Circular chromatic number and graph minor

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Abstract

Let n be an integer. This paper discusses the problem that for which rational number r there exists a graph G which has circular chromatic number r and which does not contain K_n as a minor.

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1 Introduction

The circular chromatic number of a graph was introduced by A. Vince [8] in 1988, as "the star chromatic number". For a pair of integers $p \geq q$, a (p,q)-coloring of a graph G is a mapping c from $V(G)$ to the set $\{0, 1, \cdots, p-1\}$ such that for any adjacent vertices x, y of G, $q \leq |c(x) - c(y)| \leq p - q$. The *circular chromatic number* $\chi_c(G)$ of a graph G is the infimum of the ratios p/q for which there exists a (p,q)-coloring of G.

It was shown by Vince [8] (cf. also [2] for a combinatorial proof) that for finite graphs G, the infimum in the definition above is always attained, and hence can be replaced by minimum.

Note that a $(p,1)$-coloring of a graph is just a p-coloring of G. Therefore we have $\chi_c(G) \leq \chi(G)$. On the other hand, it was proved in [8] that for any graph G we have $\chi(G) - 1 < \chi_c(G)$. Hence if we know the circular chromatic number of a graph G, then we can obtain its chromatic number by taking the ceiling of $\chi_c(G)$, i.e., $\chi(G) = \lceil \chi_c(G) \rceil$. However, two graphs of the same chromatic number may have different circular chromatic numbers. In this sense, $\chi_c(G)$ is a refinement of the parameter $\chi(G)$, and $\chi(G)$ is an approximation of $\chi_c(G)$.

Since the infimum in the definition of $\chi_c(G)$ can be replaced by minimum for finite graphs, we know that $\chi_c(G)$ is a rational for any finite graph G. On the other hand, for any rational $p/q \geq 2$, there is a finite graph G such that $\chi_c(G) = p/q$. Suppose $p/q \geq 2$ and that $(p,q) = 1$. Let G_p^q be the graph with vertex set $\{0, 1, \cdots, p-1\}$, in which ij is an edge if and only if $q \leq |i - j| \leq p - q$. Then it was shown in [8] that $\chi_c(G_p^q) = p/q$.

Given a property P of graphs, it is usually an interesting and difficult problem to determine whether or not there exists a graph G which has the property P and whose circular chromatic number is equal to a given rational number r. One such problem was discussed in [10]. For an integer g, let $P(g)$ be the property of having girth at least g. It was shown in [10] that for any integer g and for any rational number $r \geq 2$, there exists a graph G which has property $P(g)$ and which has circular chromatic number r. This result is a generalization of the result of Erdös concerning the existence of graphs with arbitrarily large girth and arbitrarily large chromatic number. Another such problem was discussed in [5, 13]. Let P be the property of being a planar graph. In [5, 13], the authors asked the problem that for which rational number r there is a planar graph G which has circular chromatic number r. It follows from the Four Color Theorem, that the number r is at most 4. It was shown in [5] that for any rational r between 2 and 3, there is a planar graph G with circular chromatic number r, and it was shown in [13] that for any rational number r between 3 and 4, there is a planar graph G with circular chromatic number r. Therefore a rational r is the chromatic number of a planar graph if and only if $r = 1$ or $2 \leq r \leq 4$.

A graph H is called a *minor* of a graph G if H is isomorpic to a graph which is obtained from a subgraph of G by contracting some edges. We say a graph G is *H-minor free* if H is not a minor of G. As a generalization of the Four Color Problem, Hadwiger conjectured that any graph G with chromatic number at least n contains K_n as a minor. Hadwiger's conjecture remains to be one of the major open problems in mathematics. The $n = 5$ case of this conjecture is equivalent to the Four Color Theorem, of which the only existing proofs rely on computer [1, 6]. The $n = 6$ case was settled in [7], where the proof relies on the Four Color Theorem and is quite complicated.

In this paper, we consider the problem that for which rational number r there is a graph G which does not contain K_n as a minor and which has circular chromatic number r. If Hadwiger conjecture is true, then for any rational number $r > n - 1$, any graph with circular chromatic number r does contain K_n as a minor. Therefore we shall only consider those rational $r \leq n - 1$. Our main result is the following theorem:

Theorem 1.1 *Suppose $n \geq 4$ is an integer and that r is a rational. If $2 \leq r \leq n - 2$, then there is a K_n-minor free graph which has circular chromatic number r.*

In case that r is between $n - 2$ and $n - 1$, it remains an open question whether or not there exists a K_n-minor free graph with circular chromatic number r. However, we have some partial results. First of all, it was shown in [13] that for any rational number between 3 and 4, there is a planar graph with circular chromatic number r. As planar graphs are K_5-minor free, we have the following result:

Theorem 1.2 *For any rational number r between 3 and 4, there exists a K_5-minor free graph G with circular chromatic number r.*

If $n = 4$, then K_4-minor free graphs have a relative simple structure. Recently, P. Hell and the author [4] proved a somehow surprising result that for a K_4-minor graph G, either $\chi_c(G) = 3$ or $\chi_c(G) \leq 8/3$.

For $n \geq 6$, we shall prove the following theorem in this paper:

Theorem 1.3 *Suppose $n \geq 6$ and $n - 2 \leq r \leq n - 1$. If the Farey sequence of r has length at most 2 and that $\alpha_2 = 2$ (see Section 2 for the definitions of Farey sequence and α_i), then there is a graph G which is K_n-minor free and which has circular chromatic number r.*

2 The proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. If $n = 4$, then the result is trivial. If $n = 5$ or 6, then the result follows from theorems of [5, 13], where it was proved that any rational number between 2 and 4 is the circular chromatic number of a planar graph, and planar graphs are known to be K_5-minor free. Thus we assume that $n \geq 7$ is a fixed integer, $r = p/q$ is a fixed rational number (where p and q are integers with $(p, q) = 1$), and that $4 \leq r \leq n - 2$. (The case that $m \leq 4$ follows from the results in [5, 13] about planar graphs.) We shall construct a K_n-minor free graph, denoted by $M(p, q)$, such that $\chi_c(M(p, q)) = p/q$.

2.1 The construction

The construction method is a modification of the method used in [14] where a sparse subgraph of G_p^q which has the same circular chromatic number as G_p^q is constructed. (That subgraph is also denoted by $M(p, q)$, but is different from the graph in this paper.)

If $q = 1$, then $r = p$ is an integer. It is easy to see that in this case we may let $M(p, q) = K_p$. Thus we assume that $m < r < m+1$ for some integer $4 \leq m \leq n - 3$.

Since $(p, q) = 1$, there exist unique integers p', q' such that $p' < p, q' < q$ and $pq' - qp' = 1$. It is straightforward to verify that $p'/q' < p/q$ and that p'/q' is the largest fraction with the property that $p'/q' < p/q$ and $p' \leq p$. Similarly, we let p'', q'' be positive integers such that $p'' < p', q'' < q'$ and $p'q'' - p''q' = 1$. Then p''/q'' is the largest fraction with the property that $p''/q'' < p'/q'$ and that $p'' \leq p'$. Repeat this process of finding smaller and smaller fractions, we shall stop at the fraction $m/1$ in a finite number of steps. Thus given any rational p/q between m and $m + 1$, there corresponds a unique sequence of fractions

$$\frac{m}{1} = \frac{p_0}{q_0} < \frac{p_1}{q_1} < \frac{p_2}{q_2} < \cdots < \frac{p_k}{q_k} = \frac{p}{q}.$$

We call the sequence $(p_i/q_i : i = 0, 1, \cdots, k)$ the *Farey sequence* of p/q. The number k is called the *length* of the Farey sequence of p/q.

For convenience, we let $p_{-1} = -1$ and $q_{-1} = 0$. Then for $i = 1, 2, \cdots, n$, we have $p_i q_{i-1} - p_{i-1} q_i = 1$ and $p_{i-1} q_{i-2} - p_{i-2} q_{i-1} = 1$. It follows that, for $1 \leq i \leq k$, $p_{i-1}(q_i + q_{i-2}) = q_{i-1}(p_i + p_{i-2})$. As p_{i-1}, q_{i-1} are co-prime,

$$\alpha_i = \frac{p_i + p_{i-2}}{p_{i-1}} = \frac{q_i + q_{i-2}}{q_{i-1}}$$

is an integer, which is greater than 1, and hence is at least 2. We call $(\alpha_1, \alpha_2, \cdots, \alpha_k)$ the *alpha sequence* of p/q, which is obviously uniquely determined by p/q. The process of deducing the alpha sequence from the rational p/q can also be reversed. In other words, each sequence $(\alpha_1, \alpha_2, \cdots, \alpha_k)$ with $\alpha_i \geq 2$ determines a rational p/q between m and $m+1$. Indeed, given the alpha sequence $(\alpha_1, \alpha_2, \cdots, \alpha_k)$, the fractions p_i/q_i can be easily determined by solving the difference equations

$$p_i = \alpha_i p_{i-1} - p_{i-2}, \qquad q_i = \alpha_i q_{i-1} - q_{i-2}, \qquad (*)$$

with the initial condition that $(p_{-1}, q_{-1}) = (-1, 0)$ and $(p_0, q_0) = (m, 1)$.

Having determined the alpha sequence of the rational p/q, we can start constructing the K_n-minor graph G which has circular chromatic number p/q. We shall indeed construct a sequence of graph G_i, for $i = 1, 2, \cdots, k$, such that $\chi_c(G_i) = p_i/q_i$, and that each of G_i is K_{m+3}-minor free (and hence K_n-minor free, as $n \geq m+3$).

Before constructing the graphs G_i, we shall recursively construct ordered graphs F_i, H_i, i.e., the vertices of F_i and H_i are linearly ordered. Let $f_i = |F_i|$ and $h_i = |H_i|$, then usually the vertices of F_i will be denoted by $(x_{i,1}, x_{i,2}, \cdots, x_{i,f_i})$ in that order, and the vertices of H_i will be denoted by $(y_{i,1}, y_{i,2}, \cdots, y_{i,h_i})$ in that order. (Sometimes we shall use simpler indices to denote the vertices, when no confusion occurs.)

For an edge $e = (x, y)$ of an ordered graph, we define the *order length* of e, denoted by $\ell(e)$, to be the positive difference between the positions of x and y.

Definition 2.1 *Suppose X and Y are disjoint ordered graphs whose vertex orderings are $(x_1, x_2, \cdots, x_s)$ and $(y_1, y_2, \cdots, y_t)$, respectively. When we say* hook X to Y with type 1 hook, *it means to add the following edges between X and Y:*

$$x_1y_t, x_sy_1, x_sy_2, x_sy_3, \cdots, x_sy_{m-1}.$$

When we say hook X to Y with type 2 hook, *it means to add the following edges between X and Y:*

$$x_1y_t, x_1y_{t-1}, \cdots, x_1y_{t-m+2}, x_sy_1.$$

In case the graph X (resp. Y) is a singleton, then in the definition of the hooks, we set $x_1 = x_2 = \cdots = x_s$ (resp. $y_1 = y_2 = \cdots = y_t$). The edge x_1y_t of either type of hooks will be called a long edge *of that hook.*

Fig. 1 below depicts the two types of hooks.

For an integer t, we let Q_t be the $(m-1)$th power of the path of length $t-1$, i.e., Q_t has vertex set $\{v_1, v_2, \cdots, v_t\}$ in which two vertices v_i and v_j are adjacent if $|i-j| \leq m-1$. The graph Q_t is considered as an ordered graph in the following, where the order of the vertices is $(v_1, v_2, \cdots, v_t)$.

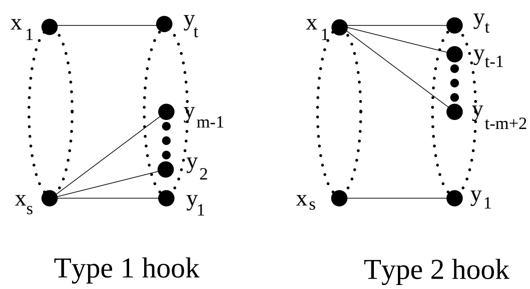

Fig. 1

First of all, we let F_1 be a singleton, let $H_1 = Q_{m\alpha_1}$, and let $F_2 = Q_{m(\alpha_1-1)}$.

For $i \geq 1$, to construct the graph H_{i+1}, we take α_{i+1} copies of F_i, denoted by $F_i^1, F_i^2, \cdots, F_i^{\alpha_{i+1}}$, and $\alpha_{i+1}-1$ copies of H_i, denoted by $H_i^1, H_i^2, \cdots, H_i^{\alpha_{i+1}-1}$ and hook them together as follows:

- If i is odd, then for $j = 1, 2, \cdots, \alpha_{i+1}-1$, we hook F_i^j to H_i^j with type 1 hook, and hook F_i^{j+1} to H_i^j with type 2 hook;

- If i is even, then for $j = 1, 2, \cdots, \alpha_{i+1}-1$, we hook H_i^j to F_i^j with type 2 hook, and hook H_i^j to F_i^{j+1} with type 1 hook.

The resulting graph is H_{i+1}.

The graph F_{i+2} is constructed in the same way as the graph H_{i+1}, but with one less copy of F_i and H_i, i.e., F_{i+2} is constructed from $\alpha_{i+1}-1$ copies of F_i and $\alpha_{i+1}-2$ copies of H_i.

The graphs H_{i+1} and F_{i+2} are regarded as ordered graphs. The order of the vertices being: the vertices of F_i^1 in order, followed by the vertices of H_i^1 in order, followed by the vertices of F_i^2 in order, etc.

Finally when i is even, we let G_i be the graph obtained by hooking H_i to F_i with type 1 hook; when i is odd, we let G_i be the graph obtained by hooking F_i to H_i with type 1 hook. We shall regard G_i as an ordered graph as well, where the order of the vertices being: thsoe of F_i in order, followed by those of H_i in order.

We note that the mapping $f : V(H_i) \mapsto V(H_i)$ defined as $f(y_{i,j}) = y_{i,h_i-j+1}$ is an automorphism of H_i; and that the mapping $g : V(F_i) \mapsto V(F_i)$ defined as $f(x_{i,j}) = x_{i,f_i-j+1}$ is an automorphism of F_i. Therefore, in the construction of G_i, when we hook F_i to H_i or H_i to F_i, it makes no difference which of the two types of hooks is used.

This finishes the construction of the graphs G_i. The graph $M(p,q)$ is equal to G_n.

2.2 K_n-minor free

Now we shall prove that the graphs G_i are K_{m+3}-minor free, and that $\chi_c(G_i) = \chi_c(G'_i) = p_i/q_i$.

To prove that each of the graphs G_i is K_{m+3}-minor free, we shall need the following lemmas which are quite obvious.

Lemma 2.1 *Suppose a graph G contains K_k as a minor, and that x is a vertex of G of degree at most $k-2$. Then there is a neighbour y of x such that the graph $G|_{xy}$, which is obtained from G by contracting the edge xy, also contains K_k as a minor.*

Let G be a graph. A *decomposition* of G by means of a subgraph H is an expression of G in the form

$$G = G_1 + G_2, \quad G_1 \cap G_2 = H.$$

In case H is a complete graph, then the expression above is called a *simplex decomposition* of G.

Lemma 2.2 *Suppose $G = G_1 + G_2$ is a simplex decomposition of G. If both G_1 and G_2 are K_k-minor free, then G is K_k-minor free.*

Suppose X is an ordered graph with vertices $\{x_1, x_2, \cdots, x_t\}$ in that order and that $t \geq m$. Let $\overline{X}$ be the graph obtained from X by adding two vertices u, v and the following edges:

$$x_1x_t,\ ux_1,\ ux_2, \cdots, ux_{m-1},\ ux_t, vx_1,\ vx_t,\ vx_{t-1}, \cdots, vx_{t-m+2},\ uv.$$

Let $\tilde{X}$ be the graph obtained from X by adding the edge x_1x_t.

Theorem 2.1 *For any $i \geq 1$, if i is odd, then the graphs $\overline{H}_i$ and $\tilde{F}_i$ are K_{m+3}-minor free; if i is even then the graphs $\tilde{H}_i$ and $\overline{F}_i$ are K_{m+3}-minor free.*

Proof. We shall prove it by induction on i. First we consider the case that $i = 1$. It is easy to see (or prove by induction) that the jth power of a path is K_{j+2}-minor free. Hence for any integer t, the graph Q_t is K_{m+1}-minor free. Thus $H_1 = Q_{m\alpha_1}$ is K_{m+1}-minor free, and hence $\overline{(}H_1)$ is K_{m+3}-minor free, as $\overline{(}H_1)$ is obtained from H_1 by adding two vertices. Similarly, $\overline{F_2}$ is K_{m+3}-minor free. We note that $\tilde{F}_1$ is obviously K_{m+3}-minor free.

Assume that Theorem 2.1 is true for $i \leq k-1$. Assume first that k is even. We consider the graph $\tilde{H}_k$. The graph H_k is obtained from α_k copies of F_{k-1}

and $\alpha_i - 1$ copies of H_{k-1}. Suppose the vertices of F_{k-1} are $x_1, x_2, \cdots, x_s$ and that the vertices of H_{k-1} are $y_1, y_2, \cdots, y_t$. We shall denote by $x_1^j, x_2^j, \cdots, x_s^j$ the vertices of the jth copy of F_{k-1}, and denote by $y_1^j, y_2^j, \cdots, y_t^j$ the vertices of the jth copy of H_{k-1}. Recall that the jth copy of F_{k-1} is hooked to the jth copy of H_i by type 1 hook, and the $(j+1)$th copy of F_{k-1} is hooked to the jth copy of H_{k-1} by type 2 hook.

We shall add to the graph $\tilde{H}_k$ the following edges:

- for $j = 1, 2, \cdots, \alpha_k$, add the edge $x_1^j x_s^j$;

- for $j = 1, 2, \cdots, \alpha_k - 1$, add the edges $x_s^j y_t^j,\ x_1^{j+1} y_1^j,\ x_s^j x_1^{j+1},\ y_1^j y_t^j$.

We shall prove that after adding these edges to $\tilde{H}_k$, the resulting graph, denoted by M_k, is still K_{m+3}-minor free.

It is straightforward to verify that for $j = 1, 2, \cdots \alpha_k$, each of the sets $\{x_1^j, x_s^j\}$ is a cut-set of the graph M_k which induces a complete subgraph of M_k, and also each of the sets $\{x_s^j, y_1^j, y_t^j, x_1^{j+1}\}$ is a cut-set of the graph M_k which induces a complete subgraph of M_k. By repeatedly applying Lemma 2.2 and the induction hypothesis, we conclude that M_k contains K_{m+3} as a minor if and only if the subgraph of M_k induced by the set

$$\{x_1^1, x_s^1, y_1^1, y_t^1, x_1^2, x_s^2, y_1^2, y_t^2, \cdots, y_1^{\alpha_k - 1}, y_t^{\alpha_k - 1}, x_1^{\alpha_k}, x_s^{\alpha_k}\}$$

contains K_{m+3} as a minor. However, this subgraph is easily seen to be K_7-minor free (hence K_{m+3}-minor free). Indeed, if the vertex x_1^1 is deleted from this subgraph, then the resulting subgraph is a subgraph of the 3rd power of a path, and hence is K_5-minor free.

The graph F_{k+1} and H_k has the same structure. The same argument shows that $\tilde{F}_{k+1}$ is still K_{m+3}-minor free.

Assume next that k is odd. We consider the graph $\overline{H}_k$. The graph H_k is obtained from α_k copies of F_{k-1} and $\alpha_i - 1$ copies of H_{k-1}. Suppose the vertices of F_{k-1} are $x_1, x_2, \cdots, x_s$ and that the vertices of H_{k-1} are $y_1, y_2, \cdots, y_t$. We shall denote by $x_1^j, x_2^j, \cdots, x_s^j$ the vertices of the jth copy of F_{k-1}, and denote by $y_1^j, y_2^j, \cdots, y_t^j$ the vertices of the jth copy of H_{k-1}. Recall that the jth copy of H_{k-1} is hooked to the jth copy of F_i by type 2 hook, and the jth copy of H_{k-1} is hooked to the $(j+1)$th copy of F_{k-1} by type 1 hook.

We shall add to the graph $\overline{H}_k$ the following edges:

- for $j = 1, 2, \cdots, \alpha_k$, add the edge $x_1^j x_s^j$;

- for $j = 1, 2, \cdots, \alpha_k - 1$, add the edges $x_1^j y_1^j,\ x_s^{j+1} y_t^j,\ y_1^j y_t^j$;

- for $j = 1, 2, \cdots, \alpha_k - 2$, add the edge $y_s^j y_1^{j+1}$;

- and finally, add the edges $ux_s^1,\ uy_1^1,\ vy_t^{\alpha_k - 1},\ vx_1^{\alpha_k}$.

We shall prove that after adding these edges to $\overline{H}_k$, the resulting graph, denoted by M_k, is still K_{m+3}-minor free.

It is straightforward to verify that

- for $j = 1, 2, \cdots \alpha_k - 1$, each of the sets $\{y_1^j, y_t^j\}$ is a cut-set of the graph M_k which induces a complete subgraph of M_k;

- for $j = 2, 3, \cdots, \alpha_k - 1$, each of the sets $\{y_t^{j-1}, x_1^j, x_s^j, y_1^j\}$ is a cut-set of the graph M_k which induces a complete subgraph of M_k;

- each of the two sets $\{u, x_1^1, x_s^1, y_1^1\}, \{v, x_1^{\alpha_k}, x_s^{\alpha_k}, y_t^{\alpha_k - 1}\}$ is a cut-set of the graph M_k which induces a complete subgraph.

By repeatedly applying Lemma 2.2 and the induction hypothesis, we conclude that M_k contains K_{m+3} as a minor if and only if the subgraph of M_k induced by the set

$$\{u, x_1^1, x_s^1, y_1^1, y_t^1, x_1^2, x_s^2, y_1^2, y_t^2, \cdots, y_1^{\alpha_k - 1}, y_t^{\alpha_k - 1}, x_1^{\alpha_k}, x_s^{\alpha_k}, v\}$$

contains K_{m+3} as a minor. However, this subgraph is easily seen to be K_7-minor free (hence K_{m+3}-minor free). Indeed, if the vertice u, v are deleted from this subgraph, then the resulting subgraph is a subgraph of the 3rd power of a path, and hence is K_5-minor free. Similarly, we can show that $\overline{F}_{k+1}$ is K_{m+3}-minor free. This completes the proof of Theorem 2.1. ∎

Note that G_i is a subgraph of H_{i+1}. Hence G_i is K_{m+3}-minor free for $1 \leq i \leq n - 1$. The same argument can prove that G_n is K_{m+3}-minor free.

Corollary 2.1 *For any $1 \leq i \leq n$, the graph G_i is K_{m+3}-minor free.*

2.3 The circular chromatic number

It remains to show that for each i the graph G_i has circular chromatic number p_i/q_i. We shall use the same idea of the proof presented in [14]. However, the graphs in consideration are not exactly the same, and hence the proof is also technically different. In the argument below, we shall omit some of the details which are straightforward and are contained in [14].

Let g_i be the number of vertices of G_i. Straightforward calculation shows that g_i satisfies the same difference equation as p_i, and also g_i has the same initial value as p_i. Therefore $|G_i| = p_i$ for $i = 1, 2, \cdots, k$.

Now we shall prove that the circular chromatic number of G_i is at most p_i/q_i. Before proving this, we need some preliminary results about the relation between the Farey sequence and the alpha sequence. We observed before that the Farey sequence is uniquely determined by the alpha sequence. The numbers p_i and q_i are obtained by solving the following difference equations:

$$p_i = \alpha_i p_{i-1} - p_{i-2}, \quad q_i = \alpha_i q_{i-1} - q_{i-2}, \qquad (*)$$

with the initial condition that $(p_{-1}, q_{-1}) = (-1, 0)$ and $(p_0, q_0) = (m, 1)$.

By repeatedly applying the equation $(*)$, we may express p_i (respectively q_i) in terms of p_j and p_{j-1} (respectively q_j and q_{j-1}) for any $0 \leq j \leq i-2$. Lemma 2.3 below, which can be proved easily by induction, gives the explicit expressions. For $1 \leq r \leq s \leq n$, we let

$$\Lambda_{r,s} = \det \begin{pmatrix} \alpha_r & 1 & 0 & \cdots & 0 & 0 \\ 1 & \alpha_{r+1} & 1 & \cdots & 0 & 0 \\ 0 & 1 & \alpha_{r+2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{s-1} & 1 \\ 0 & 0 & 0 & \cdots & 1 & \alpha_s \end{pmatrix}$$

Lemma 2.3 *For $0 \leq j \leq i-2$, we have*

$$p_i = p_j \Lambda_{j+1,i} - p_{j-1} \Lambda_{j+2,i}, \quad q_i = q_j \Lambda_{j+1,i} - q_{j-1} \Lambda_{j+2,i} \quad (**)$$

By letting $j = 0$ in $(**)$, and by using the initial condition, we have

$$p_i = m\Lambda_{1,i} + \Lambda_{2,i}, \quad q_i = \Lambda_{1,i}. \quad (***)$$

Lemma 2.4 *For $0 \leq j \leq i-2$, $p_j q_i = p_i q_j - \Lambda_{j+2,i}$.*

This is proved by induction on j, and detailed calculations can be found in [14].

Lemma 2.5 *For any $2 < t < i$, $\Lambda_{t,i} < \Lambda_{t-1,i}$.*

This is easily proved by induction, by noting that $\alpha_j \geq 2$.

Lemma 2.6 *Let $C_i = \{q_i, q_{i+1}, \cdots, p_i - q_i\}$. If $0 \le j \le i-1$, then*

$$p_j q_i \pmod{p_i} \notin C_i; \quad (p_j+1)q_i \pmod{p_i} \notin C_i;$$

but

$$(p_j - 1)q_i \pmod{p_i} \in C_i.$$

Proof. Consider first the case that $0 \le j \le i-2$. By Lemma 2.4,

$$p_j q_i = p_i q_j - \Lambda_{j+2,i},$$

and by Lemma 2.5, $(***)$ and the definition of $\Lambda_{r,s}$,

$$2 \le \alpha_i = \Lambda_{i,i} \le \Lambda_{j+2,i} < \Lambda_{1,i} = q_i.$$

Thus,

$$p_i - q_i < p_j q_i \pmod{p_i} \le p_i - 2,$$
$$0 < (p_j+1)q_i \pmod{p_i} \le q_i - 2,$$
$$p_i - 2q_i < (p_i - 1)q_i \pmod{p_i} \le p_i - q_i - 2,$$

giving the required exclusions and inclusion.

Next consider the case $j = i-1$. Since the definition of the Farey sequence gives $p_i q_{i-1} - p_{i-1} q_i = 1$, the conclusion is trivially true. ∎

Lemma 2.7 *For each i, $\chi(G_i) \le p_i/q_i$.*

Proof. We consider the graph G_i as an ordered graph, where the order of the vertices being: the vertices of F_i in order, followed by the vertices of H_i in order. Suppose the vertices of G_i are $(v_1, v_2, \cdots, v_{p_i})$ in order. Let $c(v_j) = jq_i \pmod{p_i}$. We shall show that c is a (p_i, q_i)-coloring of G_i, i.e., for every edge $e = xy$ of G_i, $|c(x) - c(y)| \in C_i$, where C_i is the set defined as in Lemma 2.6. Recall that the order length $\ell(e)$ of an edge $e = xy$ is the psoitive difference of the positions of x and y in G_i (as an ordered graph). It follows from the definition of the coloring c that $|c(x) - c(y)| = \ell(e)q_i \pmod{p_i}$. Therefore it suffices to show that for any edge e of G_i, we have

$$\ell(e)q_i \pmod{p_i} \in C_i.$$

Let $L_j = \{1, 2, \cdots, m-1\} \cup \{p_t - 1 : 0 \le t \le j-1\}$. It is not difficult to show by induction on j that the for any edge e of H_j (resp. F_j), we have $\ell(e) \in L_j$. Indeed, when H_j is constructed from copies of F_{j-1} and H_{j-1}, the edges of H_j are either those carried over from the copies of F_{j-1} and H_{j-1}, or the hooking edges. For those edges carried over from the copies of F_{j-1} and H_{j-1}, their order length remain unchanged. For those hooking edges, the long edge has order length $p_{j-1} - 1$, and the other edges have order length at most $m - 1$.

Since G_i as an ordered graph is isomorphic to the subgraph of H_{i+1} induced by the union of the first copy fo F_i and the first copy of H_i, we conclude that for any edge e of G_i, either $1 \leq \ell(e) \leq m-1$ or $\ell(e) = p_j - 1$ for some $j \leq i$. If $1 \leq \ell(e) \leq m-1$, then obviously

$$\ell(e)q_i \pmod{p_i} \in C_i,$$

as $p_i > mq_i$. If $\ell(e) = p_i - 1$, then $\ell(e)q_i \pmod{p_i} = p_i - q_i \in C_i$. If $\ell(e) = p_j - 1$ for some $j \leq i-1$, then it follows from Lemma 2.6 that $\ell(e)q_i \pmod{p_i} \in C_i$. ∎

Next we shall prove that for each i, $\chi_c(G_i) = p_i/q_i$. By Lemma 2.7, it suffices to show that $\chi_c(G_i) \geq p_i/q_i$. We shall prove it by induction on i. First we need a few lemmas.

Lemma 2.8 below was proved in [3] and also implicitly used in [8, 9].

Given a (k,d)-coloring c of a graph G. We define a directed graph $D_c(G)$ on the vertex set of G by putting a directed edge from x to y if and only if (x,y) is an edge of G and that $c(x) - c(y) = d \pmod{k}$.

Lemma 2.8 *For any graph G, $\chi_c(G) = k/d$ if and only if G is (k,d)-colorable, and that for any (k,d)-coloring c of G, the directed graph $D_c(G)$ contains a directed cycle.*

A simple calculation shows that the length of the directed cycle in $D_c(G)$ is a multiple of k, and hence is at least k

Corollary 2.2 *For any graph G, if $\chi_c(G) = k/d$ where $(k,d) = 1$, then G has a cycle of length at least k. In particular $k \leq |V(G)|$.*

Suppose $\chi_c(G_i) = p_i/q_i$, and that Δ is an (p_i, q_i)-coloring of G_i. It follows from Lemma 2.8 that there is a directed cycle of $D_\Delta(G_i)$ of length at least p_i. Since $|G_i| = p_i$, we conclude that there is a Hamiltonian cycle, say $Q = (c_1, c_2, \cdots, c_{p_i}, c_1)$, of G_i such that $\Delta(c_j) - \Delta(c_{j-1}) = q_i \pmod{p_i}$.

We say a Hamiltonian cycle $Q = (c_1, c_2, \cdots, c_t, c_1)$ of the graph G is a *good Hamiltonian cycle* (with respect to p/q) if for any edge $c_k c_\ell$ of G we have $k - \ell \neq p_i, p_i + 1$ for any $0 \leq i < n$. In particular $k - \ell \neq m, m+1$ for any edge $c_k c_\ell$ of G. Similarly a Hamiltonian path $P = (c_1, c_2, \cdots, c_t)$ of a graph G is a *good Hamiltonian path* if for any edge $c_k c_\ell$ of G, we have $k - \ell \neq p_{i'}, p_{i'} + 1$ for any $i' < n$.

Now we shall show that if $\chi_c(G_i) = p_i/q_i$, then the Hamiltonian cycle induced by any (p_i, q_i)-coloring of G_i is a good Hamiltonian cycle.

Lemma 2.9 *Suppose $\chi_c(G_i) = p_i/q_i$ and that Δ is an (p_i, q_i)-coloring of G_i. Let $Q = (c_1, c_2, \cdots, c_{p_i}, c_1)$ be the Hamiltonian cycle of G_i such that $\Delta(c_j) - \Delta(c_{j-1}) = q_i \pmod{p_i}$. Then Q is a good Hamiltonian cycle of G_i.*

Proof. Assume to the contrary that there is an edge (c_k, c_ℓ) of G_i such that $|k - \ell| = p_t$ or $p_t + 1$ for some $t \leq i - 1$. Then it follows from Lemma 2.6 that $\Delta(c_k) - \Delta(c_\ell) = (k - \ell)q_i \pmod{p_i} \notin C_i$, contrary to the assumption that Δ is a (p_i, q_i)-coloring of G_i. ∎

For any $i \leq n$, let X_i be the path of F_i given by the order of the vertices of F_i, and let Y_i be the path of H_i given by the order of the vertices of H_i. The proof of Lemma 2.7 shows that the union of X_i and Y_i is a good Hamiltonian cycle of G_i. We may conisder X_i and Y_i as the canonical good Hamiltonian paths of F_i and H_i, respectively.

It is easy to see that any good Hamiltonian cycle of G_i must the join of a good Hamiltonian path X'_i of F_i and a good Hamiltonian path Y'_i of H_i, because when i is odd, the first and the last vertices of F_i form a 2-vertex cut of G_i, and when i is even the first and the last vertices of H_i form a 2-vertex cut of G_i. Our next two lemmas shows that any such good Hamiltonian paths X'_i, Y'_i, the initial part and the terminal part of X'_i (resp. Y'_i) coincide with the corresponding part of X_i (resp. Y_i).

Lemma 2.10 *The graphs H_1 and F_2 have a unique good Hamiltonina path, up to an isomorphism.*

Proof. Each of the graphs H_1 and F_2 is of the form Q_t for some positive integer t. We shall simply prove that for any positive integer t, the graph Q_t has a unique good Hamiltonian path (with respect to any p/q), up to an isomorphism. When $t \leq m$, then Q_t is a complete graph, and there is nothing to be proved. Assume now that $t \geq m + 1$. Suppose the vertices of Q_t are $1, 2, \cdots, t$, where (x, y) is an edge if and only if $|x - y| \leq m - 1$. Let $P = (x_1, x_2, \cdots, x_t)$ be a good Hamiltonian path of Q_t. Then for any edge (x_i, x_j) of Q_t, we have $|i - j| \neq m, m + 1$. This, in particular, implies that for any $i \leq t - m$, the pair (x_i, x_{i+m}) is not an edge of Q_t. In other words, for any $i \leq t - m$, $|x_i - x_{m+i}| \geq m$.

We shall assume that $x_1 < x_{m+1}$, hence $x_1 \leq x_{m+1} - m$ (the case that $x_1 > x_{m+1}$ is parallel). Because $x_2 \leq x_1 + m - 1$ and $x_{m+2} \geq x_{m+1} - m + 1$ (as (x_1, x_2) and (x_{m+1}, x_{m+2}) are edges of Q_t), we conclude that $x_2 \leq x_{m+2} + m - 2$. Since $|x_2 - x_{m+2}| \geq m$, we conclude that $x_2 \leq x_{m+2} - m$. Repeating this argument, we can prove that $x_i \leq x_{i+m} - m$ for all $i \leq t - m$.

This implies that $\{x_1, x_2, \cdots, x_m\} = \{1, 2, \cdots, m\}$, for otherwise there would exist an $x \leq m$ and an $i \geq 1$ such that $x_{i+m} = x$ and hence $1 \leq x_i \leq x_{i+m} - m = x - m \leq 0$, an obvious contradiction.

Suppose $x_i = m+1$. Then $i \geq m+1$, by the previous paragraph. Since $x_{i-m} \leq x_i - m = 1$, we conclude that $x_{i-m} = 1$. Now by induction, it is easy to prove that if $x_i = m+j$ then $x_{i-m} = j$. This implies that $x_{j+m} = x_j + m$ for all $1 \leq j \leq t-m$. Now we shall show that $x_1 < x_2 < \cdots < x_{t-m}$. Otherwise $x_i > x_{i+1}$ for some $i \leq t-m-1$. Then $1 \leq x_{i+m+1} - x_i \leq m-1$, and hence $x_i x_{i+m+1}$ is an edge of Q_t, contrary to the assumption that P is a good Hamiltonian path. Now it follows easily that $x_i = i$ for all $i \leq t$. This completes the proof of Lemma 2.10. ∎

Lemma 2.11 *Suppose X_i' and Y_i' are good Hamiltonian paths of F_i and H_i respectively, such that the union of X_i' and Y_i' is a good Hamiltonian cycle of G_i. Then the first and the last vertex of X_i' (resp. Y_i') coincide with the first and last vertex of X_i (resp. Y_i). Moreover, if i is even (resp. odd), then the first and last m vertices of X_i' (resp. Y_i') coincide with the first and the last m vertices of X_i (resp. Y_i), up to an isomorphism.*

Proof. The first half of this lemma is more or less trivial. We shall only prove that when i is even (resp. odd), the first and the last m vertices of X_i' (resp. Y_i') are the same as that of X_i (resp. Y_i) and are of the same order as in X_i (resp. Y_i). We shall prove this by induction on i. When $i = 1, 2$ this follows from Lemma 2.10. Suppose the lemma is true for all $j < i$. We shall prove it for i. First consider the case that i is odd. We shall prove that the first and the last m vertices of Y_i' coincide with the first and the last m vertices of Y_i, up to an isomorphism.

The graph H_i is constructed from copies of F_{i-1} and H_{i-1}. The first vertex and the last vertex of any of the copies of H_{i-1} form a 2-vertex cut of H_i. Therefore the Hamiltoinan path Y_i' must be the concatenation of good Hamiltonian paths of the copies of F_{i-1} and the copies of H_{i-1}. Let X_{i-1}' be the good Hamiltonian path of the first copy of F_{i-1}, and let Y_{i-1}' be the good Hamiltonian path of the first copy of H_{i-1}. It is easy to see that the union of X_{i-1}' and Y_{i-1}' must be a good Hamiltonian cycle of G_{i-1} (note that the union of the first copy of F_{i-1} and the first copy of H_{i-1} is a copy of G_{i-1}). Therefore by the induction hypotheses, the first m vertices of X_{i-1}' coincide with the first m vertices of X_{i-1}. Now the first m vertices of X_{i-1}' are the first m vertices of Y_i', and the first m vertices of X_{i-1} are the first m vertices Y_i. Thus we have proved that the first m vertices of Y_i' coincide with the first m vertices of Y_i. The same argument can be used to prove that the last m vertices of Y_i' coincide with the last m vertices of Y_i. The case i is even can be proved by the same method, and we omit the details. This completes the proof of Lemma 2.11. ∎

Applying Lemmas 2.8, 2.9 and 2.11, and by the remark follwoing the proof of Lemma 2.9, we have the following lemma:

Lemma 2.12 *Suppose $\chi_c(G_i) = p_i/q_i$ for some i. Let Δ be any (p_i, q_i)-coloring of G_i. Then the colors of the first vertex and the last vertex of F_i (resp. H_i) are uniquely determined by the colors of the first vertex and the last vertex of H_i (resp. F_i). Moreover, when i is even (resp. odd) then the colors of the first m vertices and the last m vertices of F_i (resp. H_i) are uniquely determined by the colors of the first vertex and the last vertex of H_i (resp. F_i).*

To prove that $\chi_c(G_i) \geq p_i/q_i$ (and hence $\chi_c(G_i) = p_i/q_i$), we need another gadget. If $i \geq 2$ is even, let T_i be the graph obtained by hooking F_{i-1} to F_i by Type 1 hook. If $i \geq 2$ is odd, let T_i be the graph obtained by hooking F_i to F_{i-1} by Type 1 hook.

Theorem 2.2 *For each $i \geq 2$, $\chi_c(G_i) = p_i/q_i$ and $\chi_c(T_i) > p_{i-1}/q_{i-1}$. Moreover, $\chi_c(G_1) = p_1/q_1$.*

Proof. First we prove that $\chi_c(G_1) = p_1/q_1$. By Lemma 2.7, it suffices to show that $\chi_c(G_1) \geq p_1/q_1$. It is easy to verify that $\chi(G_1) = m + 1$. Hence $\chi_c(G_1) > m$. Suppose $\chi_c(G_1) = k/d > m$, then $k \leq |V(G_1)| = p_1$ by Corollary 2.2. Therefore $k/d \geq p_1/q_1$, because it follows from the construction of the Farey sequence that any fraction a/b strictly between $m = p_0/q_0$ and p_1/q_1 must have numerator $a > p_1$.

Next we show that $\chi_c(T_2) > p_1/q_1$. Again it is easy to verify that $\chi(T_2) = m + 1$. Suppose $\chi_c(T_2) = k/d > m$. As $|V(T_2)| < p_1$ (because $|V(F_2)| < |V(H_1)|$), we know that $k < p_1$. Therefore $k/d > p_1/q_1$, because by the construction of the Farey sequence, any fraction a/b strictly between m and p_1/q_1 has numerator $a > p_1$ (note that $k/d \neq p_1/q_1$).

Now assume that $i \geq 2$, $\chi_c(T_i) > p_{i-1}/q_{i-1}$ and that $\chi_c(G_{i-1}) = p_{i-1}/q_{i-1}$. We shall prove that $\chi_c(G_i) = p_i/q_i$.

Assume to the contrary that $\chi_c(G_i) = k/d < p_i/q_i$. Then $k \leq p_i$ and hence $k/d \leq p_{i-1}/q_{i-1}$, because by the construction of the Farey sequence, any fraction a/b strictly between p_{i-1}/q_{i-1} and p_i/q_i has numerator $a > p_i$. Since $\chi_c(G_{i-1}) = p_{i-1}/q_{i-1}$ and that G_{i-1} is a subgraph of G_i, it follows that $\chi_c(G_i) = p_{i-1}/q_{i-1}$.

Let Δ be a (p_{i-1}, q_{i-1})-coloring of G_i. First we consider the case that i is odd. Then G_i is obtained by hooking F_i to H_i. Now H_i is constructed from α_i copies of F_{i-1} and $\alpha_i - 1$ copies of H_{i-1}. The first copy of H_{i-1} is hooked to the first copy of F_{i-1} by Type 1 hooks and hooked to the second copy of F_{i-1} by Type 2 hooks. The subgraph of H_i induced by the union of the first copy of F_{i-1} and the first copy of H_{i-1} is a copy of G_{i-1}. It is not difficult to verify that the subgraph of H_i induced by the first copy of H_{i-1} and the second copy of F_{i-1} is also isomorphic to G_{i-1}. (For this purpose, one only needs to observe that each of the graphs F_j and H_j has an automorphism which reverses the order of the vertices, i.e., the mapping h defined as $h(x_{j,s}) = x_{j,f_j - s}$ is an automorphism of F_j, and $h(y_{j,s}) = y_{j,h_j - s}$ is an automorphism of H_j.)

By using the induction hypotheses that $\chi_c(G_{i-1}) = p_{i-1}/q_{i-1}$, and by applying Lemma 2.12 to each of the two copies of G_{i-1}, we conclude that the first and last m vertices of the first copy of F_{i-1} are colored the same way (under Δ) as the first and the last m vertices of the second copy of F_{i-1}.

Repeating the same argument, we can prove that the first and the last m vertices of the first copy of F_{i-1} are colored the same way as the first and the last m vertices of the last copy of F_{i-1}. This implies that the restriction of Δ to the union of F_i and the first copy of F_{i-1} of H_i is indeed a (p_{i-1}, q_{i-1})-coloring of T_i (recall that T_i is obtaioned by hooking F_i to F_{i-1} by Type 1 hook). This is contrary to our assumption that $\chi_c(T_i) > p_{i-1}/q_{i-1}$.

Finally, assuming that $i \geq 2$, $\chi_i(G_i) = p_i/q_i$ and that $\chi_c(T_i) > p_{i-1}/q_{i-1}$, we shall prove that $\chi_c(T_{i+1}) > p_i/q_i$.

Assume to the contrary that $\chi_c(T_{i+1}) = k/d \leq p_i/q_i$. Since $|F_{i+1}| < |H_i|$, hence $|T_{i+1}| < |G_i| = p_i$. It follows from Corollary 2.2 that $k < p_i$. As p_{i-1}/q_{i-1} is the largest fraction satisfying the property that $p_{i-1} < p_i$ and $p_{i-1}/q_{i-1} \leq p_i/q_i$, we conclude that $\chi_c(T_{i+1}) \leq p_{i-1}/q_{i-1}$.

We consider two cases:

Case 1: $\alpha_i = 2$. In this case $F_{i+1} = F_{i-1}$, and hence $T_{i+1} = T_i$. By induction hypothesis, $\chi_c(T_i) > p_{i-1}/q_{i-1}$.

Case 2: $\alpha_i > 2$. In this case F_{i+1} consists of $\alpha_i - 1$ copies of F_{i-1} and $\alpha_i - 2$ copies of H_{i-1}. The union of any copy of F_{i-1} and the consecutive copy of H_{i-1} induces a copy of G_{i-1}. Therefore we must have $\chi_c(T_{i+1}) = p_{i-1}/q_{i-1}$. Using the same argument as before (cf. the proof of the fact that $\chi_c(G_i) = p_i/q_i$), we conclude that for any (p_{i-1}, q_{i-1})-coloring Δ of T_{i+1}, the restriction of Δ to the union of F_i and the first copy of F_{i-1} in F_{i+1} is indeed a (p_{i-1}, q_{i-1})-coloring of T_i, contrary to our assumption that $\chi_c(T_i) > p_{i-1}/q_{i-1}$. This completes the proof of Theorem 2.2. ∎

3 Rational numbers between $n-2$ and $n-1$

Up to now, we have proved Theorem 1.1, which asserts the existence of a K_n-minor free graph of circular chromatic number r for any $2 \leq r \leq n-2$. For those rational r between $n-2$ and $n-1$, it is generally unknown whether or not there exists a K_n-minor free graph G with $\chi_c(G) = r$. However, for $n = 4$, P. Hell and the author [4] has recently proved the surprising result that for any K_4-minor free graph G, we have either $\chi_c(G) = 3$ or $\chi_c(G) \leq 8/3$.

In other words, there is a gap among the rationals that are the circular chromatic numbers of K_4-minor free graphs. For $n \geq 6$, we do not know if such gaps exists. Even for $n = 4$, we do not have a complete answer to the question that "which rational is the circular chromatic number of a K_4-minor free graph?"

Assume that $n \geq 6$. We shall prove Theorem 1.3, which says that if r is a rational between $n-2$ and $n-1$ whose Farey sequence either has length 1, or has length 2 and that $\alpha_2 = 2$, then there is a K_n-minor free graph G with $\chi_c(G) = r$.

Proof of Theorem 1.3: When the Farey sequence of r has length 1, then $r = n-2+1/d$ for some integer $d \geq 1$. Let $t = (n-2)d+1$. Let G be the graph with vertex set $V = \{1, 2, \cdots, t\}$ and edge set $E = \{ij : 1 \leq |i-j| \leq n-3\} \cup \{1t\}$. It is easy to verify that $\chi(G) = n-1$, and that $c(j) = dj \pmod{t}$ is a (t, d)-coloring of G, and hence $\chi_c(G) \leq t/d = r$. On the other hand, any fraction p/q between $n-2$ and r must have $p > t$. Therefore $\chi_c(G) = r$ by Corollary 2.2.

It is straightforward to verify that G is K_n-minor free.

Finally we assume that the Farey sequence of r has length 2, and that the alpha sequence is $(\alpha_1, 2)$. Suppose the Farey sequence of r is $(p_0/q_0, p_1/q_1, p_2/q_2)$. Then $p_0 = n-2, q_0 = 1, p_1 = (n-2)\alpha_1 + 1, q_1 = \alpha_1, p_2 = (2\alpha_1 - 1)(n-2) + 2$ and $q_2 = 2\alpha_1 - 1$.

We construct the graph G as follows:

First let H_1 be the $(n-3)$rd power of a path with $n-1$ vertices (or equivalently $H_1 = K_{n-1} - e$, where e is an edge of K_{n-1}). Let F_1 be the $(n-3)$rd power of a path of length $(\alpha_1 - 1)(n-2)$. Both H_1 and F_1 are considered as ordered graphs as before. Then we take two copies of F_1 and one copy of H_1 and hook H_1 to the first copy of F_1 by type 2 hook, and hook H_1 to the second copy of F_1 by type 1 hook. (Note that we should take $m = n-2$ in the definition of hooks.) Finally we add one more vertex u, and connected u to the first $n-3$ vertices of the first copy of F_1, and to the last vertex of the second copy of F_1.

Figure 2 belows shows the graph G, in the case that $n = 6$ and $r = 14/3$ (hence the alpha sequence is $(2, 2)$).

It is straightforward to verify that $|F_1| + |H_1| = p_1$ and $|G| = p_2$. With the same argument as in Section 2.3, we can prove that $\chi_c(G) = r$, by first proving that for $i = 1, 2, \cdots, \alpha_2 - 1$, the subgraph induced by the union of the first copy of F_1 and the first copy of H_1, as well the subgraph induced by the union of the first copy of H_1 and the second copy of F_1 has circular chromatic number p_1/q_1. Moreover, for any (p_1, q_1)-coloring of $G - u$, the first copy of F_1 and the second copy of F_1 are colored the same way. This means that such a coloring cannot be extended to a (p_1, q_1)-coloring of G,

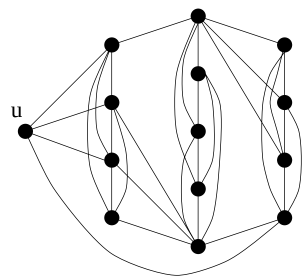

Fig. 2

and hence $\chi_c(G) > p_1/q_1$. It is easy to produce a (p_2, q_2)-coloring of G (cf. Lemma 2.7), hence $\chi_c(G) = p_2/q_2$ (because any fraction strictly between p_1/q_1 and p_2/q_2 has a numerator greater than p_2). We shall omit the details, and refer readers to the proof of Section 2.3.

Now we shall show that G is K_n-minor free. Suppose the vertices of the jth copy of F_1 is $x_1^j, x_2^j, \cdots, x_s^j$, for $j = 1, 2$, and that the vertices of H_1 is $y_1, y_2, \cdots, y_t$. Let G' be the graph obtained from G by adding the following edges:

$$uy_1, x_1^1 y_1, y_t x_s^2, y_1 y_t.$$

Then each of the following sets is a cut-set of G' which induces a complete subgraph:

$$\{y_1, y_t\}, \{u, x_1^1, y_1\}, \{y_t, x_s^2\}.$$

By applying Lemma 2.2, it is now straightforward to show that G' does not contain K_n as a minor. Hence G is K_n-minor free. This completes the proof of Theorem 1.3.

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