The circular chromatic number of induced subgraphs

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Abstract

This note presents an infinite family of graphs G for which $\chi_c(G) = 4$ and for each vertex x of G, $\chi_c(G-x) = 8/3$. This gives a negative answer to a question asked in [8].

Keywords: Circular chromatic number, subgraph, orientation.

Suppose $G = (V, E)$ is a graph and $k \geq 2d$ are positive integers. A (k,d)-coloring of G is a mapping $f : V \to \{0, 1, \cdots, k-1\}$ such that for any edge xy of G, $d \leq |f(x) - f(y)| \leq k - d$. The *circular chromatic number* $\chi_c(G)$ of a graph G is defined as $\chi_c(G) = \min\{k/d : \text{there exists a } (k,d)\text{-coloring of } G\}$. It is known [1, 5, 8] that $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$, so $\chi_c(G)$ is a refinement of $\chi(G)$.

As $\chi(G) - 1 \leq \chi(G-x) \leq \chi(G)$ for every vertex x of G, it follows that $\chi_c(G) - 2 < \chi_c(G-x) \leq \chi_c(G)$. How small can $\chi_c(G-x)$ be? It was shown in [7] that for any $\epsilon > 0$ there is a graph G which contains a vertex x such that $\chi_c(G-x) < \chi_c(G) - 2 + \epsilon$. But it is natural to ask whether every graph G must contain a vertex x with $\chi_c(G-x) \geq \chi_c(G) - 1$. This question was posed in [7] and remained open until now: in this paper we give a negative answer by constructing an infinite family of graphs G with $\chi_c(G) = 4$ and $\chi_c(G-x) = 8/3$ for every vertex x of G.

A corresponding question can be asked regarding edge deletion. In this case it was proved already in [7] that $\chi_c(G-e) \geq \chi_c(G) - 1$ for every edge e of G. Recently, Hajiabolhassan and Zhu [4] strengthened this inequality to $\chi_c(G-e) \geq \chi(G) - 1$. From this result it follows that any graph G with $\chi_c(G) > n$ for an integer n has a subgraph H with $\chi_c(H) = n$. We cannot always take H

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to be an induced subgraph here, since this would require that any graph G has a vertex x for which $\chi_c(G-x) \geq \chi(G)-1$, and by our construction this might not be possible.

Let C_{2k+1} be the $(2k+1)$-cycle and P_n be the path with n edges. The *Cartesian product* $P_n \Box C_{2k+1}$ has vertices $x_{i,j} : i = 0, 1, \cdots, n, j = 0, 1, \cdots, 2k$, and in which $x_{i,j} \sim x_{i',j'}$ if either $j = j'$ and $i = i' \pm 1$, or $i = i'$ and $j = j' \pm 1 \pmod{2k+1}$.

Let $G(n,k)$ be the graph obtained from $P_n \Box C_{2k+1}$ by identifying $x_{0,j}$ with $x_{n,2k+1-j}$ for $j = 0, 1, \cdots, 2k$. (Note that the calculation in the second coordinate of the index is modulo $2k+1$. Thus $2k+1-0=0$, i.e., $x_{0,0}$ is identified with $x_{n,0}$.)

Theorem 1 *For any $k \geq 1$ and $n \geq 2$, $\chi_c(G(n,k)) = 4$. If $k \geq 2$ and $n \geq 6$, then for any vertex x of $G(n,k)$, $\chi_c(G(n,k)-x) = 8/3$.*

Proof. The graph $G = G(n,k)$ is actually a quadrangulation of the Klein bottle, and it follows from a result in [2] that $\chi_c(G) = 4$. We also note that the special case $k = 1$ of the first part of the theorem was proved by Zhou [6], where the proof is quite long. For the completeness of this note, we give a direct proof, as the argument is also needed to prove that $\chi_c(G-x) = 8/3$ for any vertex x of G.

As G is 4-regular, it follows from Brooks' theorem that G is 4-colorable. Thus $\chi_c(G) \leq 4$. Assume to the contrary that $\chi_c(G) = r < 4$. By a result in [3], G has an orientation D such that for any cycle C of G, $|C|/|C^+(D)| \leq r$ and $|C|/|C^-(D)| \leq r$.

Such an orientation D of G induces an orientation D' of $P_n \Box C_{2k+1}$, in which the circuit $C(0) = (x_{0,0}, x_{0,1}, x_{0,2}, \cdots, x_{0,2k})$ and the circuit $C(n) = (x_{n,0}, x_{n,2k}, x_{n,2k-1}, \cdots, x_{n,1})$ have the same orientation (because they are a single circuit in G). Thus $|C(0)^+(D)| = |C(n)^+(D)|$ and $|C(0)^-(D)| = |C(n)^-(D)|$.

For $i = 0, 1, \cdots, n-1$ and $j = 0, 1, \cdots, 2k$, let $C(i,j)$ be the circuit $C(i,j) = (x_{i,j}, x_{i,j+1}, x_{i+1,j+1}, x_{i+1,j})$. (The addition in the second coordinate of the indices is taken modulo $2k+1$). Since $|C|/|C(i,j)^+(D)| < 4$ and $|C|/|C(i,j)^-(D)| < 4$, it follows that $|C(i,j)^+(D)| = |C(i,j)^-(D)| = 2$. Therefore

$$\sum_{i=0}^{n-1} \sum_{j=0}^{2k} (|C(i,j)^+(D)| - |C(i,j)^-(D)|) = 0.$$

In the summation above, each edge of $P_n \Box C_{2k+1}$ other than those edges in $C(0)$ and in $C(n)$ is contained in two of the circuits $C(i,j)$, once as a forward edge, and once as a backward edge.

Therefore each of these edges contributes 0 to the summation. The edges in $C(0)$ and $C(n)$ are counted once, in the direction of $C(0)$ and $C(n)$, respectively. So

$$\begin{aligned} 0 &= \sum_{i=0}^{n-1}\sum_{j=0}^{2k}(|C(i,j)^+(D)| - |C(i,j)^-(D)|) \\ &= (|C(0)^+(D)| - |C(0)^-(D)|) + (|C(n)^+(D)| - |C(n)^-(D)|) \\ &= 2(|C(0)^+(D)| - |C(0)^-(D)|). \end{aligned}$$

Hence $|C(0)^+(D)| = |C(0)^-(D)|$. But this is impossible, as $|C(0)^+(D)| + |C(0)^-(D)| = 2k+1$ is odd, and each of $|C(0)^+(D)|, |C(0)^-(D)|$ is an integer.

Next we show that for any vertex x of G, $\chi_c(G-x) = 8/3$. For $i \geq 1$, the mapping ϕ defined as

$$\phi(x_{i',j}) = \begin{cases} x_{i'-i,j}, & \text{if } i \leq i' \leq n-1 \\ x_{i'-i+n,2k+1-j}, & \text{otherwise} \end{cases}$$

is an automorphism of $G(n,k)$. Also the mapping ψ defined as $\psi(x_{i,j}) = x_{i,2k+1-j}$ is an automorphism of $G(n,k)$. Therefore, we may assume the deleted vertex $x = x_{1,j^*}$ for some $0 \leq j^* \leq k$.

First we show that $\chi_c(G-x) \geq 8/3$. Assume $\chi_c(G-x) = r < 8/3$. Again let D be an orientation of $G-x$ for which $|C|/|C^+(D)| \leq r$ and $|C|/|C^-(D)| \leq r$ for each circuit C of G. Similarly let D' be the orientation of $P_n \Box C_{2k+1} - x_{1,j^*}$ induced by D. We consider the circuits $C_{i,j}$ as defined in the proof above, except that the four circuits $C_{0,j^*-1}, C_{0,j^*}, C_{1,j^*-1}, C_{1,j^*}$ are replaced by a single circuit C^* which is the symmetric difference of $C_{0,j^*-1}, C_{0,j^*}, C_{1,j^*-1}, C_{1,j^*}$. Now $|C^*| = 8$. Since $|C^*|/|(C^*)^+(D)| < 8/3$ and $|C^*|/|(C^*)^-(D)| < 8/3$, it follows that $|(C^*)^+(D)| = |(C^*)^-(D)| = 4$. The rest of the argument is the same as in the previous paragraph, which leads to the same contradiction.

It remains to prove that $\chi_c(G-x) \leq 8/3$. If $n \geq 8$, then the mapping f defined as

$$f(x_{i,j}) = \begin{cases} x_{i,j}, & \text{if } i \leq n-3 \\ x_{i-2,j}, & \text{if } i \geq n-2 \end{cases}$$

is a homomorphism from $G(n,k) - x$ to $G(n-2,k) - x$. Therefore, it suffices to consider the case that $n=6$ and $n=7$. If $k \geq 3$ and $j^* \neq k-1, k, k+1, k+2$, then let f be the mapping defined as

$$f(x_{i,j}) = \begin{cases} x_{i,j}, & \text{if } 0 \leq j \leq k; \\ x_{i,j-2}, & \text{if } k+1 \leq j \leq 2k \end{cases}$$

It is straightforward to verify that f is a homomorphism from $G(n,k) - x_{1,j^*}$ to $G(n,k-1) - f(x_{1,j^*})$. If $k \geq 5$ and $j^* \in \{k-1, k, k+1, k+2\}$, then let f be the mapping defined as

$$f(x_{i,j}) = \begin{cases} x_{i,j}, & \text{if } j = 0, 1; \\ x_{i,j-2}, & \text{if } 2 \leq j \leq 2k-2; \\ x_{i,0}, & \text{if } j = 2k-1; \\ x_{i,2k-4}, & \text{if } j = 2k. \end{cases}$$

Then f is a homomorphism from $G(n,k) - x_{1,j^*}$ to $G(n,k-2) - f(x_{1,j^*})$. Therefore it suffices to show the following:

1. $G(6,2) - x_{1,j^*}$ is $(8,3)$-colorable for $j^* = 0,1,2$.
2. $G(7,2) - x_{1,j^*}$ is $(8,3)$-colorable for $j^* = 0,1,2$.
3. $G_{6,3} - x_{1,j^*}$ is $(8,3)$-colorable for $j^* = 2,3$.
4. $G_{7,3} - x_{1,j^*}$ is $(8,3)$-colorable for $j^* = 2,3$.
5. $G_{6,4} - x_{1,j^*}$ is $(8,3)$-colorable for $j^* = 3$.
6. $G_{7,4} - x_{1,j^*}$ is $(8,3)$-colorable for $j^* = 3$.

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Figures 1, 2, 3 should be here.

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Figures 1-3 give the required $(8,3)$-colorings of $G - x$ for each of the above cases. ∎

Although we have an infinite family of graphs G such that $\chi_c(G-x) < \chi_c(G) - 1$ for each vertex x, all these graphs G have the same circular chromatic number, and all the subgraphs $G - x$ also have the same circular chromatic number. It would be interesting to know whether these are the exception cases, or there are other examples. In particular, it would be interesting to answer the following questions.

Question 1 *Are there n-chromatic graphs G for some $n \geq 5$ such that for any vertex x, $\chi_c(G) - \chi_c(G-x) > 1$?*

Question 2 *Are there graphs G such that for every vertex x of G, $\chi_c(G) - \chi_c(G-x) \geq 2 - \epsilon$ for some $\epsilon < 2/3$? Or even for any $\epsilon > 0$?*

Question 3 *Are there graphs G for which $\chi_c(G) \neq \chi(G)$ and yet there is a vertex x of G such that $\chi_c(G) - \chi_c(G-x) > 1$?*

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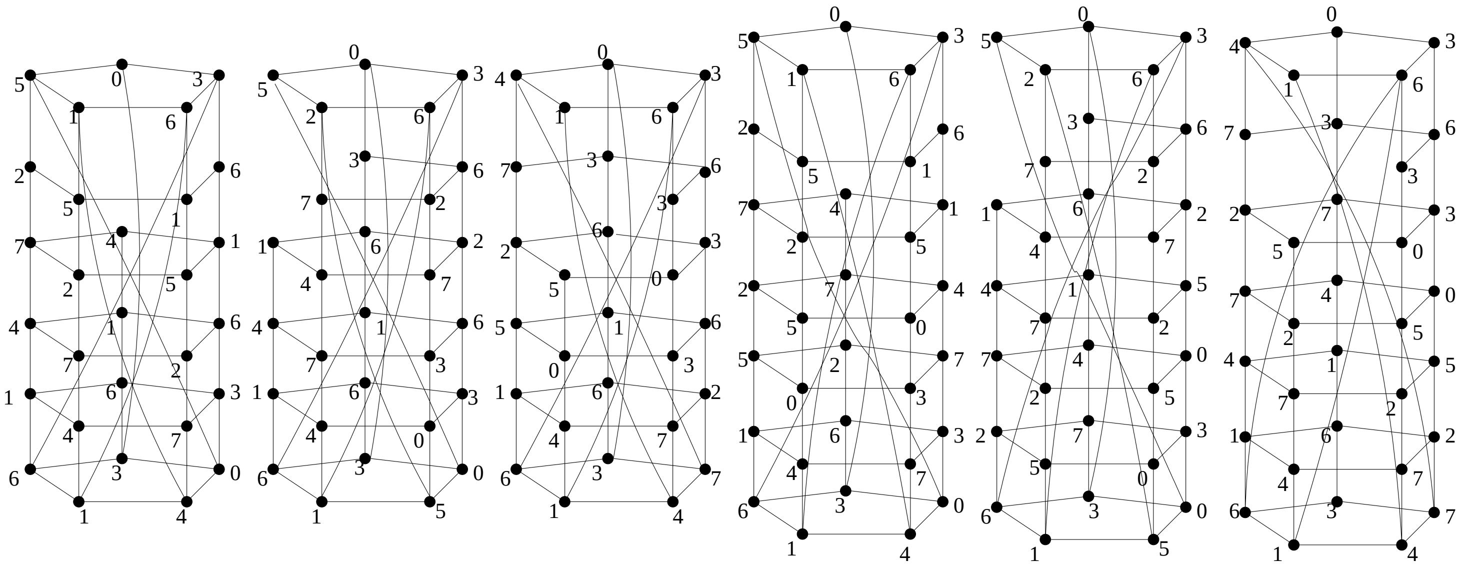

Figure 1: $(8,3)$-coloring of $G(6,2)-x$ and $G(7,2)-x$

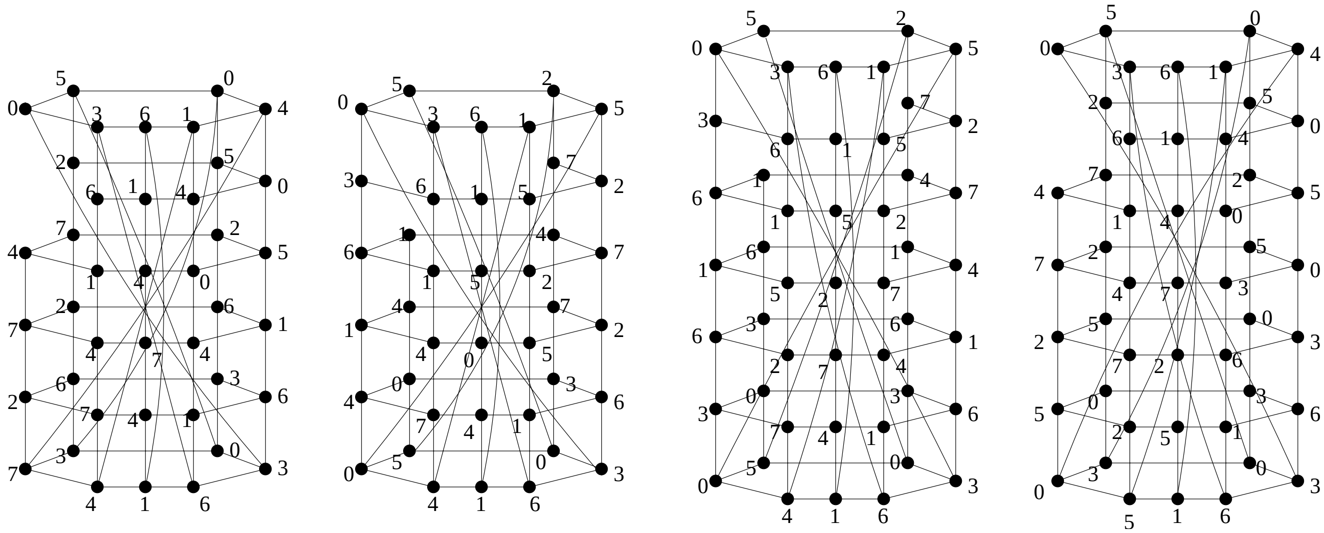

Figure 2: $(8,3)$-coloring of $G(6,3)-x$ and $G(7,3)-x$

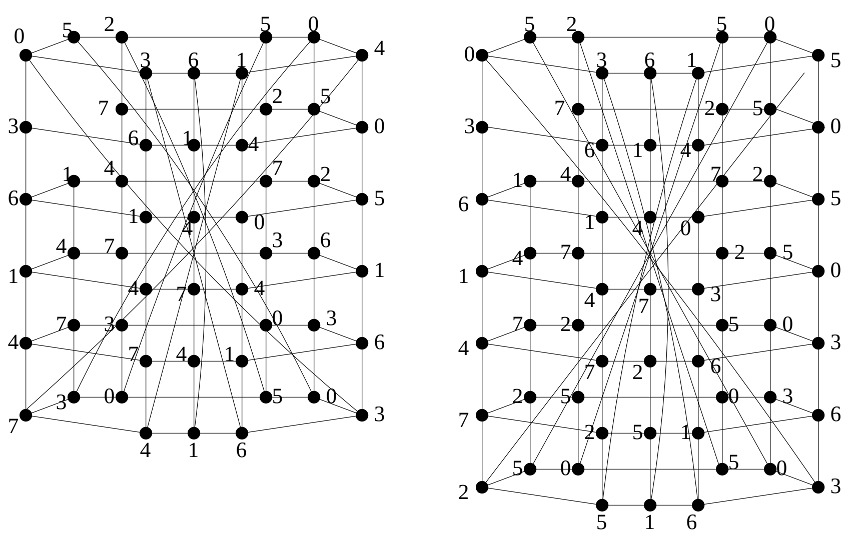

Figure 3: $(8,3)$-coloring of $G(6,4)-x$ and $G(7,4)-x$