it is easy to show that $\chi_c(M(W_{2n+1})) \leq 4.5 < \chi(M(W_{2n+1}))$, and the argument as in the proof of Theorem 15 shows that $\chi_c(M(W_{2n+1})) \geq 4.5$. Hence $\chi_c(M(W_{2n+1})) = 4.5$, which was conjectured in [9].

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