The circular chromatic number of induced subgraphs

Xuding Zhu*
Department of Applied Mathematics
National Sun Yat-sen University
Kaohsiung 80424, Taiwan

Oct., 2002

Abstract

This note presents an infinite family of graphs G for which $\chi_c(G) = 4$ and for each vertex x of G, $\chi_c(G - x) = 8/3$. This gives a negative answer to a question asked in [8].

Keywords: Circular chromatic number, subgraph, orientation.

Suppose $G = (V, E)$ is a graph and $k \geq 2d$ are positive integers. A (k, d)-coloring of G is a mapping $f : V \to \{0, 1, \cdots, k-1\}$ such that for any edge xy of G, $d \leq |f(x) - f(y)| \leq k - d$. The *circular chromatic number* $\chi_c(G)$ of a graph G is defined as $\chi_c(G) = \min\{k/d :$ there exists a (k, d)-coloring of $G\}$. It is known [1, 5, 8] that $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$, so $\chi_c(G)$ is a refinement of $\chi(G)$.

It is obvious that the deletion of any vertex or edge decrease the chromatic number of a graph by at most 1. For circular chromatic number, it is proved in [7] that the deletion of one edge decrease the circular chromatic number of a graph by at most 1. Recently, Hajiabolhassan and Zhu [4] improved this result by showing that $\chi_c(G - e) \geq \chi(G) - 1$ for any edge e of G. This result has the following interesting corollary.

Theorem 1 *If G has circular chromatic number $\chi_c(G) > n$ for an integer n, then G has a subgraph H with $\chi_c(H) = n$.*

A natural question is whether G has an induced subgraph H with $\chi_c(H) = n$. This question is equivalent to ask if every graph G has a vertex x such that $\chi_c(G - x) \geq \chi(G) - 1$. It was shown

*Supported in part by the National Science Council under grant NSC89-2115-M-110-012.

in [7] that for any graph G and for any vertex x of G, $\chi_c(G-x) > \chi(G) - 2$. Moreover, this bound is sharp. For any $\epsilon > 0$, there is a graph G and a vertex x of G such that $\chi_c(G-x) \leq \chi_c(G) - 2 + \epsilon$. The example graph with $\chi_c(G-x) \leq \chi_c(G) - 2 + \frac{1}{t}$ is obtained by taking the circular complete graphs $K_{(nt+1)/t}$ (which has vertices $\{0, 1, 2, \cdots, nt\}$ and $i \sim j$ if and only if $t \leq |i-j| \leq nt+1-t$) and adding to it a universal vertex u. The resulting graph G has circular chromatic number $n+2$, however, $\chi_c(G-u) = \chi_c(K_{(nt+1)/t}) = n + \frac{1}{t}$. For such example graphs, if one delete any vertex other than the universal vertex u, the decrease of the circular chromatic number is 1. The following question was asked quite a few times [4, 7, 8], however, it remained open:

Question 1 *Is it true that for any graph G there is a vertex x such that $\chi_c(G-x) \geq \chi_c(G) - 1$?*

In this note we give a negative answer to Question 1 by constructing an infinite family of graphs G with $\chi_c(G) = 4$, and for any vertex x of G, $\chi_c(G-x) = 8/3$.

Let C_{2k+1} be the $(2k+1)$-cycle and P_n be the path with n edges. The *Cartesian product* $P_n \square C_{2k+1}$ has vertices $x_{i,j} : i = 0, 1, \cdots, n, j = 0, 1, \cdots, 2k$, and in which $x_{i,j} \sim x_{i',j'}$ if either $j = j'$ and $i = i' \pm 1$, or $i = i'$ and $j = j' \pm 1 \pmod{2k+1}$.

Let $G(k,n)$ be the graph obtained from $P_n \square C_{2k+1}$ by identifying $x_{0,j}$ with $x_{n,2k+1-j}$ for $j = 0, 1, \cdots, 2k$. (Note that the calculation in the second coordinate of the index is modulo $2k+1$. Thus $2k+1-0 = 0$, i.e., $x_{0,0}$ is identified with $x_{n,0}$.)

Theorem 2 *For any $k \geq 1$ and $n \geq 2$, $\chi_c(G(k,n)) = 4$. If $k \geq 2$ and $n \geq 6$, then for any vertex x of $G(k,n)$, $\chi_c(G(k,n) - x) = 8/3$.*

Proof. The graph $G = G(k,n)$ is actually a quadrangulation of the Klein bottle, and it follows from a result in [2] that $\chi_c(G) = 4$. We also note that the special case $k = 1$ of the first part of the theorem was proved by Zhou [6], where the proof is quite long. For the completeness of this note, we give a direct proof, as the argument is also needed to prove that $\chi_c(G-x) = 8/3$ for any vertex x of G.

As G is 4-regular, it follows from Brook's theorem that G is 4-colorable. Thus $\chi_c(G) \leq 4$. Assume to the contrary that $\chi_c(G) = r < 4$. By a result in [3], G has an orientation D such that for any circuit C of G, $|C|/|C^+| \leq r$ and $|C|/|C^-| \leq r$. In this note, a circuit is denoted by a cyclic sequence of distinct vertices $C = (v_0, v_1, \cdots, v_{m-1})$ such that $v_i v_{i+1}$ is an edge of G for $i = 0, 1, \cdots, m-1$ (the addition in the indices is taken modulo m) and $C^+ = \{v_i v_{i+1} : v_i \to v_{i+1}$ is an arc of $D\}$ is the set of forward edges of C, and $C^- = \{v_i v_{i+1} : v_{i+1} \to v_i$ is an arc of $D\}$ is the set of backward edges of C.

Such an orientation D of G induces an orientation D' of $P_n \Box C_{2k+1}$, in which the circuit $C(0) = (x_{0,0}, x_{0,1}, x_{0,2}, \cdots, x_{0,2k})$ and the circuit $C(n) = (x_{n,0}, x_{n,2k}, x_{n,2k-1}, \cdots, x_{n,1})$ have the same orientation (because they are a single circuit in G). Thus $|C(0)^+| = |C(n)^+|$ and $|C(0)^-| = |C(n)^-|$.

For $i = 0, 1, \cdots, n-1$ and $j = 0, 1, \cdots, 2k$, let $C(i,j)$ be the circuit $C(i,j) = (x_{i,j}, x_{i,j+1}, x_{i+1,j+1}, x_{i+1,j})$. (The addition in the second coordinate of the indices is taken modulo $2k+1$). Since $|C|/|C(i,j)^+| < 4$ and $|C|/|C(i,j)^-| < 4$, it follows that $|C(i,j)^+| = |C(i,j)^-| = 2$. Therefore

$$\sum_{i=0}^{n-1} \sum_{j=0}^{2k} (|C(i,j)^+| - |C(i,j)^-|) = 0.$$

In the summation above, each edge of $P_n \Box C_{2k+1}$ other than those edges in $C(0)$ and in $C(n)$ is contained in two of the circuits $C(i,j)$, once as a forward edge, and once as a backward edge. Therefore each of these edges contributes 0 to the summation. The edges in $C(0)$ and $C(n)$ are counted once, in the direction of $C(0)$ and $C(n)$, respectively. So

$$\begin{aligned} 0 &= \sum_{i=0}^{n-1} \sum_{j=0}^{2k} (|C(i,j)^+| - |C(i,j)^-|) \\ &= (|C(0)^+| - |C(0)^-|) + (|C(n)^+| - |C(n)^-|) \\ &= 2(|C(0)^+| - |C(0)^-|). \end{aligned}$$

Hence $|C(0)^+| = |C(0)^-|$. But this is impossible, as $|C(0)^+| + |C(0)^-| = 2k+1$ is odd, and each of $|C(0)^+|, |C(0)^-|$ is an integer.

Next we show that for any vertex x of G, $\chi_c(G-x) = 8/3$. For $i \geq 1$, the mapping ϕ defined as $\phi(x_{i',j}) = x_{i'-i,j}$ for $i \leq i' \leq n-1$ and $\phi(x_{i',j}) = x_{i'-i+n,2k+1-j}$ is an automorphism of $G(k,n)$. Also the mapping ψ defined as $\psi(x_{i,j}) = x_{i,2k+1-j}$ is an automorphism of $G(k,n)$. Therefore, we may assume the deleted vertex $x = x_{1,j}$ for some $0 \leq j \leq k$.

First we show that $\chi_c(G-x) \geq 8/3$. Assume $\chi_c(G-x) = r < 8/3$, where $x = x_{1,j}$. Again let D be an orientation of $G-x$ for which $|C|/|C^+| \leq r$ and $|C|/|C^-| \leq r$. Similarly let D' be the orientation of $P_n \Box C_{2k+1} - x_{1,j}$ induced by D. We consider the circuits $C_{i,j}$ as defined in the proof above, except that the four circuits $C_{0,j-1}, C_{0,j}, C_{1,j-1}, C_{1,j}$ are replaced by a single circuit C^* which is the symmetric difference of $C_{0,j-1}, C_{0,j}, C_{1,j-1}, C_{1,j}$. Now $|C^*| = 8$. Since $|C^*|/|(C^*)^+| < 8/3$ and $|C^*|/|(C^*)^-| < 8/3$, it follows that $|(C^*)^+| = |(C^*)^-| = 4$. The rest of the argument is the same as in the previous paragraph, which leads to the same contradiction.

It remains to prove that $\chi_c(G-x) \leq 8/3$. If $k \geq 3$, then the mapping f defined as $f(x_{i,j}) = x_{i,j}$ for $j \leq 2k-2$ and $f(x_{i,2k-1}) = x_{i,2k-3}$ and $f(x_{i,2k}) = x_{i,2k-2}$ is a homomorphism from $G(k,n)$ to $G(k-1,n)$. If $n \geq 8$, then the mapping f defined as $f(x_{i,j}) = x_{i,j}$ for $i \leq n-3$ and $f(x_{i,n-2}) = x_{i,n-4}$ and $f(x_{i,n-1}) = x_{i,n-3}$ is a homomorphism from $G(k,n)$ to $G(k,n-2)$. Therefore it suffices to show that $G(2,6) - x$ is $(8,3)$-colorable for any vertex x of $G(2,6)$, and $G(2,7) - x$ is $(8,3)$-colorable for any vertex x of $G(2,7)$. As mentioned above, by symmetry, we may assume that $x = x_{1,j}$ for some $0 \leq j \leq 2$. Figures 1 and 2 below give $(8,3)$-colorings of $G - x$, for $x = x_{1,0}, x_{1,1}$ and $x_{1,2}$, and for $G = G(2,6)$ and $G = G(2,7)$, respectively. ∎

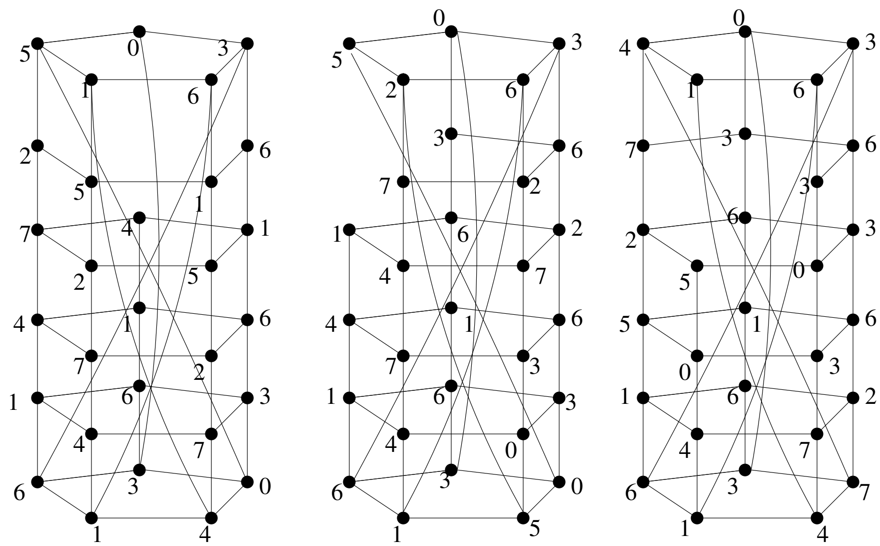

Figure 1: $(8,3)$-coloring of $G(2,6) - x$

Although the infinite family of graphs presented here give a negative answer to Question 1, there are still many questions remain unanswered.

Question 2 1. *Are there n-chromatic graphs G for some $n \geq 5$ such that for any vertex x, $\chi_c(G) - \chi_c(G-x) > 1$?*

2. *Is there a graph G such that for every vertex x of G, $\chi_c(G) - \chi_c(G-x) \geq 2 - \epsilon$ for some $\epsilon < 2/3$? Or even for any $\epsilon > 0$?*

3. *Does there exist a graph G for which $\chi_c(G) \neq \chi(G)$ and yet there is a vertex x of G such that $\chi_c(G) - \chi_c(G-x) > 1$?*

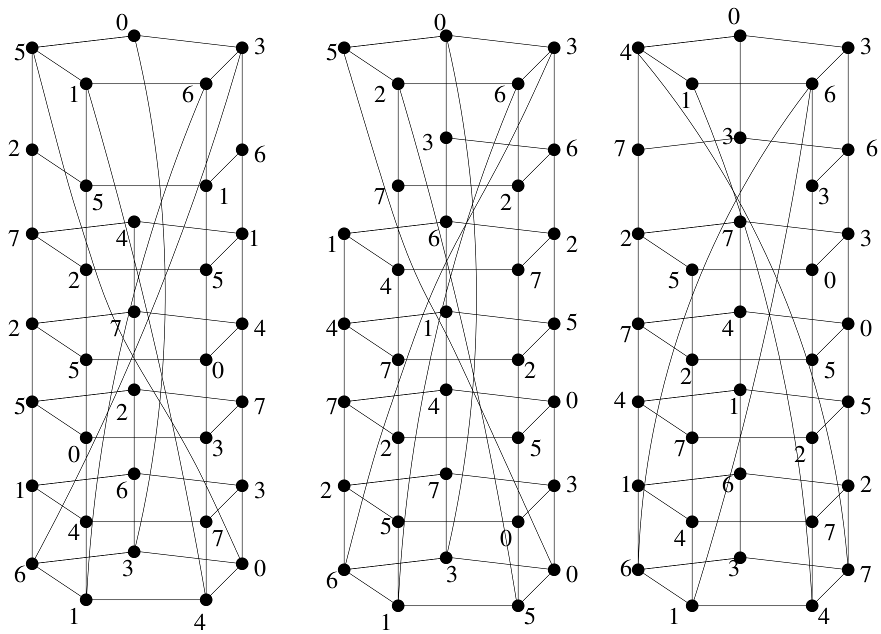

Figure 2: $(8,3)$-coloring of $G(2,7)-x$

References

[1] J. A. Bondy and P. Hell, *A note on the star chromatic number*, J. Graph Theory 14 (1990) 479-482.

[2] M. DeVos, L. Goddyn, B. Mohar, D. Vertigan and X. Zhu, *Coloring-flow duality of graphs on surfaces*, manuscript, 2002.

[3] L. A. Goddyn, M. Tarsi, and C. Q. Zhang, *On (k,d)-colorings and fractional nowhere-zero flows*, J. Graph Theory, 28(1998), 155-161.

[4] H. Hajiabolhassan and X. Zhu, *The circulr chromatic number of subgraphs*, manuscript, 2001.

[5] A. Vince, *Star chromatic number*, J. Graph Theory 12 (1988) 551-559.

[6] B. Zhou, *Some theorems concerning the star chromatic number of a graph*, J. Comb. Theory (B), 70(1997), 245-258.

[7] X. Zhu, *Star chromatic numbers and products of graphs*, J. Graph Theory **16** (1992), 557-569.

[8] X. Zhu, *Circular chromatic number, a survey*, Discrete Mathematics, 229(1-3)(2001), 371-410.