Construction of uniquely H-colorable graphs

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Abstract

We shall prove that for any graph H which is a core, if $\chi(G)$ is large enough, then $H \times G$ is uniquely H-colorable. We also give a new construction of triangle free graphs which are uniquely n-colorable.

All graphs considered in this paper are finite. Suppose G and H are graphs. An *H-coloring* of G is a mapping f from $V(G)$ to $V(H)$ such that $f(x)f(y)$ is an edge of H whenever xy is an edge of G. An H-coloring of G is also called a *homomorphism* from G to H. We say G is H-colorable if there exists an H-coloring of G. Two graphs G and H are *homomorphically equivalent* if G admits a homomorphism to H (i.e., G is H-colorable) and H admits a homomorphism to G. It is easy to see that a graph G is K_n-colorable if and only if G is n-colorable. Indeed, a K_n-coloring of G corresponds to an n-coloring of G.

A graph H is a core if H is not H'-colorable for any proper subgraph H' of H. A graph G is said to be uniquely H-colorable, if there exists an H-coloring f of G such that $f(V(G)) = V(H)$ and for any other H-coloring f' of G, f' is the composition $f \circ \sigma$ of f with an automorphism σ of H. Again, when $H = K_n$, the concept of uniquely K_n-colorable coincides with the concept of uniquely n-colorable.

Suppose H is a core. Let $C(H)$ be the graph whose vertices are all the mappings from $V(H)$ to $V(H)$ which are not automorphisms of H. Two such mappings f and g are adjacent in $C(H)$ if for every edge xy of H, $f(x)g(y)$

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is also an edge of H. First we note that $C(H)$ has no loops. Indeed, if f is adjacent to f, then for any edge xy of H, $f(x)f(y)$ is also an edge of H, i.e., f is a homomorphism of H to H. Since H is a core, any homomorphism of H to H is an automorphism. As the vertices of $C(H)$ are not automorphisms of H, it follows that $C(H)$ has no loops.

For graphs G and H, the categorical product $H \times G$ has vertex set $\{(x, y) : x \in V(H), y \in V(G)\}$. Two vertices (x, y) and (x', y') are adjacent in $H \times G$ if and only if x and x' are adjacent in H, y and y' are adjacent in G.

Theorem 1 *If G is a connected graph which does not admit a homomorphism to $C(H)$, then $H \times G$ is uniquely H-colorable.*

Proof. The mapping defined as $\phi(h, x) = h$ is a homomorphism of $H \times G$ to H. Suppose there is another homomorphism ψ from $H \times G$ to H, which is not a composition of ϕ with an automorphism of H. Let x be any vertex of G. Consider the mapping ψ_x from $V(H)$ to $V(H)$ defined as $\psi_x(h) = \psi(h, x)$. If ψ_x is an automorphism of H, then for any vertex y adjacent to x, we must have $\psi(h, y) = \psi(h, x)$ for all $h \in V(H)$, for otherwise, suppose $\psi(h^*, y) \neq \psi(h^*, x)$, then the mapping f defined as $f(h) = \psi(h, x)$ for $h \neq h^*$, and $f(h^*) = \psi(h^*, y)$ would be a homomorphism from H to H, which is not one to one, contrary to the assumption that H is a core.

Since G is connected, this implies that $\psi_x = \psi_{x'}$ for all $x' \in V(G)$. As ψ_x is an automorphism of H, it follows that ψ is a composition of ϕ with the automorphism ψ_x of H, contrary to choice of ψ.

Therefore for any vertex x of G, ψ_x is not an automorphism of H, i.e., ψ_x is a vertex of $C(H)$. It is easy to see that if xy is an edge of G, then $\psi_x\psi_y$ is an edge of $C(H)$, i.e., $f : V(G) \mapsto C(H)$ defined as $f(x) = \psi_x$ is a homomorphism from G to $C(H)$, contrary to the assumption that G does not admit a homomorphism to $C(H)$. ∎

It is well-known, and also easy to see that $C(K_n)$ is homomorphically equivalent to K_n. Indeed, the constant mappings in $C(K_n)$ induces a copy of K_n in $C(K_n)$, which shows that K_n admits a homomorphism to $C(K_n)$. On the other hand, for any mapping $g \in C(K_n)$, there are two vertices $i \neq j$ of K_n such that $g(i) = g(j)$. Choose any such two vertices i, j and let $f(g) = g(i) = g(j)$. Then it is easy to verify that f is a homomorphism from $C(K_n)$ to K_n. Therefore a graph G admits a homomorphism to $C(K_n)$ if and only if G is n-colorable. Thus the following result of Greenwell and Lovász [2] follows from Theorem 1:

Corollary 1 *([2]) If G is not n-colorable, then $K_n \times G$ is uniquely n-colorable.*

It is easy to see that the odd girth of $H \times G$ is at least as large as the odd girth of G, and that if G has chromatic number larger than the chromatic number of $C(H)$, then G does not admit a homomorphism to $C(H)$. Since there are many known methods to construct graphs of arbitrarily large girth and arbitrarily large chromatic number (cf. [4, 5, 7, 8, 9]), we have the following corollary:

Corollary 2 *If H is a core, then for any integer g, there is a graph G with odd girth at least g such that G is uniquely H-colorable.*

In [13], we have given a probabilistic proof of a stronger result, namely, if H is a core then there exist uniquely H-colorable graphs of arbitrarily large girth. An explicit construction of such graphs for general H remains open.

For a pair of integers k, d such that $k \geq 2d$, let G_k^d be the graph which has vertices $\{0, 1, \cdots, k-1\}$ and in which ij is an edge if and only if $d \leq |i-j| \leq k-d$. A G_k^d-coloring of a graph G is also called a (k,d)-*coloring* of G. The *circular chromatic number* $\chi_c(G)$ of a graph G is defined to be the infimum of the ratios k/d for which there exists a (k,d)-coloring of G. The circular chromatic number is a generalization of the chromatic number, introduced by A. Vince [11] in 1988, as the star chromatic number. It is known [11] that for any graph G we have $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$. Therefore $\chi_c(G)$ is indeed a refinement of $\chi(G)$, and thus $\chi(G)$ is an integral approximation of $\chi_c(G)$. For a given rational number $r \geq 2$, the question whether there exist graphs of arbitrarily large girth and circular chromatic number r was investigated in [13], and a positive answer was obtained by using probabilistic method. Recently, Kirsch [3] gave constructive proof of the result that there exist graphs G of arbitrarily large odd girth with $\chi_c(G) = r$, by using the categorical product of graphs. We note that this result also follows from Corollary 2.

Corollary 3 *([3]) For any rational $k/d \geq 2$ and any integer g, there is a graph G of odd girth at least g such that $\chi_c(G) = k/d$.*

Proof. Suppose $k/d \geq 2$, and that k, d are coprime. It was proved in [1, 11, 14] that G_k^d does not admit homomorphisms to any of its proper subgraphs, i.e., G_k^d is a core. It was proved in [10] that if a graph G is uniquely G_k^d-colorable, then $\chi_c(G) = k/d$. Thus Corollary 3 follows from Corollary 2. ∎

Next we present a different method of constructing triangle free graphs which are uniquely n-colorable. Earlier methods of constructing such graphs were given by Nešetřil [7], and Greenwell and Lovász [2]. The method given here is a slight modification of the method given by Zhou in [12] that constructs triangle free graphs with circular chromatic number n for any integer n. Note that uniquely n-colorable graphs have circular chromatic number n ([10]), therefore the result below is a strengthening of the above mentioned result of [12].

Given an integer m let $\mathcal{G}_i(m)$ be the sets of graphs constructed recursively as follows:

The set $\mathcal{G}_1(m)$ consists of all the graphs which have at least m vertices and no edges. Suppose $\mathcal{G}_i(m)$ is defined, and the vertex set of each graph $G \in \mathcal{G}_i(m)$ is the union of i independent sets $V_1 \cup V_2 \cup \cdots V_i$. Then for each graph $G \in \mathcal{G}_i(m)$, let G' be any graph constructed as follows: for each independent set X of G such that $|X \cap V_j| = 2$ for $j = 1, 2, \cdots, i$, add vertices $x_1, x_2, \cdots, x_{k(X)}$, where $k(X) \geq 1$, and join each x_l to all the $2i$ vertices of X. Here the number $k(X)$ is an arbitrary integer which is at least 1, and $k(X)$ can be different from $k(Y)$ when $X \neq Y$. All the added vertices form an independent set of $G_{i+1}(m)$, which is denoted by V_{i+1}. The set $\mathcal{G}_{i+1}(m)$ consists of all the graphs G' constructed from graphs $G \in \mathcal{G}_i(m)$ in such a way.

We shall prove that when m is large enough, then each graph $G \in \mathcal{G}_n(m)$ is uniquely n-colorable. To be precise, we have the following result:

Theorem 2 *If $m \geq 4n(n-2)+1$, then for any graph $G \in \mathcal{G}_n(m)$, G is triangle free and uniquely n-colorable graph.*

Proof. It can be proved easily by induction on n that each graph G in $\mathcal{G}_n(m)$ is triangle free. We shall also prove by induction on n that each graph G in $\mathcal{G}_n(m)$ is uniquely n-colorable. When $n = 1$ or 2, this is trivial.

Suppose the statement above is false and that n is the minimum integer for which there is a graph $G \in \mathcal{G}_n(m)$ which is not uniquely n-colorable, where $m \geq 4n(n-2)+1$.

Note that there is a trivial n-coloring of G, i.e., the coloring c defined as $c(x) = i$ for all $x \in V_i$. Since G is not uniquely n-colorable, there is another n-coloring c' of G.

By the pigeon hole principle, there is an index j such that $|V_1 \cap c'^{-1}(j)| \geq 4(n-2)+1$. Without loss of generality, we assume that $|V_1 \cap c'^{-1}(1)| \geq 4(n-2)+1$. Let $V_1^* = V_1 \cap c'^{-1}(1)$. For $j = 2, 3, \cdots, n$, we define the sets V_j^* as follows:

$$V_2^* = \{x \in V_2 : |N(x) \cap V_1^*| \geq 1\},$$

where $N(x)$ is the set of vertices of G adjacent to x.

Suppose $V_{j'}^*$ is defined for all $j' < j$. Then

$$V_j^* = \{x \in V_j : |N(x) \cap V_{j'}^*| = 2 \text{ for } j' = 2, \cdots, j-1 \text{ and } |N(x) \cap V_1^*| \geq 1\}.$$

We denote by H the subgraph of G induced by the subset $\cup_{j=2}^n V_j^*$ of vertices.

First we show that H is a member of $\mathcal{G}_{n-1}(4(n-1)(n-3)+1)$.

It follows trivially from the definition that $|V_2^*| > 4(n-1)(n-3)+1$. To prove that $H \in \mathcal{G}_{n-1}(4(n-1)(n-3)+1)$, it suffices to show the following:

(1): for any integer $2 \leq i \leq n-1$, and for any independent set X of $V_2^* \cup V_3^* \cup \cdots \cup V_i^*$ such that $|X \cap V_j^*| = 2$ for $j = 2, 3, \cdots, i$, there exists a vertex $x \in V_{i+1}^*$ such that x is adjacent to each element of X;

(2): each vertex of V_{i+1}^* is adjacent to exactly two vertices of V_j^* for $j = 2, 3, \cdots, i$, and that these $2(i-1)$ vertices form an independent set of H.

Statement (2) follows from the definition. To prove (1), we first show that there are two vertices of V_1, say u and v, such that at least one of u, v is in V_1^* and that $X \cup \{u, v\}$ is an independent set of G. This follows from the fact that each vertex of X is adjacent to at most two vertices of V_1^* and that $|V_1^*| \geq 4(n-2)+1$.

By the way that we construct the graph G, there is a vertex $x \in V_{i+1}$ which is adjacent to each vertex of $X \cup \{u, v\}$. By definition, $x \in V_{i+1}^*$. This proves that H is indeed a member of $\mathcal{G}_{n-1}(m)$. Hence H is uniquely $(n-1)$-colorable by the induction hypothesis.

As each vertex of H is adjacent to a vertex of $V_1 \cap c'^{-1}(1)$, none of the vertices of H is colored by color 1 by c'. Therefore the restriction of c' to H is the unique $(n-1)$-coloring of H. Without loss of generality, we may assume that $c'(x) = i$ for each $x \in V_i^*$ and for $i = 2, 3, \cdots, n$.

Let x be any vertex of V_1. We shall show that for each $i = 2, 3, \cdots, n$, there is a vertex $z \in V_i^*$ such that x is adjacent to z. By the definition of V_i^*, this amounts to proving that there is an independent set X such that $x \in X$ and $X \cap V_1^* \neq \emptyset$, $|X \cap V_1| = 2$ and $|X \cap V_j^*| = 2$ for $j = 2, 3, \cdots, i-1$. The existence of such an independent set is trivial. Indeed, we may choose any $y \in V_1^* - \{x\}$. Let

$S_2 = \{v \in V_2^* : v$ is not adjacent to x and $y\}$.

Then $s_2 = |S_2| \geq \binom{|V_1^*|-2}{2}$. Arbitrarily take two vertices, say x_2, y_2, of S_2. Let

$S_3 = \{v \in V_3^* : v$ is not adjacent to any of $\{x, y, x_2, y_2\}\}$.

An argument similar to that in the third previous paragraph shows that $s_3 = |S_3| \geq \binom{s_2 - 2}{2}$. Repeat this process, we will find an independent set $X = \{x, y, x_2, y_2, x_3, y_3, \cdots, x_{i-1}, y_{i-1}\}$ such that $x_j, y_j \in V_j^*$ and $y \in V_1^*$. Hence there is a vertex $z \in V_i^*$ such that z is adjacent to each vertex of X, and hence adjacent to x. Therefore $c'(x) \neq i$ for $i = 2, 3, \cdots, n$, which implies that $c'(x) = 1$. As x is an arbitrary vertex of V_1, we conclude that $V_1^* = V_1$, and hence $V_j^* = V_j$ for $j = 1, 2, 3, \cdots, n$. This implies that $c = c'$, contrary to our assumption that c' is another n-coloring of G. ∎

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