

# Construction of $K_n$ -minor free graphs with given circular chromatic number

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## Abstract

For each integer  $n \geq 5$  and each rational number  $r$  in the interval  $[2, n-1]$ , we construct a  $K_n$ -minor free graph  $G$  with  $\chi_c(G) = r$ . This answers a question asked by Zhu [16]. In case  $n = 5$ , the constructed graphs are actually planar. Such planar graphs were first constructed in [5] (for  $r \in [2, 3]$ ) and [12] (for  $r \in [3, 4]$ ). However, our construction and proof are much simpler.

## 1 Introduction

Suppose  $r \geq 1$  is a real number. An  $r$ -coloring of a graph  $G$  is a mapping  $c : V(G) \rightarrow [0, r)$  such that for any adjacent vertices  $x, y$  of  $G$ ,  $1 \leq |c(x) - c(y)| \leq r - 1$ . We say  $G$  is  $r$ -colorable if there exists an  $r$ -coloring of  $G$ . The *circular chromatic number*  $\chi_c(G)$  (also known as the *star chromatic number* and denoted by  $\chi^*(G)$ ) of a graph  $G$  is defined as

$$\chi_c(G) = \inf\{r : G \text{ is } r\text{-colorable}\}.$$

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The infimum in the definition can always be attained (cf. [1, 2, 8]), so it can be replaced by the minimum.

It is easy to verify (cf [16]) that if  $r$  is an integer, then a graph  $G$  is  $r$ -colorable if and only if there is a mapping  $f : V(G) \rightarrow \{0, 1, \dots, r - 1\}$  such that for every edge  $xy$  of  $G$  we have  $f(x) \neq f(y)$  (i.e., it is the same as the  $r$ -colorability in the traditional sense). If  $r \leq r'$  then a graph  $G$  is  $r$ -colorable certainly implies that  $G$  is  $r'$ -colorable. Therefore we have

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G).$$

So  $\chi_c(G)$  can be viewed as a refinement of  $\chi(G)$ , and it contains more information about the structure of the graph than  $\chi(G)$  does.

There are a few equivalent definitions of  $\chi_c(G)$ . In [8], the parameter  $\chi_c(G)$  is defined through  $(k, d)$ -colorings of a graph. It can also be defined through  $r$ -circular colorings of a graph. Readers are referred to [16] for the proof of the equivalence of these definitions, and for a comprehensive survey on this subject. In this paper, we shall only use the  $r$ -coloring defined above.

A graph  $H$  is called a *minor* of a graph  $G$ , if  $H$  can be obtained from  $G$  by deleting some vertices and edges, and contracting some edges. A graph  $G$  is called  *$H$ -minor free* if  $H$  is not a minor of  $G$ . The well-known Hadwiger's conjecture asserts that for any positive integer  $n$ , any  $K_n$ -minor free graph  $G$  is  $(n - 1)$ -colorable. If this conjecture is true, then for any  $K_n$ -minor free graph  $G$ , we have  $\chi_c(G) \leq n - 1$ . On the other hand, for any graph  $G$  with at least one edge we have  $\chi_c(G) \geq 2$ . This paper is motivated by the following problem which was posed in [12]:

**Question 1.1** *Is it true that for any rational numbers  $2 \leq r \leq n - 1$ , there exists a  $K_n$ -minor free graph  $G$  with  $\chi_c(G) = r$  ?*

This question has been studied in [4, 6, 14, 12]. In particular, the answer is NO if  $n = 4$ . It was proved in [4] that for any  $K_4$ -minor free graph  $G$ ,  $\chi_c(G)$  is either equal to 3, or  $\chi_c(G) \leq 8/3$ . So for  $8/3 < r < 3$ , there is no  $K_4$ -minor free graph with  $\chi_c(G) = r$ . On the other hand, it was proved in [6] that for any  $2 \leq r \leq 8/3$ , there exists a  $K_4$ -minor free graph  $G$  with  $\chi_c(G) = r$ .

For  $n = 5$ , Question 1.1 is answered in affirmative in [12]. It was proved in [5] that for any  $2 \leq r \leq 3$ , there is a planar graph  $G$  with  $\chi_c(G) = r$ ; and proved in [12] that for any  $3 \leq r \leq 4$ , there is a planar graph  $G$  with  $\chi_c(G) = r$ . Since planar graphs are  $K_5$ -minor free we conclude that for every  $2 \leq r \leq 4$ , there is a  $K_5$ -minor free graph  $G$  with  $\chi_c(G) = r$ .

In [14], it was proved that if  $n \geq 6$ , then for any  $2 \leq r \leq n - 2$ , there is a  $K_n$ -minor free graph  $G$  with  $\chi_c(G) = r$ .

In this paper, we shall answer Question 1.1 in affirmative for  $n \geq 5$ .

**Theorem 1.1** *If  $n \geq 5$ , then for any rational number  $2 \leq r \leq n - 1$ , there is a  $K_n$ -minor free graph  $G$  with  $\chi_c(G) = r$ .*

In case  $n = 5$ , for  $2 \leq r \leq 4$ , we shall actually construct a planar graph  $G$  with  $\chi_c(G) = r$ . Although planar graphs with circular chromatic number  $r$  (for  $2 \leq r \leq 4$ ) have been constructed before [5, 12], our construction and proof are considerably simpler than the previous constructions and proofs.

## 2 Two terminal graphs and the labeling method

To prove Theorem 1.1, we shall actually construct the required graphs. The construction and the proof that the constructed graphs do have the required properties are based on the labeling method of calculating the circular chromatic number of graphs. First we need the definition of a two-terminal graph and the label set of a two-terminal graph.

**Definition 2.1** *A two-terminal graph is a triple  $(G; x, y)$  such that  $G$  is a graph,  $x, y$  are two (not necessarily distinct) vertices of  $G$ . The two specified vertices  $x$  and  $y$  are called the terminals of  $G$ . Suppose  $n \geq 3$  is an integer. We say a two-terminal graph  $(G; x, y)$  is strongly  $K_n$ -minor free if  $G + xy$  (i.e., add an edge  $xy$  to  $G$ ) is  $K_n$ -minor free.*

Suppose  $(G; x, y)$  is a two-terminal graph and  $r$  is a real number. We are interested in the possible images of  $y$  of an  $r$ -coloring  $f$  of  $G$  for which  $f(x) = 0$ . We define the label set  $L_r((G; x, y))$  as follows:

**Definition 2.2** *Suppose  $(G; x, y)$  is a two-terminal graph and  $r \geq 1$  is a real number. The label set  $L_r((G; x, y))$  is defined as*

$$L_r((G; x, y)) = \{t \in [0, r) : \text{there exists an } r\text{-coloring } f \text{ of } G \text{ with } f(x) = 0, f(y) = t\}.$$

When the terminals  $x, y$  are clear from the context, we write  $L_r(G)$  for  $L_r((G; x, y))$ .

**Lemma 2.1** For any two-terminal graph  $(G; x, y)$ ,  $t \in L_r(G)$  if and only if  $r - t \bmod r \in L_r(G)$ .

**Proof.** If  $f$  is an  $r$ -coloring of  $G$  with  $f(x) = 0$  and  $f(y) = t$ , then  $g(v) = r - f(v) \bmod r$  defines an  $r$ -coloring with  $g(x) = 0$  and  $g(y) = r - t \bmod r$ . ■

It is obvious that if  $G$  is  $r$ -colorable, then  $G$  has an  $r$ -coloring  $f$  such that  $f(x) = 0$ . Therefore  $L_r(G) \neq \emptyset$  if and only if  $G$  is  $r$ -colorable. Thus  $\chi_c(G)$  is the minimum  $r$  for which  $L_r(G) \neq \emptyset$ . So one may calculate the circular chromatic number by determining the label set  $L_r(G)$ . We call this method of determining the circular chromatic number of a graph the *labeling method*.

For two-terminal series-parallel graphs  $G$  (see definition in [4, 6]), the label set  $L_r(G)$  can be easily calculated recursively. For general two-terminal graphs  $G$ , it is difficult to determine the label set  $L_r(G)$ . However, if  $G$  is built on some simple structured graphs by using simple graph operations (similar to series and parallel operations), then it may still be possible to determine its label set. This is indeed what we shall do in our proofs. First we define some “simple” graph operations on two-terminal graphs.

- (Parallel construction): Let  $(G; x, y)$  and  $(G'; x', y')$  be two disjoint two-terminal graphs. Define  $G''$  to be the graph obtained from the union of  $G$  and  $G'$  by identifying  $x$  and  $x'$  into a single vertex  $x''$ , and identifying  $y$  and  $y'$  into a single vertex  $y''$ . The two-terminal graph  $(G''; x'', y'')$  is said to be obtained from  $(G; x, y)$  and  $(G'; x', y')$  by *the parallel construction*.
- (Series construction): Let again  $(G; x, y)$  and  $(G'; x', y')$  be two disjoint two-terminal graphs. Define  $G''$  to be the graph obtained from the union of  $G$  and  $G'$  by identifying  $y$  and  $x'$  into a single vertex. Then the two-terminal graph  $(G''; x, y')$  is said to be obtained from  $(G; x, y)$  and  $(G'; x', y')$  by *the series construction*.
- ( $k^+$ -series construction): Let  $(G; x, y)$  and  $(G'; x', y')$  be two disjoint two-terminal graphs, and  $(G''; x, y')$  be obtained from  $(G; x, y)$  and  $(G'; x', y')$  by the series construction defined above. Add  $k$  new vertices  $x_1, x_2, \dots, x_k$  which induces a  $K_k$ , and join each  $x_i$  to  $x, y$  and  $y'$ . Denote the new graph by  $G^*$ . Then  $(G^*; x, y')$  is said to be obtained from  $(G; x, y)$  and  $(G'; x', y')$  by *the  $k^+$ -series construction*.
- (Diamond construction): Let  $(G; x, y)$  be a two-terminal graph. Define  $G'$  to be the graph obtained from  $G$  by adding two new vertices  $x', y'$ ,

and by joining each of  $x', y'$  to each of  $x, y$  by an edge. Then the two-terminal graph  $(G'; x', y')$  is said to be obtained from  $(G; x, y)$  by the *diamond construction*.

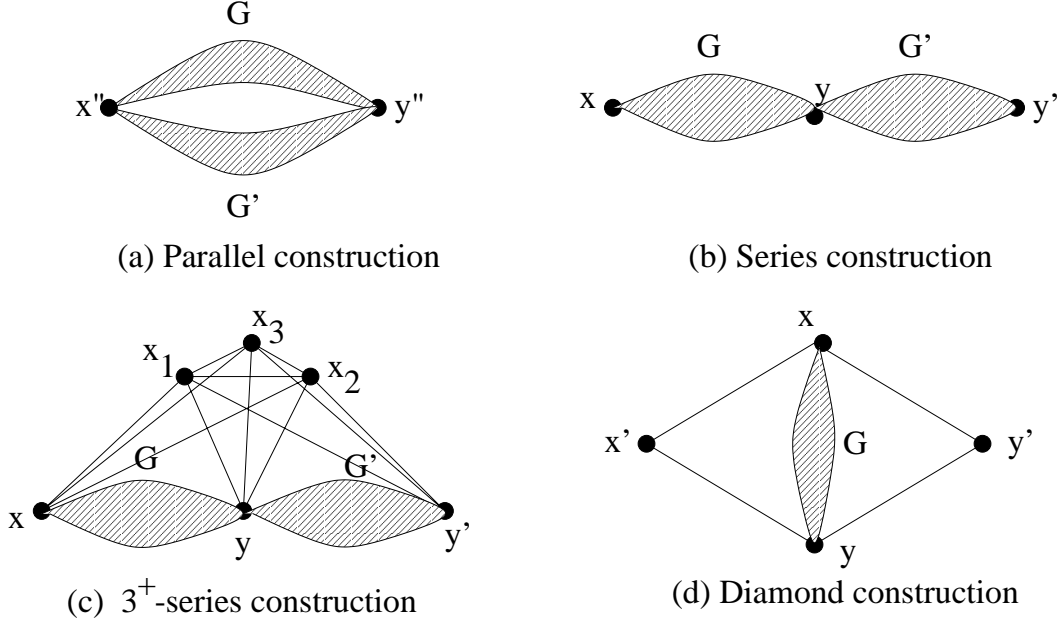


Figure 1: Illustration of the graph constructions

Suppose  $r \geq 2$  is a real number. We shall view the real numbers in the set  $[0, r)$  forming a circle (by identifying 0 and  $r$ ), and denote this circle by  $C^r$ . For two points  $a, b \in C^r$ , the distance between  $a$  and  $b$  is equal to  $\min\{|a - b|, r - |a - b|\}$ , i.e., it is equal to the length of the shorter arc of  $C^r$  joining  $a$  and  $b$ . Thus for any  $r$ -coloring  $f$  of a graph  $G$ , the points assigned to two adjacent vertices of  $G$  are of distance at least 1.

For two distinct real numbers  $a, b \in [0, r)$ , we denote by  $[a, b]_r$  the interval of  $C^r$  goes from  $a$  to  $b$  along the clockwise (i.e., the increasing) direction. To be precise, if  $a < b$  then the interval  $[a, b]_r$  contains all the real numbers  $t$  such that  $a \leq t \leq b$ ; if  $a > b$  then the interval  $[a, b]_r$  contains all the real numbers  $t$  such that either  $a \leq t < r$  or  $0 \leq t \leq b$ . When the real number  $r$  is clear from the context, we shall write  $[a, b]$  for  $[a, b]_r$ .

For  $r = 3.5$ , the intervals  $[0.5, 2]$  and  $[2.5, 0.2]$  are depicted in Fig. 1. The length  $\ell([a, b]_r)$  of an interval  $[a, b]_r$  of  $C^r$  is just the geometric length of that interval on  $C^r$ , i.e., if  $a < b$  then  $\ell([a, b]_r) = b - a$ ; if  $a > b$  then  $\ell([a, b]_r) = b + r - a$ . If  $A \subset [0, r)$  is the union of disjoint intervals  $X_1, X_2, \dots, X_k$ , then let  $\ell(A) = \sum_{i=1}^k \ell(X_i)$ .

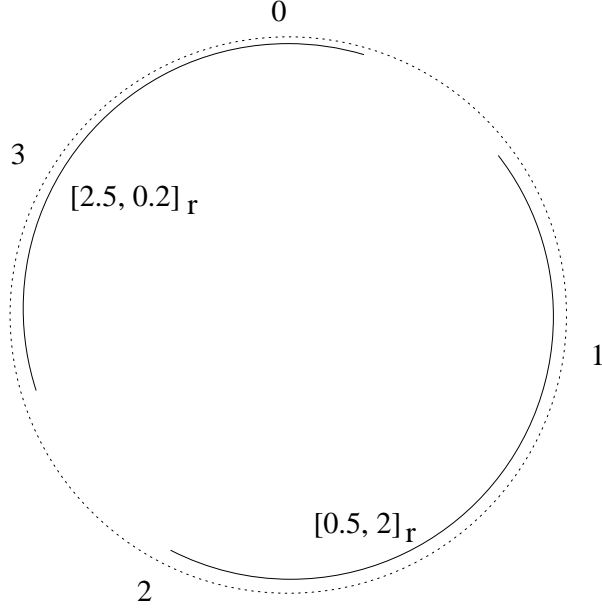


Figure 2: Illustration of the circle  $C^r$  for  $r = 3.5$  and intervals of  $C^r$

For two subsets  $A, B$  of  $C^r$ , we define

$$A + B = \{t : \exists a \in A, b \in B, a + b \bmod r = t\}.$$

For simplicity, we write  $t + B$  for  $\{t\} + B$  ( $t \in [0, r)$ ).

**Lemma 2.2** 1. If  $G''$  is obtained from  $G$  and  $G'$  by the parallel construction then

$$L_r(G'') = L_r(G) \cap L_r(G').$$

2. If  $G''$  is obtained from  $G$  and  $G'$  by the series construction then

$$L_r(G'') = L_r(G) + L_r(G').$$

3. If  $G''$  is obtained from  $G$  and  $G'$  by the  $k^+$ -series construction then

$$L_r(G'') = \{t : \exists s \in L_r(G), s' \in L_r(G'), s + s' \bmod r = t, \\ \text{moreover, the set } [1, r - 1] \cap [t + 1, t - 1] \cap [s + 1, s - 1] \\ \text{contains } k \text{ points pairwise of distance at least } 1.\}$$

**Proof.** (1) and (2) are obvious.

(3): Suppose  $f$  is an  $r$ -coloring of  $G''$  with  $f(x) = 0$ . Assume  $f(y) = s$  and  $f(y') = t$ . Then certainly  $s \in L_r(G)$  and  $t - s \bmod r \in L_r(G')$ . Since

each  $x_i$  is adjacent to  $x, y, y'$ , we conclude that  $f(x_1), f(x_2), \dots, f(x_k)$  are  $k$  points in the set  $[1, r-1] \cap [t+1, t-1] \cap [s+1, s-1]$  which are pairwise of distance at least 1.

Conversely, if  $s \in L_r(G)$ ,  $t - s \bmod r \in L_r(G')$  and

$$[1, r-1] \cap [t+1, t-1] \cap [s+1, s-1]$$

contains  $k$  points which are pairwise of distance at least 1, then the mapping  $f(x) = 0, f(y) = s$  and  $f(y') = t$  can be extended to an  $r$ -coloring of  $G''$ . ■

As shown in Lemma 2.2, to calculate the label set  $L_r(G)$  of a graph  $G$ , one frequently needs to add two subsets of  $C^r$ . When the two subsets are intervals of  $C^r$ , then their sum can be easily determined. Lemma 2.3 below is quoted from [6].

**Lemma 2.3** *Suppose  $A = [a, b]_r$  and  $B = [c, d]_r$ . If  $\ell([a, b]_r) + \ell([c, d]_r) \geq r$  then  $A + B = C^r$ ; if  $\ell([a, b]_r) + \ell([c, d]_r) < r$  then  $A + B = [a + c, b + d]$ , where the summations are carried out modulo  $r$ .*

Note that if  $A = \emptyset$  then for any set  $B$ ,  $A + B = \emptyset$ .

### 3 The proof of Theorem 1.1

The remaining of this paper is devoted to the proof of Theorem 1.1. We shall divide the proof into two parts:

**A** : For every  $n \geq 5$ , for every rational  $n - 2 \leq r \leq n - 1$ , there is a  $K_n$ -minor free graph  $G$  with  $\chi_c(G) = r$ .

**B** : For each  $2 \leq r \leq 3$ , there is a  $K_5$ -minor free graph  $G$  with  $\chi_c(G) = r$ .

Observe that this suffices for the proof of Theorem 1.1. Assume that (A) and (B) hold. Let  $n \geq 5$  and  $r \in [2, n-1]$ . Then  $r \in [m-2, m-1]$  for some  $m \leq n$ . If  $m = 4$  then by (B) there is a  $K_5$ -minor free (and hence  $K_n$ -minor free) graph  $G$  with  $\chi_c(G) = r$ ; if  $m \geq 5$ , then by (A) there is a  $K_m$ -minor free (and hence  $K_n$ -minor free) graph  $G$  with  $\chi_c(G) = r$ .

We prove (A) in this section and leave the proof of (B) to the next section. Whenever we write a fraction  $p/q$ , we assume that  $p, q > 0$  and  $(p, q) = 1$  unless otherwise specified. It is well-known that for each fraction  $p/q$  with

$p, q > 1$  (and  $(p, q) = 1$ ), there are unique integers  $0 < a < p$  and  $0 < b < q$  such that  $pb - aq = 1$ . We call the fraction  $a/b$  the *lower parent* of  $p/q$ , denote it by  $p_l(p/q) = a/b$ . Let  $a' = p - a$  and  $b' = q - b$ . The fraction  $a'/b'$  is called the *upper parent* of  $p/q$ , and denoted by  $p_u(p/q) = a'/b'$ . Observe that

$$p_l(p/q) < p/q < p_u(p/q).$$

Moreover, since  $a'b - ab' = (p - a)b - a(q - b) = pb - qa = 1$ , so if  $a < a'$  then  $p_l(a'/b') = a/b$ . It is also easy to see that in case  $a > a'$  then  $p_l(a'/b') < a/b$ .

**Theorem 3.1** *If  $n \geq 5$ , then for each rational  $n - 2 < p/q < n - 1$ , there is a  $K_n$ -minor free graph  $H$  with  $\chi_c(H) = p/q$ .*

The following lemma is well-known and will be used in proving a graph is  $K_n$ -minor free:

**Lemma 3.1** *Suppose  $G$  is a graph and  $X$  is a vertex cut of  $G$  which induces a complete graph. Assume  $G_1, G_2, \dots, G_m$  are the components of  $G - X$ . Then  $G$  contains  $K_n$  as a minor if and only if there is an index  $1 \leq j \leq m$  such that the subgraph of  $G$  induced by  $V(G_j) \cup X$  contains  $K_n$  as a minor.*

What we shall use in our proofs is the following special case of Lemma 3.1:

**Corollary 3.1** *Suppose  $G$  is a graph and  $\{x, y\}$  is a 2-vertex cut of  $G$ . Assume  $G_1, G_2, \dots, G_m$  are the components of  $G - \{x, y\}$ . For each  $i$ , let  $H_i$  be obtained from the subgraph of  $G$  induced by  $V(G_i) \cup \{x, y\}$  by adding the edge  $xy$ . If each  $H_i$  is  $K_n$ -minor free, then  $G$  is  $K_n$ -minor free.*

**Lemma 3.2** *If  $n \geq 5$ , then for each integer  $k \geq 1$ , there is a strongly  $K_n$ -minor free two-terminal graph  $(H_k; x, y)$  such that for  $n - 2 \leq r < ((k - 1)(n - 2) + 1)/(k - 1)$ ,*

$$L_r(H_k) = [(n - 2)k - (k - 1)r, kr - (n - 2)k].$$

Moreover, for  $r < n - 2$ ,

$$L_r(H_k) = \emptyset.$$

**Proof.** Let  $K_{n-1}$  be the complete graph with vertices  $x, y, x_1, x_2, \dots, x_{n-3}$ , and let  $H_1 = K_{n-1} - e$ , where  $e = xy$ . Consider the two-terminal graph  $(H_1; x, y)$ , which is strongly  $K_n$ -minor free. For  $n - 2 \leq r < n - 1$ , the label



set  $L_r(H_1)$  can be determined as follows: Since  $H_1 + xy = K_{n-1}$  which is not  $r$ -colorable, we conclude that for any  $r$ -coloring  $f$  of  $H_1$  with  $f(x) = 0$ ,  $f(y) = t$ , we have  $t \in (r-1, 1)$ . Since  $f(x_1), f(x_2), \dots, f(x_{n-3})$  are  $n-3$  points in  $[1, r-1] \cap [t+1, t+r-1]$ , which are pairwise of distance at least 1. So  $[1, r-1] \cap [t+1, t+r-1]$  is an interval of length at least  $n-4$ . Therefore  $t \in [n-2, r-(n-2)]$ . Conversely for any  $t \in [n-2, r-(n-2)]$ , let  $f(x) = 0, f(y) = t$ , then  $f$  can be extended to an  $r$ -coloring of  $H_1$ . Therefore

$$L_r(H_1) = [n-2, r-(n-2)].$$

Since  $H_1$  contains a copy of  $K_{n-2}$ , so for  $r < n-2$ ,  $L_r(H_1) = \emptyset$ .

For  $k \geq 2$ , let  $H_k$  be obtained from  $H_{k-1}$  and  $H_1$  by the series construction. By applying Corollary 3.1, it is easy to see that  $H_k$  are strongly  $K_n$ -minor free. We shall prove by induction that for  $n-2 \leq r < ((n-2)(k-1)+1)/(k-1)$ , we have

$$L_r(H_k) = [(n-2)k - (k-1)r, kr - (n-2)k].$$

For  $k=1$ , the conclusion has been proved already. Assume for  $n-2 \leq r < ((n-2)(k-2)+1)/(k-2)$ ,

$$L_r(H_{k-1}) = [(n-2)(k-1) - (k-2)r, (k-1)r - (n-2)(k-1)].$$

Note that  $((n-2)(k-1)+1)/(k-1) < ((n-2)(k-2)+1)/(k-2)$ . So for  $n-2 \leq r < ((n-2)(k-1)+1)/(k-1)$ ,

$$\begin{aligned} L_r(H_k) &= L_r(H_{k-1}) + L_r(H_1) \\ &= [(n-2)(k-1) - (k-2)r, (k-1)r - (n-2)(k-1)] \\ &\quad + [n-2, r-(n-2)]. \end{aligned}$$

The sum of the lengths of these two intervals is

$$2((k-1)r - (n-2)(k-1)) + 2(r - (n-2)) = 2kr - 2(n-2)k.$$

It follows from the condition  $r < ((n-2)(k-1)+1)/(k-1)$  that  $2kr - 2(n-2)k < r$  (note that this is not true if  $n=4$  and  $k=2$ , which explains why  $n=4$  is different). By Lemma 2.2,

$$L_r(H_k) = [(n-2)k - (k-1)r, kr - (n-2)k].$$

■

**Lemma 3.3** *If  $n \geq 5$ , then for each rational  $n-2 < p/q < n-1$ , there is a strongly  $K_n$ -minor free two-terminal graph  $(G; x, y)$  such that for  $p_l(p/q) \leq r < (p-n+2)/(q-1)$ ,*

$$L_r(G) = [p-1 - (q-1)r, qr - p + 1].$$

Moreover, for  $r < p_l(p/q)$ ,

$$L_r(G) = \emptyset.$$

**Proof.** We shall prove this lemma by induction on the denominator  $q$ . If  $p/q = (k(n-2) + 1)/k$  for some integer  $k$ , then  $p_l(p/q) = (n-2)/1$ . We let  $G$  be the graph  $H_k$  defined in the proof of Lemma 3.2. The conclusion follows from Lemma 3.2. If  $q = 2$  then  $p = 2(n-2) + 1$ , which is the case discussed above.

Assume now that  $p/q \neq (k(n-2) + 1)/k$  for any integer  $k$  and  $q \geq 3$ , and assume that Lemma 3.3 is true for all  $p'/q'$  with  $q' < q$ . Let  $a/b = p_l(p/q)$ . Then  $b/a \neq (n-2)/1$  (i.e.,  $b \geq 2$ ). By our induction hypothesis, there is a strongly  $K_n$ -minor free two terminal graph  $(G'; x', y')$  such that for  $p_l(a/b) \leq r < (a-n+2)/(b-1)$ ,

$$L_r(G') = [a-1-(b-1)r, br-a+1].$$

Let  $Q$  be obtained from  $G'$  and  $K_2$  by the parallel construction. Then  $Q$  is strongly  $K_n$ -minor free. By Lemma 2.2, for  $r < a/b$ ,  $L_r(Q) = \emptyset$ , and for  $a/b \leq r < (a-n+2)/(b-1)$ ,

$$\begin{aligned} L_r(Q) &= L_r(G') \cap L_r(K_2) \\ &= [a-1-(b-1)r, br-a+1] \cap [1, r-1] \\ &= [1, br-a+1] \cup [a-1-(b-1)r, r-1]. \end{aligned}$$

Fig. 3 below illustrate the intersection of  $L_r(G')$  and  $L_r(K_2)$ .

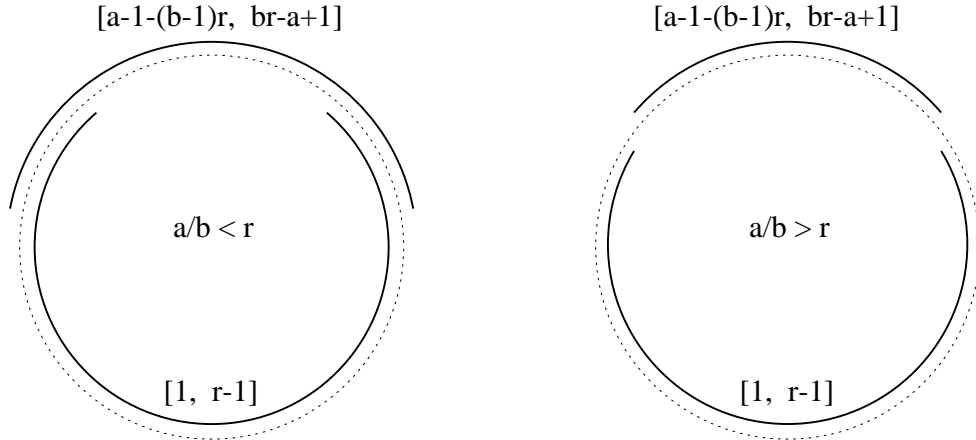


Figure 3: Illustration of the intersection of  $L_r(G') \cap L_r(K_2)$

Let  $G''$  be obtained from two copies of  $Q$  by an  $(n-4)^+$ -series construction. By applying Corollary 3.1, we conclude that  $G''$  is  $K_n$ -minor free (note that the two terminal of each copy of  $Q$  is a cut set of  $G''$ ).

We shall prove that for  $a/b \leq r < (a - n + 2)/(b - 1)$ ,

$$L_r(G'') = [a - (b - 1)r, br - a].$$

By Lemma 2.2,  $t \in L_r(G'')$  if and only if there are  $s, s' \in L_r(Q)$  such that  $t = s + s' \pmod r$  and the set  $[1, r - 1] \cap [t + 1, t - 1] \cap [s + 1, s - 1]$  contains  $n - 4$  points which are pairwise of distance at least 1. If  $s \in [1, br - a + 1]$  and  $s' \in [a - 1 - (b - 1)r, r - 1]$  then  $t \in [a - 1 - (b - 1)r + s, s - 1]$ . In this case,

$$[1, r - 1] \cap [t + 1, t - 1] \cap [s + 1, s - 1] = [s + 1, t - 1] \text{ or } [s + 1, r - 1].$$

Since  $r < (a - n + 2)/(b - 1)$ , we have

$$r - 1 - (s + 1) \geq t - 1 - (s + 1) \geq (a - 1 - (b - 1)r + s) - 1 - (s + 1) > n - 5.$$

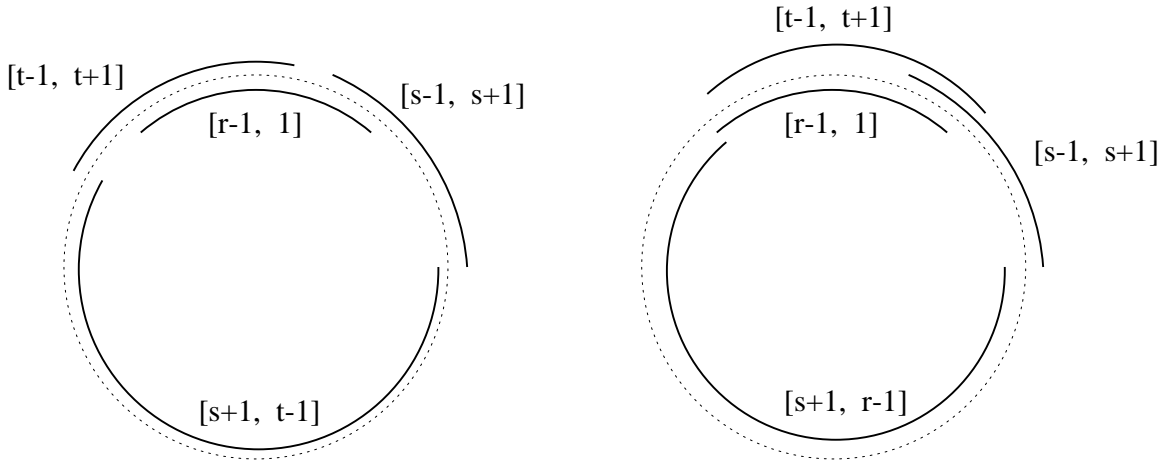


Figure 4: Illustration of the interval  $[s + 1, t - 1]$  for  $s \in [1, br - a + 1]$  and  $s' \in [a - 1 - (b - 1)r, r - 1]$  or  $s' \in [1, br - a + 1]$  and  $s \in [a - 1 - (b - 1)r, r - 1]$

This means the set  $[s + 1, t - 1]$  is an interval of length at least  $n - 5$ . Hence it contains  $n - 4$  points which are pairwise of distance at least 1. Therefore,  $t \in L_r(G'')$ .

If  $s \in [a - 1 - (b - 1)r, r - 1]$  and  $s' \in [1, br - a + 1]$  then  $t \in [1 + s, (b - 1)r - a + 1 + s]$ . In this case,

$$[1, r - 1] \cap [t + 1, t - 1] \cap [s + 1, s - 1] = [t + 1, s - 1] \text{ or } [1, s - 1].$$

Since  $r < (a - n + 2)/(b - 1)$ , we have

$$s - 1 - 1 \geq s - 1 - (t + 1) \geq s - 1 - ((b - 1)r - a + 1 + s + 1) > n - 5.$$

For the same reason as above,  $t \in L_r(G'')$ .

If  $s \in [1, br - a + 1]$  and  $s' \in [1, br - a + 1]$  then  $t \in [1 + s, br - a + 1 + s]$ . In this case,

$$[1, r - 1] \cap [t + 1, t - 1] \cap [s + 1, s - 1] = [t + 1, r - 1].$$

Since  $r < n - 1$ , we have  $r - 1 - (t + 1) \leq r - 1 - (2 + 1) < n - 5$ . This means the set  $[s + 1, t - 1]$  is an interval of length less than  $n - 5$ . Hence it does not contain  $n - 4$  points which are pairwise of distance at least 1. Therefore,  $t \notin L_r(G'')$ .

If  $s \in [a - 1 - (b - 1)r, r - 1]$  and  $s' \in [a - 1 - (b - 1)r, r - 1]$  then  $t \in [a - 1 - (b - 2)r + s, s - 1]$ . In this case,

$$[1, r - 1] \cap [t + 1, t - 1] \cap [s + 1, s - 1] = [1, t - 1].$$

Since  $r < n - 1$ , we have  $t - 1 - 1 \leq r - 2 - 2 < n - 5$ . For the same reason as above,  $t \notin L_r(G'')$ .

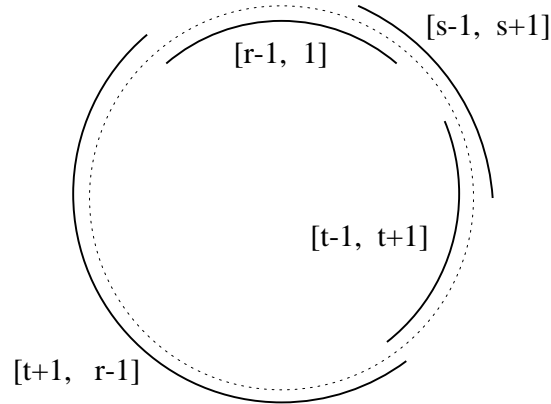


Figure 5: Illustration of the interval  $[s + 1, t - 1]$  for  $s, s' \in [1, br - a + 1]$  or  $s, s' \in [a - 1 - (b - 1)r, r - 1]$

Therefore

$$L_r(G'') = [a - (b - 1)r, br - a].$$

Now we consider two case. If  $q = b + 1$  then  $p = a + n - 1$  and  $p_u(p/q) = (n - 1)/1$ . In this case, we let  $G$  be obtained from  $G''$  and  $H_1$  by the series construction. By Corollary 3.1,  $G$  is strongly  $K_n$ -minor free. By Lemma 2.2, for  $a/b \leq r < (p - n + 2)/(q - 1) \leq (a - n + 2)/(b - 1)$ ,

$$L_r(G) = L_r(G'') + L_r(H_1)$$

$$\begin{aligned}
&= [a - (b - 1)r, br - a] + [n - 2, r - n + 2] \\
&= [(a + n - 1) - 1 - ((b + 1) - 1)r, (b + 1)r - (a + n - 1) + 1] \\
&= [p - 1 - (q - 1)r, qr - p + 1].
\end{aligned}$$

If  $q \geq b + 2$  then let  $a''/b'' = (p - a)/(q - b) = p_u(p/q)$ . By the induction hypothesis, there is a strongly  $K_n$ -minor free two-terminal graph  $G^*$  such that for  $p_l(a''/b'') \leq r < (a'' - n + 2)/(b'' - 1)$ ,

$$L_r(G^*) = [a'' - 1 - (b'' - 1)r, b''r - a'' + 1].$$

Let  $G$  be obtained from  $G''$  and  $G^*$  by the series construction. By Lemma 3.1,  $G$  is strongly  $K_n$ -minor free. By Corollary 2.2, for  $a/b \leq r < (p - n + 2)/(q - 1) \leq (a - n + 2)/(b - 1)$ ,

$$\begin{aligned}
L_r(G) &= L_r(G'') + L_r(G^*) \\
&= [a - (b - 1)r, br - a] + [a'' - 1 - (b'' - 1)r, b''r - a'' + 1] \\
&= [(a + a'') - 1 - (b + b'' - 1)r, (b + b'')r - (a + a'') + 1] \\
&= [p - 1 - (q - 1)r, qr - p + 1].
\end{aligned}$$

This completes the proof of Lemma 3.3. ■

**Proof of Theorem 3.1:** Suppose  $n - 2 < p/q < n - 1$ . By Lemma 3.3, there is a strongly  $K_n$ -minor free two terminal graph  $G$  with

$$L_r(G) = [p - 1 - (q - 1)r, qr - p + 1].$$

Let  $H$  be obtained from  $G$  and  $K_2$  by the series construction. Then  $H$  is a  $K_n$ -minor free. Since  $L_r(K_2) = [1, r - 1]$ , by using Lemma 2.2, it is easy to verify that  $L_r(H) \neq \emptyset$  if and only if  $r \geq p/q$ . Therefore  $\chi_c(H) = p/q$ . This completes the proof of Theorem 3.1.

## 4 Proof for the case $2 \leq r \leq 3$

The case  $2 \leq r \leq 3$  of Theorem 1.1 was already known. It was first proved in [5], and then a simpler proof was given in [11]. In this section, we give an alternate proof, which is based on the same idea of labeling as the proof of the main part of Theorem 1.1. However, the technique is different from the proof of part (A).

Assume  $2 \leq r \leq 3$ . We shall construct a planar graph (which is then a  $K_5$ -minor free graph)  $G$  with  $\chi_c(G) = r$ .

We say a two-terminal graph  $(G; x, y)$  is *strongly planar* if  $G + xy$  is a planar graph.

**Lemma 4.1** *Suppose  $p/q \in (2, 3)$  and that  $p_l(p/q) = a/b$ . There is a strongly planar two-terminal graph  $(G; x, y)$  such that for each  $a/b \leq r < (p-2)/(q-1)$ ,*

$$L_r(G) = [p-3 - (q-2)r, (q-1)r - p + 3].$$

Moreover, if  $r < a/b$  then

$$L_r(G) = \emptyset.$$

**Proof.** First we consider the case  $p/q = (2k+1)/k$  for some  $k \geq 2$ . In this case,  $p_l(p/q) = 2/1$ . Let  $G = P_{2k-2}$  be the path of length  $2k-2$  with the two ends as the two terminals. It is obvious that for  $r < 2$ ,  $L_r(G) = \emptyset$ .

Now we shall prove by induction that for  $j \geq 1$ , for  $2 \leq r < \frac{2j+1}{j}$ ,

$$L_r(P_{2j}) = [2j - (j-1)r, jr - 2j].$$

It follows from the definition  $L_r(P_1) = [1, r-1]$ . Since  $P_2$  is obtained from two copies of  $P_1$  by the series construction, we have

$$L_r(P_2) = L_r(P_1) + L_r(P_1) = [2, r-2].$$

Assume that for  $2 \leq r < \frac{2j+1}{j}$ ,

$$L_r(P_{2j}) = [2j - (j-1)r, jr - 2j].$$

Since  $P_{2(j+1)}$  is obtained from  $P_{2j}$  and  $P_2$  by the series construction, we have

$$L_r(P_{2(j+1)}) = L_r(P_{2j}) + L_r(P_2) = [2j - (j-1)r, jr - 2j] + [2, r-2].$$

The length of  $[2j - (j-1)r, jr - 2j]$  is equal to  $2(jr - 2j)$  and the length of  $[2, r-2]$  is equal to  $2(r-2)$ . So the sum of the two lengths is equal to  $(2j+2)r - (4j+4)$ . Assume that

$$r < \frac{2j+3}{j+1},$$

which implies that

$$(2j+2)r < 4j+6.$$

Therefore

$$(2j+2)r - (4j+4) < 2 < r.$$

Therefore

$$[2j - (j-1)r, jr - 2j] + [2, r-2] = [2(j+1) - jr, (j+1)r - 2(j+1)].$$

So for any  $k \geq 1$ , we have

$$\begin{aligned} L_r(P_{2k-2}) &= [2k - 2 - (k - 2)r, (k - 1)r - (2k - 2)] \\ &= [p - 3 - (q - 2)r, (q - 1)r - p + 3]. \end{aligned}$$

We shall prove that Lemma 4.1 is true for  $p/q \in (2, 3)$  by induction on the denominator  $q$ . If  $q = 2$  or  $p_l(p/q) = 2/1$ , then  $p = 2q + 1$ , which has been proved above. Assume  $q \geq 3$ , and assume that  $p_l(p/q) = a/b \neq 2/1$ . By the induction hypothesis, there is a strongly planar two-terminal graph  $G'$  such that for  $p_l(a/b) \leq r < (a - 2)/(b - 1)$ , we have

$$L_r(G') = [a - 3 - (b - 2)r, (b - 1)r - a + 3].$$

First we observe that  $(p - a)/(q - b) > p/q > 2$ , which implies that  $(p - 2)/(q - 1) < (a - 2)/(b - 1)$ . So for  $a/b \leq r < (p - 2)/(q - 1)$ , the equality

$$L_r(G') = [a - 3 - (b - 2)r, (b - 1)r - a + 3]$$

holds.

Let  $G''$  be obtained from  $G'$  and  $P_1$  by the series construction. By noting that  $r < (a - 2)/(b - 1)$ , we have

$$\begin{aligned} L_r(G'') &= L_r(G') + L_r(P_1) \\ &= [a - 2 - (b - 1)r, br - a + 2]. \end{aligned}$$

Let  $G^l$  be obtained from  $G''$  by the diamond construction.

**Claim**  $L_r(G^l) = [a - (b - 1)r, br - a]$ .

**Proof of the claim:** First we show that  $[a - (b - 1)r, br - a] \subseteq L_r(G^l)$ . By symmetry, it suffices to show that  $[0, br - a] \subseteq L_r(G^l)$ . For any  $0 \leq t \leq br - a$ , we define an  $r$ -coloring  $f$  of  $G^l$  as follows:  $f(x') = 0$ ,  $f(y') = t$  and  $f(x) = 1 + br - a$ ,  $f(y) = r - 1$ . Since  $f(y) - f(x) \in L_r(G'')$ , we know that  $f$  can be extended to an  $r$ -coloring of  $G''$ , and hence an  $r$ -coloring of  $G^l$ .

Conversely, assume that  $t \notin [a - (b - 1)r, br - a]$ . Again by symmetry, we may assume that  $br - a < t \leq r/2$ . We shall prove that the mapping  $f(x') = 0$  and  $f(y') = t$  cannot be extended to an  $r$ -coloring of  $G^l$ . Assume to the contrary that  $f$  is extended to an  $r$ -coloring of  $G^l$ . Since  $L_r(G'') = [a - 2 - (b - 1)r, br - a + 2]$ ,  $|f(x) - f(y)|_r \geq a - 2 - (b - 1)r$ . If any of  $f(x)$ ,  $f(y)$  is in the interval  $[0, t]$ , then  $t \geq 2$ , contrary to the assumption that  $t \leq r/2 < 2$ . Without loss of generality, assume that  $f(x) < f(y)$ . Then

the circle  $C^r$  is divided into 4 arcs  $[0, t], [t, f(x)], [f(x), f(y)], [f(y), 0]$ . So the total length of the circle is

$$\begin{aligned}
r &= t + (f(x) - t) + (f(y) - f(x)) + (r - f(y)) \\
&\geq t + 1 + (a - 2 - (b - 1)r) + 1 \\
&> br - a + 1 + (a - 2 - (b - 1)r) + 1 \\
&= r
\end{aligned}$$

which is an obvious contradiction. This completes the proof of the claim.

Let  $p_u(p/q) = a'/b'$ . If  $a'/b' < 3$ , then by induction hypothesis, there exists a graph  $G^u$  such that

$$L_r(G^u) = [a' - 3 - (b' - 2)r, (b' - 1)r - a' + 3], \text{ for } p_l\left(\frac{a'}{b'}\right) \leq r < \frac{a' - 2}{b' - 1}$$

and

$$L_r(G^u) = \emptyset \text{ for } r < p_l\left(\frac{a'}{b'}\right).$$

If  $a'/b' = 3$  then let  $G^u = K_1$  (with the two-terminals being the same vertex) and with  $L_r(G^u) = \{0\}$ .

We construct  $G$  from  $G^l$  and  $G^u$  by the series construction. It is obvious that  $G$  is strongly planar and

$$\begin{aligned}
L_r(G) &= L_r(G^l) + L_r(G^u) \\
&= [a - (b - 1)r, br - a] + [a' - 3 - (b' - 2)r, (b' - 1)r - a' + 3] \\
&= [(a + a') - 3 - (b + b' - 2)r, (b + b' - 1)r - (a + a') + 3] \\
&= [p - 3 - (q - 2)r, (q - 1)r - p + 3].
\end{aligned}$$

This completes the proof of Lemma 4.1. ■

**Theorem 4.1** *For each rational number  $2 < p/q < 3$ , there is a strongly planar two-terminal graph  $H$  with  $\chi_c(H) = p/q$ .*

**Proof.** By Lemma 4.1, there is a strongly planar two-terminal graph  $(G; x, y)$  such that for  $p_l(p/q) \leq r < (p - 2)/(q - 1)$ ,

$$L_r(G) = [p - 3 - (q - 2)r, (q - 1)r - p + 3],$$

and for  $r < a/b$ ,

$$L_r(G) = \emptyset.$$



Let  $H$  be obtained from  $G$  and  $P_3$  by the parallel construction. Then  $H$  is planar and for  $p_l(p/q) \leq r < (p-2)/(q-1)$ ,

$$L_r(H) = L_r(G) \cap L_r(P_3).$$

Since  $L_r(P_3) = [3-r, 2r-3]$ , it is easy to verify that  $L_r(H) \neq \emptyset$  if and only if  $r \geq p/q$ . So  $\chi_c(H) = p/q$ . ■

**Remark** For  $3 < p/q < 4$ , the  $K_5$ -minor free graph  $H$  with  $\chi_c(H) = p/q$  constructed in the proof of Theorem 3.1 is actually a planar graph. Together with Theorem 4.1, we have presented an alternate proof of the results of [5] and [12], i.e., every rational number between 2 and 4 is the circular chromatic number of a planar graph. This proof is based on a different idea and is much simpler than the original proofs.

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